Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **42** (1), 26-41 (2022).

Extinction properties of solutions for a parabolic equation with a parametric variable exponent nonlinearity

Rabil Ayazoglu (Mashiyev) * · **Ebubekir Akkoyunlu**

Received: 04.05.2021 / Revised: 30.10.2021 / Accepted: 30.11.2021

Abstract. *In this paper, we study a class of $p(\cdot)$-Laplace equation including nonstandard growth nonlinearity in a bounded smooth domain with homogeneous Dirichlet boundary condition. We establish the conditions of non-extinction and extinction are studied of global weak solutions in finite time for any initial data u_0. Moreover, we show the global existence results for $N \geq 1$ with constant p for any initial data u_0.*

Keywords. Parabolic equation, $p(\cdot)$-Laplacian, variable exponent, parametric, non-extinction, extinction, global existence.

Mathematics Subject Classification (2010): 35K35, 35B40, 35K57

1 Introduction

We discuss and determine the non-extinction and extinction for the following parabolic equation involving the $p(\cdot)$-Laplacian operator with parametric variable exponent growth nonlinearity:

$$\begin{cases} u_t = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + \lambda u^{q(x)}, (x,t) \in Q_T, \\ u(x,t) = 0, (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a smooth bounded domain with a smooth boundary $\partial\Omega$, $Q_T := \Omega \times (0,T)$, $\lambda > 0$ is a real parameter, T denotes the maximal existence time of solutions, u_0 is continuous and nonnegative in Ω. Moreover, variable exponents q is measurable and p is log-Hölder continuous (see [7]), that is, there exists a constant $C > 0$ such that, for all $x, y \in \Omega$ and

$$|p(x) - p(y)| \leq \frac{C}{|\ln|x-y||} \tag{1.2}$$

$\star$ Corresponding author

R. Ayazoglu (Mashiyev)
Institute of Mathematics and Mechanics of ANAS, Baku, Azerbaijan
Faculty of Education, Bayburt University, Bayburt, Turkey
E-mail: rabilmashiyev@gmail.com

E. Akkoyunlu
Faculty of Education, Bayburt University, Bayburt, Turkey
E-mail: eakkoyunlu@bayburt.edu.tr

for $|x-y| \leq \frac{1}{2}$.

Let p, q satisfy that

$$1 < p^{-} := \inf_{x\in\Omega} p(x) \leq p(x) \leq \sup_{x\in\Omega} p(x) := p^{+} < 2, \tag{1.3}$$

and

$$0 < q^{-} := \inf_{x\in\Omega} q(x) \leq q(x) \leq \sup_{x\in\Omega} q(x) := q^{+} < 1. \tag{1.4}$$

We denote by $P(\Omega)$ the set of all measurable real functions defined on Ω and $C^{0,\frac{1}{|\ln(.)|}}(\overline{\Omega}) := P_{\ln}(\Omega)$ the set of all $p \in P(\Omega)$ satisfying the conditions (1.2) and (1.3).

Nonlinear parabolic equations with nonstandard growth conditions of the type (1.1) appear in various applications such as the mathematical modeling of heat and mass transfer in nonhomogeneous media, in description of the filtration processes, in the processes of recovery of digital images (see [1, 14, 18–20] and the references therein for an account of such models in the stationary case). For the sake of presentation, we will regard problem (1.1) as the mathematical model of a diffusion process.

The questions we address in this paper are already studied for the evolutional p-Laplacian equation

$$u_t = \Delta_p u \equiv \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right),\ p \in (1,\infty). \tag{1.5}$$

It is well known that Eq. (1.5) is degenerate if $p > 2$ or singular if $1 < p < 2$, since the modulus of ellipticity is degenerate $(p > 2)$ or blows up $(1 < p < 2)$ at points where $\nabla u = 0$, and therefore there is no classical solution in general. Unlike the linear case, for $p \neq 2$ the solutions of the Dirichlet problem for Eq. (1.5) are localized either in space, or in time. More precisely, the following alternative holds: if u is a solution of the Dirichlet problem for Eq. (1.5) with $p \neq 2$, then either

1) $1 < p < 2$ (fast diffusion)$\Longrightarrow \exists T_1 : u \equiv 0$ for all $t \geq T_1$,

2) $p > 2$ (slow diffusion) and $u_0 \equiv 0$ in

$$B_r(x_0) = \left\{x \in \mathbb{R}^N : |x-x_0| < r\right\} \Longrightarrow \exists t^*(x_0) : u(x_0,t) \equiv 0$$

for all $t \in [0, t^*(x_0)]$. These properties complement each other: the former is called extinction in a finite time, the latter is usually referred to as finite speed of propagation of disturbances from the data. If $p > 2$ and the support of the initial function u_0 is compact in Ω, then the support of the solution is expanding with time and eventually covers the whole of ΩRecently, many paper studied for parabolic problems with nonstandart growth (see [2–6, 9, 10, 12, 16]).

Note that, problem (1.1) appears in a lot of applications to describe the evolution of diffusion processes, in particular, fast diffusion for $1 < p(\cdot) < 2$. In combustion theory, for instance, the function $u(.,t)$ represents the temperature, the term $\Delta_{p(\cdot)}u \equiv \operatorname{div}\left(|\nabla u|^{p(\cdot)-2}\nabla u\right)$ represents the thermal diffusion, and $u^{q(.)}$ is a source.

When $p(\cdot) \equiv p$ and $q(\cdot) \equiv q$ are constants in problem (1.1), the problem (1.1) is turning the following p-Laplacian parabolic equation:

$$\begin{cases} u_t = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + \lambda u^q, (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), x \in \Omega, \end{cases} \tag{1.6}$$

where $\Omega \in \mathbb{R}^N$, $N \geq 2$ is an open bounded domain with smooth boundary. In [23], the authors investigated the problem (1.6) with $1 < p < 2$, and λ, $q > 0$. They showed that if $q > p-1$, then any bounded and non-negative weak solution of problem (1.6) vanishes in finite time for appropriately small initial data u_0. They showed that $q = p - 1$ is the critical exponent of extinction for the weak solution. Furthermore, for $1 < p < 2$ and $q = p - 1$ they proved the extinction and non-extinction conditions.

In [15], the authors emphasized that the small condition on the initial data u_0 in [23] can be removed for the case $p - 1 < q < 1$. Accurate estimates of the decay of the solution were also obtained.

In [21], Tian and Mu dealt with the extinction of solutions of the initial-boundary value problem of the p-Laplacian equation

$$u_t = \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) + \lambda u^q$$

in a bounded domain of $\mathbb{R}^N$ with $N \geq 2$. For $1 < p < 2$, $\lambda > 0$, $q > 0$ and $0 \leq u_0 \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$ the authors showed that $q = p - 1$ is the critical exponent of extinction for the weak solution.

Problem (1.6) with $p > 1$ and $q > 0$ has been investigated extensively in recent years. For $1 < p \leq 2$, the conditions on quenching or extinction were studied in [11,17,22].

In this note, we establish the non-extinction and extinction results for a nonlinear parabolic problem involving $p(\cdot)$-Laplacian operator subject to homogeneous Dirichlet boundary conditions. Namely, we prove energy estimate and the comparison principle of the ordinary differential equation to study the non-extinction or extinction of solutions for any initial data u_0, also establish the precise decay estimates of solution. Moreover, we show the global existence result for $N \geq 1$ with constant p for any initial data u_0.

Let $h : \Omega \to (1, \infty)$ be a measurable function in Ω. We define the Lebesgue space with variable exponent as usual,

$$L^{h(\cdot)}(\Omega) := \left\{ u : u \in P(\Omega), \int_\Omega |u(x)|^{h(x)}\, dx < +\infty \right\}.$$

The set $L^{h(\cdot)}(\Omega)$ equipped with the Luxemburg norm

$$\|u\|_{L^{h(\cdot)}(\Omega)} := \|u\|_{h(\cdot)} = \inf\left\{ \gamma > 0 : \int_\Omega \left|\frac{u(x)}{\gamma}\right|^{h(x)} dx \leq 1 \right\},$$

becomes a Banach space. The modular of $L^{h(\cdot)}(\Omega)$, which is the mapping $\rho_{h(\cdot)} : L^{h(\cdot)}(\Omega) \to \mathbb{R}$, is defined

$$\rho_{h(\cdot)}(u) := \int_\Omega |u(x)|^{h(x)}\, dx < +\infty.$$

We define the Sobolev space with a variable exponent $W^{1,h(\cdot)}(\Omega)$ as a linear space of functions $u \in L^{h(\cdot)}(\Omega)$, such that $\nabla u \in L^{h(\cdot)}(\Omega)$ with the norm

$$\|u\|_{W^{1,h(\cdot)}(\Omega)} = \|u\|_{h(.)} + \|\nabla u\|_{h(\cdot)}, \, u \in W^{1,h(\cdot)}(\Omega).$$

Note that $C^{0,1}(\overline{\Omega}) \hookrightarrow C^{0,\frac{1}{|\ln(\cdot)|}}(\overline{\Omega})$. Also, when $h \in C^{0,\frac{1}{|\ln(\cdot)|}}(\overline{\Omega})$, then $W_0^{1,h(\cdot)}(\Omega) := \overline{C_0^\infty(\overline{\Omega})}^{W^{1,h(\cdot)}(\Omega)}$. Furthermore, for all $u \in W_0^{1,h(\cdot)}(\Omega)$, we can define an equivalent norm $\|u\|_{W_0^{1,h(\cdot)}(\Omega)}$ such that

$$\|u\|_{W_0^{1,h(\cdot)}(\Omega)} := \|u\|_0 = \|\nabla u\|_{h(\cdot)}.$$

Moreover, it is well known that if $1 < h^- \leq h^+ < \infty$, then spaces $\left(L^{h(\cdot)}(\Omega), \|.\|_{h(\cdot)}\right)$ and $\left(W_0^{1,h(\cdot)}(\Omega), \|.\|_{W_0^{1,h(\cdot)}(\Omega)}\right)$ are separable and reflexive Banach spaces. We refer to [7] for further properties of variable exponent Lebesgue-Sobolev spaces.

We could get the following properties:

Proposition 1.1 *(see [7]). If $1 < h^- \leq h^+ < \infty$ is satisfied, then for any $u \in L^{h(\cdot)}(\Omega)$ the following inequalities are provided.*

$(i)\min\left\{\|u\|_{h(\cdot)}^{h^-}, \|u\|_{h(\cdot)}^{h^+}\right\} \leq \rho_{h(\cdot)}(u) \leq \max\left\{\|u\|_{h(\cdot)}^{h^-}, \|u\|_{h(\cdot)}^{h^+}\right\}$;

$(ii)\ \|u\|_{h(\cdot)}^{h^-} - 1 \leq \rho_{h(\cdot)}(u) \leq \|u\|_{h(\cdot)}^{h^+} + 1.$

Proposition 1.2 *(Hölder-type inequality, see [7]). Let $h \in L_+^\infty(\Omega)$.*

(i)The conjugate space of $L^{h(\cdot)}(\Omega)$ is $L^{h'(\cdot)}(\Omega)$, where $1/h(x) + 1/h'(x) = 1$ for almost every (a.e.) $x \in \Omega$. Moreover, the following inequality hold

$$\left|\int_\Omega u(x)v(x)dx\right| \leq 2\|u\|_{h(\cdot)}\|v\|_{h'(\cdot)},$$

for all $u \in L^{h(\cdot)}(\Omega)$ and $v \in L^{h'(\cdot)}(\Omega)$.

(ii) If $p_1, p_2 \in L_+^\infty(\Omega)$, $p_1(x) \leq p_2(x)$ for any $x \in \overline{\Omega}$, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$, and the embedding is continuous.

Proposition 1.3 *(see [8],[13]). Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $p \in P_{\ln}(\Omega)$. Let $q : \Omega \to [1, +\infty)$ be a measurable and bounded function and suppose that $q(x) \leq p^*(x) = Np(x)/(N - p(x))_+$ for a.e. $x \in \Omega$. Then $W^{1,p(\cdot)}(\Omega)$ is continuously embedded in $L^{q(\cdot)}(\Omega)$. In addition, assume that $\operatorname{ess\,inf}_{x\in\Omega}\{p^*(x) - q(x)\} > 0$. Then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is compact.*

In particular, if $p^- > \frac{2N}{N+2}$, then there exists a positive constant K_0 such that

$$\|u\|_2 \leq K_0\|u\|_{W_0^{1,p(\cdot)}(\Omega)}, \ \forall u \in W_0^{1,p(\cdot)}(\Omega). \tag{1.7}$$

We further, set

$$K = \max\{1, K_0\}, \tag{1.8}$$

where K_0 is the embedding constant of the (1.7).

Definition 1.1 *We define a function $u \in L^\infty(0, T; W_0^{1,p(\cdot)}(\Omega)\} \cap C\left([0, T], L^2(\Omega)\right))$ with $u_t \in L^2(0, T; L^2(\Omega))$ to be a weak solution of problem (1.1), if it satisfies the initial condition $u(., 0) := u_0 \in L^\infty(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)$, and*

$$(u_t, v) + \left(|\nabla u|^{p(x)-2}\nabla u, \nabla v\right) = \left(\lambda |u|^{q(x)}, v\right),$$

for all $v \in W_0^{1,p(\cdot)}(\Omega)$, and for a.e. $t \in (0, T)$.

Definition 1.2 *Let $u = u(t)$ be a global solution of problem (1.1), we say that u vanishes in finite time if there exists a $t_0 \in (0, +\infty)$ such that $\lim_{t\to t_0^-} u(t)(x) = 0$ for a.e. $x \in \Omega$.*

Definition 1.3 *A function* $u \in L^{\infty}(0,T;W_0^{1,p(\cdot)}(\Omega)\} \cap C\left([0,T], L^2(\Omega)\right))$ *with* $u_t \in L^2(0,T;L^2(\Omega))$ *is called to be a weak upper solution of problem* (1.1) *provided that for any* $T>0, \lambda>0$ *and any* $0 \leq \upsilon \in E$

$$\begin{cases} \int_{Q_T} u_t \upsilon dx dt + \int_{Q_T} |\nabla u|^{p(x)-2} \nabla u \nabla \upsilon dx dt \geq \lambda \int_{Q_T} u^{q(x)} \upsilon dx dt, \\ \quad u(x,t) \geq 0,\ x \in \partial\Omega \times (0,T), \\ \quad u(x,0) \geq u_0(x),\ x \in \Omega, \end{cases}$$

where $E = \left\{ u \in L^{\infty}(0,T;W_0^{1,p(\cdot)}(\Omega)\} \cap C\left([0,T], L^2(\Omega)\right)) : u|_{\partial\Omega} = 0 \right\}$.

Similarly, a weak lower solution u is defined by replacing "$\geq$" as "$\leq$" in the above inequalities. Furthermore, if u is a weak upper solution as well as a weak lower solution, then we call it a weak solution of problem (1.1) (see for example [17]).

2 Main Results

Let us introduce the functions

$$E(t) = \int_{\Omega} \frac{|\nabla u(x,t)|^{p(x)}}{p(x)} dx - \lambda \int_{\Omega} \frac{|u(x,t)|^{q(x)+1}}{q(x)+1} dx, \tag{2.1}$$

for all $u \in W_0^{1,p(\cdot)}(\Omega)$, and

$$F(t) = \int_{\Omega} u^2 dx, \tag{2.2}$$

for all $t>0$.

Multiplying the Eq. (1.1) by u_t, integrating by parts and using the fact that

$$E'(t) = \frac{d}{dt}E(t) = -\int_{\Omega} u_t^2 dx \leq 0,$$

which implies that $E(t) \leq E(0)$. ($E(t)$-nonincreasing).

Our main results can now be stated as follows.

Theorem 2.1 *(Non-extinction of global weak solutions). Assume that* $p \in P_{\ln}(\Omega)$, $q \in P(\Omega)$, $0 \leq u_0 \in L^{\infty}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)$ *and the following conditions* (1.3) *and* (1.4) *hold.*

i) If $\lambda \in \left(\frac{p^+}{q^-+1} + p^+E(0), \frac{p^+}{q^-+1}\right)$, $\lambda \neq 1$ *and* $\frac{-1}{q^-+1} < E(0) < 0$, *the non-negative weak solution of problem* (1.1) *does not go extinct in finite time for any initial data* u_0. *Furthermore, we have the following estimate:*

$$\|u(t)\|_2^2 \geq \min\left\{ \|u_0\|_2^2, \left(\frac{A_1}{A_0}\right)^{\frac{2}{q^++1}} \right\},$$

for $0<t<T$, *where* A_0 *and* A_1 *are positive constants which will be determined later.*

ii) If $\lambda = \frac{p^+}{q^-+1}$, $\lambda \neq 1$ *and* $E(0)<0$, *the non-negative weak solution of problem* (1.1) *does not go extinct in finite time for any initial data* u_0. *Furthermore, we have the following estimate:*

$$\|u(t)\|_2^2 \geq \|u_0\|_2^2 - 2p^+E(0)t,$$

for $0<t<T$.

$iii)$ Assume that

$$1 < q^- + 1 \leq q^+ + 1 \leq p^+ < 2.$$

If $\lambda = 1$ and $E(0) < 0$, the non-negative weak solution of problem (1.1) does not go extinct in finite time for any initial data u_0. Furthermore, we have the following estimate:

$$\|u(t)\|_2^2 \geq \min\left\{\|u_0\|_2^2, \left(\frac{D_1}{D_0}\right)^{\frac{2}{q^+ + 1}}\right\},$$

for $0 < t < T$, where D_0 and D_1 are positive constants which will be determined later.

Theorem 2.2 *(Extinction of global weak solutions). Assume that $p \in P_{\ln}(\Omega)$, $q \in P(\Omega)$, $0 \leq u_0 \in L^\infty(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)$ and the following condition holds*

$$\frac{2N}{N+2} < p^- \leq p^+ < q^- + 1 \leq q^+ + 1 < 2, \tag{2.3}$$

then the non-negative weak solution of problem (1.1) vanishes in finite time for any initial data u_0. More precisely speaking, we have the following estimates

$$\begin{cases} \|u(t)\|_2^{2-p^-} \leq \|u_0\|_2^{2-p^-} + \Im(\|u_0\|_2)\, t - F(\|u_0\|_2)\, t, t \in (0, T_0), \\ \|u(t)\|_2 \equiv 0, t \in [T_0, +\infty), \end{cases}$$

where

$$\Im(\|u_0\|_2) := 2\lambda\left(2 - p^-\right)(|\Omega| + 1)^{(1-q^-)/2} \max\left\{\|u_0\|_2^{q^- - p^- + 1}, \|u_0\|_2^{q^+ - p^- + 1}\right\},$$

$$F(\|u_0\|_2) := \left(2 - p^-\right) K^{p^+} \min\left\{1, \|u_0\|_2^{p^+ - p^-}\right\},$$

and

$$\lambda \in \left(0, \frac{\min\left\{1, \|u_0\|_2^{p^+ - p^-}\right\}}{2(|\Omega| + 1)^{(1-q^-)/2} K^{p^+} \max\left\{\|u_0\|_2^{q^- - p^- + 1}, \|u_0\|_2^{q^+ - p^- + 1}\right\}}\right),$$

$$T_0 = \frac{\|u_0\|_2^{2-p^-}}{F(\|u_0\|_2) - \Im(\|u_0\|_2)},$$

and K is a constant given in (1.8).

Theorem 2.3 *(Extinction of global weak solutions). Assume that $p \in P_{\ln}(\Omega)$, $q \in P(\Omega)$ with (2.3) and $0 \leq u_0 \in L^\infty(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)$. Then the non-negative weak solution of problem (1.1) vanishes in finite time for any initial data u_0, $B > 0$ and*

$$\begin{cases} \|u(t)\|_2^2 \leq Be^{-\sigma t}, t \in [0, T_1), \\ \|u(t)\|_2 \leq \left(\|u_0\|_2^{2-p^-} - \frac{K_1(2-p^-)}{2} t\right)^{\frac{1}{2-p^-}}, t \in [T_1, T_2), \\ \|u(t)\|_2 \equiv 0, t \in [T_2, +\infty) \end{cases}$$

for some T_1, where

$$K_1 = 2K^{p^+} - 4\lambda(|\Omega| + 1)^{(1-q^-)/2}\left(Be^{-\sigma T_1}\right)^{\frac{q^- + 1 - p^+}{2}} > 0,$$

with

$$\lambda \in \left(0, \frac{K^{p^+} \min\left\{\|u_0\|_2^{p^-}, \|u_0\|_2^{p^+}\right\}}{2\left(|\Omega|+1\right)^{(1-q^-)/2} \max\left\{\|u_0\|_2^{q^-+1}, \|u_0\|_2^{q^++1}\right\}}\right),$$

$$\sigma = \frac{K^{p^+}}{\min\left\{\|u_0\|_2^{2(1-p^-)}, \|u_0\|_2^{2(1-p^+)}\right\}},$$

and

$$T_2 = \frac{2\left\|u(.,T_1)\right\|_2^{2-p^-}}{K_1\left(2-p^-\right)},$$

and K is a constant given in (1.8).

In this Theorem, we give some global existence results of the solution of problem (1.1) for $N \geq 1$ with constant $p(x) \equiv p$ by making use of sub and super solution techniques. Let $\varphi(x)$ satisfies the following elliptic problem:

$$\begin{cases} -\text{div}\left(|\nabla\varphi(x)|^{p-2}\nabla\varphi(x)\right) = 1 \text{ in } x \in \Omega, \\ \varphi(x) = 1 \text{ on } x \in \partial\Omega. \end{cases} \tag{2.4}$$

By using the result in [24], we can see that the above nonlinear problem has a unique solution, and the following inequalities hold:

$$M := \sup_{x\in\Omega}\varphi(x) < +\infty, \varphi(x) > 1 \text{ and } \nabla\varphi\cdot\nu < 0, x \in \partial\Omega,$$

where ν is the unit outer normal vector on $\partial\Omega$ and M is a positive constant.

Theorem 2.4 *(Global existence). Let $u(x,t)$ be the solution of problem* (1.1).
(i) *For any initial data u_0, if $p > q^+ + 1$ and $\lambda > 0$, then $u(x,t)$ exists globally;*
(ii) *For any initial data u_0, if $p < q^+ + 1$ and $\lambda > 0$, then $u(x,t)$ exists globally;*
(iii) *For any initial data u_0, if $p = q^+ + 1$ and*

$$0 < \lambda \leq M^{-q^+},$$

then $u(x,t)$ exists globally.

Theorem 2.1 implies that, when $1 < q^- + 1 \leq q^+ + 1 < p^+ < 2$, the nonlinear diffusion dominates the property of weak solutions, which have some positive lower bound at any finite time, provided that $0 < q^- \leq q^+ < 1$. The condition $\frac{2N}{N+2} < p^+ < q^- + 1 < 2$ in Theorem 2.2, Theorem 2.3 means that the effect of reaction on the solutions is higher than the diffusion.

3 Proof of the Results

Now, we give some lemmas, which will be needed for proof of the Theorem 2.1.

Lemma 3.1 *(lemma 1.2 in [11]). Suppose that constants $d > 0$, $\alpha > 0$, $\beta > 0$ and h is a nonnegative and absolutely continuous function satisfying that*

$$h'(t) + \alpha h^d(t) \geq \beta, t \in (0,+\infty).$$

Then there exists an estimate as follows:

$$h(t) \geq \min\left\{h(0), \left(\frac{\beta}{\alpha}\right)^{1/d}\right\}.$$

Proof of Theorem 2.1. Multiplying the Eq. (1.1) by u, integrating over Ω and from (2.1) with $E(t) \leq E(0) < 0$, we have

$$\begin{aligned} F'(t) &= 2\int_\Omega uu_t dx = 2\lambda \int_\Omega |u|^{q(x)+1} dx - 2\int_\Omega |\nabla u|^{p(x)} dx \\ &\geq 2\left(\lambda - \frac{p^+}{q^- + 1}\right)\int_\Omega |u|^{q(x)+1} dx - 2p^+ E(t) \\ &\geq 2\left(\lambda - \frac{p^+}{q^- + 1}\right)\int_\Omega |u|^{q(x)+1} dx - 2p^+ E(0). \end{aligned} \tag{3.1}$$

We consider the following three cases:

i) Let $\lambda \in \left(\frac{p^+}{q^-+1} + p^+ E(0), \frac{p^+}{q^-+1}\right)$ such that $\frac{-1}{q^-+1} < E(0) < 0$. Since $q(x) + 1 < 2$, $\forall x \in \Omega$ and by Proposition 1.1 (i), Proposition 1.2 (ii), we obtain

$$\begin{aligned} &\int_\Omega u^{q(x)+1} dx \leq \|u\|_{q(\cdot)+1}^{q^+ +1} + 1 \\ &\leq B^{q^+ +1} \|u\|_2^{q^+ +1} + 1 = B^{q^+ +1} F^{\frac{q^+ +1}{2}}(t) + 1, \end{aligned} \tag{3.2}$$

where B is the embedding constant of the embedding $L^2(\Omega) \hookrightarrow L^{q(\cdot)+1}(\Omega)$. From (3.1) and (3.2), we obtain

$$F'(t) + 2\left(\frac{p^+}{q^- + 1} - \lambda\right)\left(B^{q^+ +1} F^{\frac{q^+ +1}{2}}(t) + 1\right) \geq -2p^+ E(0),$$

so

$$F'(t) + A_0 F^{\frac{q^+ +1}{2}}(t) \geq A_1, \tag{3.3}$$

where

$$A_0 = 2B^{q^+ +1}\left(\frac{p^+}{q^- + 1} - \lambda\right) > 0,$$

and

$$A_1 = 2\left(\lambda - \frac{p^+}{q^- + 1}\right) - 2p^+ E(0) > 0$$

with $\frac{-1}{q^-+1} < E(0) < 0$. Lemma 3.1 and (3.3) imply

$$F(t) \geq \min\left\{F(0), \left(\frac{A_1}{A_0}\right)^{\frac{2}{q^+ +1}}\right\}, t > 0.$$

Since $F(0) = \|u_0\|_2^2 > 0$, we derive $F(t) > 0$ for all $t \in (0, T)$.

ii) Let $\lambda = \frac{p^+}{q^-+1}$. Using (3.3) with $E(0) < 0$, it easily follows that

$$F(t) \geq F(0) - 2p^+ E(0)t > 0$$

for all $t \in (0, T)$.

iii) *If* $\lambda = 1$, $q^- + 1 < p^+$ and $E(0) < 0$. Using (3.3), we obtain

$$F'(t) + 2\left(\frac{p^+ - q^- - 1}{q^- + 1}\right) B^{q^+ +1} F^{\frac{q^+ +1}{2}}(t) \geq 2\left(\frac{p^+ - q^- - 1}{q^- + 1}\right) - 2p^+ E(0),$$

then

$$F'(t) + D_0 F^{\frac{q^+ + 1}{2}}(t) \geq D_1, \tag{3.4}$$

By Lemma 3.1 and (3.4) imply

$$\|u(t)\|_2^2 \geq \min\left\{\|u_0\|_2^2, \left(\frac{D_1}{D_0}\right)^{\frac{2}{q^+ + 1}}\right\},$$

where

$$D_0 = 2B^{q^+ + 1}\left(\frac{p^+ - q^- - 1}{q^- + 1}\right) > 0,$$

and

$$D_1 = 2\left(\frac{p^+ - q^- - 1}{q^- + 1}\right) - 2p^+ E(0) > 0.$$

The above three cases imply $\|u(.,t)\|_2^2 = F(t) > 0$ for all $t > 0$. Then for any $s > 1$, by interpolation inequality, we obtain

$$\|u\|_2 \leq \|u\|_s^{\frac{1}{2}} \|u\|_{s'}^{\frac{1}{2}},$$

where $s' = s/(s-1) > 1$, which combines with $\|u(.,t)\|_2 > 0$ imply that every $s > 1$, there does not exist $T^* > 0$ such that

$$\lim_{t \to T^*} \|u\|_s = 0.$$

Thus the proof of Theorem 2.1 is complete.

Proof of Theorem 2.2

In order to obtain the extinction properties of weak solutions, we introduce an auxiliary lemma on the ordinary differential inequality as follows.

Lemma 3.2 *Assume that* $0 < l_1 \leq l_2 < r_1 \leq r_2 \leq 1$ *and* $\alpha \geq 0$, $\beta \geq 0$ *and* φ *is a nonnegative and absolutely continuous function, which satisfies*

$$\begin{array}{l} \varphi'(t) + \alpha \min\left\{\varphi^{l_1}(t), \varphi^{l_2}(t)\right\} \leq \beta \max\left\{\varphi^{r_1}(t), \varphi^{r_2}(t)\right\}, t \geq 0, \\ \varphi(0) > 0,\ \beta \max\left\{\varphi^{r_1 - l_1}(0), \varphi^{r_2 - l_1}(0)\right\} < \alpha \min\left\{1, \varphi^{l_2 - l_1}(0)\right\}, \end{array}$$

then φ *holds*

$$\begin{cases} \varphi(t) \leq \left[\varphi^{1-l_1}(0) - \alpha_0 (1 - l_1) t\right]^{\frac{1}{1-l_1}}, 0 < t < T_0, \\ \quad \varphi(t) \equiv 0, t \geq T_0, \end{cases}$$

where

$$\alpha_0 = \alpha \min\left\{1, \varphi^{l_2 - l_1}(0)\right\} - \beta \max\left\{\varphi^{r_1 - l_1}(0), \varphi^{r_2 - l_1}(0)\right\} > 0,$$

and

$$T_0 = \alpha_0^{-1} (1 - l_1)^{-1} \varphi^{1 - l_1}(0) > 0.$$

Proof of Lemma 3.2. For $t \geq 0$, we have

$$\varphi'(t) \leq -\alpha\varphi^{l_1}(t)\left[\min\left\{1, \varphi^{l_2-l_1}(t)\right\} - \frac{\beta}{\alpha}\max\left\{\varphi^{r_1-l_1}(t), \varphi^{r_2-l_1}(t)\right\}\right]. \tag{3.5}$$

Since

$$\frac{\beta \max\left\{\varphi^{r_1-l_1}(0), \varphi^{r_2-l_1}(0)\right\}}{\min\left\{1, \varphi^{l_2-l_1}(0)\right\}} < \alpha,$$

there exists a sufficiently small constant $\varepsilon > 0$ such that

$$\frac{\beta \max\left\{\varphi^{r_1-l_1}(t), \varphi^{r_2-l_1}(t)\right\}}{\min\left\{1, \varphi^{l_2-l_1}(t)\right\}} < \alpha, t \in [0, \varepsilon],$$

and $\varphi(t)$ is decreasing in $[0, \varepsilon]$. Noticing that $r_1 - l_1 > 0$ and $r_2 - l_1 > 0$. Therefore, we have

$$\alpha > \frac{\beta \max\left\{\varphi^{r_1-l_1}(\varepsilon), \varphi^{r_2-l_1}(\varepsilon)\right\}}{\min\left\{1, \varphi^{l_2-l_1}(\varepsilon)\right\}} > 0.$$

From (3.5) we obtain

$$\varphi'(t) \leq -\alpha_0\varphi^{l_1}(t), \tag{3.6}$$

where

$$\alpha_0 = \alpha \min\left\{1, \varphi^{l_2-l_1}(0)\right\} - \beta \max\left\{\varphi^{r_1-l_1}(0), \varphi^{r_2-l_1}(0)\right\} > 0.$$

Then integrating (3.6) from 0 to t, we have

$$\varphi^{1-l_1}(t) \leq \varphi^{1-l_1}(0) - \alpha_0 (1 - l_1) t.$$

Thus, from $\varphi(t) \geq 0$, we get

$$\begin{cases} \varphi(t) \leq \left[\varphi^{1-l_1}(0) - \alpha_0 (1 - l_1) t\right]^{\frac{1}{1-l_1}}, 0 < t < T_0, \\ \varphi(t) \equiv 0, t \geq T_0, \end{cases}$$

where

$$T_0 = \alpha_0^{-1} (1 - l_1)^{-1} \varphi^{1-l_1}(0) > 0.$$

Thus the proof of Lemma 3.2 is complete.

Proof of Theorem 2.2. By using (3.1), we have

$$F'(t) + 2\int_\Omega |\nabla u|^{p(x)}\, dx = 2\lambda \int_\Omega |u|^{q(x)+1}\, dx.$$

Furthermore, by using (2.2), Proposition 1.1 and Proposition 1.3 we obtain

$$\begin{aligned} &2\int_\Omega |\nabla u|^{p(x)}\, dx \\ &\geq 2\min\left\{\|u\|_0^{p^-}, \|u\|_0^{p+}\right\} \geq \alpha_1 \min\left\{\|u\|_2^{p^-}, \|u\|_2^{p^+}\right\} \\ &= \alpha_1 \min\left\{F^{\frac{p^-}{2}}(t), F^{\frac{p^+}{2}}(t)\right\}, \end{aligned} \tag{3.7}$$

where

$$\alpha_1 = 2K^{-p^+}.$$

By Proposition 1.2 we have

$$
\begin{aligned}
2\lambda \int_{\Omega} |u|^{q(x)+1}\, dx &\le 4\lambda \left\| |u|^{q(.)+1} \right\|_{\frac{2}{q(.)+1}} \|1\|_{\frac{2}{1-q(\cdot)}} \\
&\le \lambda\beta_1 \max\left\{ \|u\|_2^{q^-+1}, \|u\|_2^{q^++1} \right\} \\
&= \lambda\beta_1 \max\left\{ F^{\frac{q^-+1}{2}}(t), F^{\frac{q^++1}{2}}(t) \right\}, \qquad (3.8)
\end{aligned}
$$

where

$$
\beta_1 = 4\left(|\Omega| + 1\right)^{(1-q^-)/2}.
$$

By (3.7) and (3.8), we arrive at the following relation

$$
F'(t) + \alpha_1 \min\left\{ F^{\frac{p^-}{2}}(t), F^{\frac{p^+}{2}}(t) \right\} \le \lambda\beta_1 \max\left\{ F^{\frac{q^-+1}{2}}(t), F^{\frac{q^++1}{2}}(t) \right\} \qquad (3.9)
$$

with $0 < \frac{p^-}{2} \le \frac{p^+}{2} < \frac{q^-+1}{2} \le \frac{q^++1}{2} < 1$. By using Lemma 3.2, we obtain

$$
F'(t) \le -\alpha_0 F^{\frac{p^-}{2}}(t), \qquad (3.10)
$$

where

$$
\alpha_0 = \alpha_1 \min\left\{ 1, F^{\frac{p^+-p^-}{2}}(0) \right\} - \lambda\beta_1 \max\left\{ F^{\frac{q^--p^-+1}{2}}(0), F^{\frac{q^+-p^-+1}{2}}(0) \right\} > 0,
$$

with

$$
0 < \lambda < \frac{\alpha_1 \min\left\{ 1, F^{\frac{p^+-p^-}{2}}(0) \right\}}{\beta_1 \max\left\{ F^{\frac{q^--p^-+1}{2}}(0), F^{\frac{q^+-p^-+1}{2}}(0) \right\}},
$$

that is

$$
F(t) \le \left(F^{\frac{2-p^-}{2}}(0) - \frac{\alpha_0\left(2-p^-\right)}{2} t \right)^{\frac{2}{2-p^-}}, t \ge 0. \qquad (3.11)
$$

Thus, from $F(t) \ge 0$ with $F(0) > 0$, we get

$$
\begin{aligned}
&F^{\frac{2-p^-}{2}}(t) \le F^{\frac{2-p^-}{2}}(0) \\
&-\frac{2-p^-}{2}\left(\alpha_1 \min\left\{ 1, F^{\frac{p^+-p^-}{2}}(0) \right\} + \lambda\beta_1 \max\left\{ F^{\frac{q^--p^-+1}{2}}(0), F^{\frac{q^+-p^-+1}{2}}(0) \right\} \right) t
\end{aligned}
$$

for $t \in (0, T_0)$, and

$$
F(t) \equiv 0
$$

for $t \in [T_0, +\infty)$, where

$$
T_0 = \frac{2F^{\frac{2-p^-}{2}}(0)}{\left(2-p^-\right)\left(\alpha_1 \min\left\{ 1, F^{\frac{p^+-p^-}{2}}(0) \right\} - \lambda\beta_1 \max\left\{ F^{\frac{q^--p^-+1}{2}}(0), F^{\frac{q^+-p^-+1}{2}}(0) \right\} \right)}.
$$

Thus the proof of Theorem 2.2 is complete.

Proof of Theorem 2.3

We introduce an auxiliary lemma on the ordinary differential inequality as follows.

Lemma 3.3 *Assume that* $0 < l_1 \leq l_2 < r_1 \leq r_2 \leq 1$*, and* $\eta > 0$*,* $\mu > 0$ *and* $\varphi(t) \geq 0$ *is a solution of the differential inequality*

$$\begin{cases} \varphi'(t) + \eta \min\left\{\varphi^{l_1}(t), \varphi^{l_2}(t)\right\} \leq \mu \max\left\{\varphi^{r_1}(t), \varphi^{r_2}(t)\right\}, t \geq 0, \\ \varphi(0) = \varphi_0 > 0, \end{cases} \tag{3.12}$$

where $\eta > 0$ *and*

$$\mu \leq \frac{\min\left\{\varphi_0^{l_1}, \varphi_0^{l_2}\right\}}{\max\left\{\varphi_0^{r_1}, \varphi_0^{r_2}\right\}} \left(\eta - \sigma \min\left\{\varphi_0^{1-l_1}, \varphi_0^{1-l_2}\right\}\right),$$

and

$$\sigma = \frac{\eta}{2 \min\left\{\varphi_0^{1-l_1}, \varphi_0^{1-l_2}\right\}}.$$

Then there exists $B > 0$ *such that*

$$0 \leq \varphi(t) \leq Be^{-\sigma t}, t \geq 0.$$

Proof of Lemma 3.3. Since $\varphi(t) \equiv 0$ is a subsolution of (3.12), we only need to choose σ, B properly such that $\varphi(t) = Be^{-\sigma t}$ is a supersolution of (3.12). In fact, we first choose $B = \varphi(0) = \varphi_0 > 0$. Then, we obtain

$$\begin{aligned} &-\sigma Be^{-\sigma t} + \eta \min\left\{B^{l_1} e^{-\sigma l_1 t}, B^{l_2} e^{-\sigma l_2 t}\right\} \\ &\geq \mu \max\left\{B^{r_1} e^{-\sigma r_1 t}, B^{r_2} e^{-\sigma r_2 t}\right\}, \forall t \geq 0. \end{aligned}$$

Then

$$-\sigma Be^{-\sigma t} + \eta \min\left\{B^{l_1}, B^{l_2}\right\} e^{-\sigma l_2 t} \geq \mu \max\left\{B^{r_1}, B^{r_2}\right\} e^{-\sigma r_1 t},$$

that is

$$\eta \min\left\{B^{l_1}, B^{l_2}\right\} e^{-\sigma l_2 t} \geq \mu \max\left\{B^{r_1}, B^{r_2}\right\} e^{-\sigma r_1 t} + \sigma Be^{-\sigma t},$$

or

$$\eta \min\left\{B^{l_1}, B^{l_2}\right\} e^{\sigma(r_1 - l_2)t} \geq \mu \max\left\{B^{r_1}, B^{r_2}\right\} + \sigma Be^{-\sigma(1-r_1)t},$$

we only demand that

$$e^{\sigma(r_1 - l_2)t} \geq \frac{\mu \max\left\{B^{r_1}, B^{r_2}\right\} + \sigma B}{\eta \min\left\{B^{l_1}, B^{l_2}\right\}}, \forall t \geq 0,$$

since $0 < l_1 \leq l_2 < r_1 \leq r_2 \leq 1$. For this purpose, we need

$$\frac{\mu \max\left\{\varphi_0^{r_1}, \varphi_0^{r_2}\right\} + \sigma\varphi_0}{\eta \min\left\{\varphi_0^{l_1}, \varphi_0^{l_2}\right\}} \leq 1,$$

that is

$$\begin{aligned} \mu &\leq \frac{\eta \min\left\{\varphi_0^{l_1}, \varphi_0^{l_2}\right\} - \sigma\varphi_0}{\max\left\{\varphi_0^{r_1}, \varphi_0^{r_2}\right\}} \\ &= \frac{\min\left\{\varphi_0^{l_1}, \varphi_0^{l_2}\right\}}{\max\left\{\varphi_0^{r_1}, \varphi_0^{r_2}\right\}} \left(\eta - \sigma \min\left\{\varphi_0^{1-l_1}, \varphi_0^{1-l_2}\right\}\right). \end{aligned}$$

Therefore, we only need to choose

$$\sigma = \frac{\eta}{2\min\left\{\varphi_0^{1-l_1}, \varphi_0^{1-l_2}\right\}}.$$

Thus the proof of Lemma 3.3 is complete.

Proof of Theorem 2.3. By (3.9) we have

$$F'(t) + \alpha_1 \min\left\{F^{\frac{p^-}{2}}(t), F^{\frac{p^+}{2}}(t)\right\} \le \lambda\beta_1 \max\left\{F^{\frac{q^-+1}{2}}(t), F^{\frac{q^++1}{2}}(t)\right\}, \tag{3.13}$$

where

$$\alpha_1 = 2K^{-p^+},$$

and

$$\beta_1 = 4\left(|\Omega| + 1\right)^{(1-q^-)/2}.$$

By Lemma 3.3, there exist $\sigma > 0$, $B > 0$, such that

$$0 \le F(t) \le Be^{-\sigma t}, t \ge 0,$$

provided that

$$\lambda \le \frac{\alpha_1 \min\left\{F^{\frac{p^-}{2}}(0), F^{\frac{p^+}{2}}(0)\right\}}{\beta_1 \max\left\{F^{\frac{q^-+1}{2}}(0), F^{\frac{q^++1}{2}}(0)\right\}}.$$

Furthermore, there exists T_1, for $t \in [T_1, +\infty)$

$$\begin{aligned}
&\alpha_1 - \frac{\lambda\beta_1 \max\left\{F^{\frac{q^-+1}{2}}(t), F^{\frac{q^++1}{2}}(t)\right\}}{\min\left\{F^{\frac{p^-}{2}}(t), F^{\frac{p^+}{2}}(t)\right\}} \\
&\ge \alpha_1 - \frac{\lambda\beta_1 \max\left\{\left(Be^{-\sigma T_1}\right)^{\frac{q^-+1}{2}}, \left(Be^{-\sigma T_1}\right)^{\frac{q^++1}{2}}\right\}}{\min\left\{\left(Be^{-\sigma T_1}\right)^{\frac{p^-}{2}}, \left(Be^{-\sigma T_1}\right)^{\frac{p^+}{2}}\right\}} \\
&= \alpha_1 - \frac{\lambda\beta_1 \left(Be^{-\sigma T_1}\right)^{\frac{q^-+1}{2}}}{\left(Be^{-\sigma T_1}\right)^{\frac{p^+}{2}}} \\
&= \alpha_1 - \lambda\beta_1 \left(Be^{-\sigma T_1}\right)^{\frac{q^-+1-p^+}{2}} := K_1 > 0
\end{aligned}$$

holds, where

$$\sigma = \frac{\alpha_1}{2\min\left\{F^{1-p^-}(0), F^{1-p^+}(0)\right\}}.$$

Therefore, when $t \in [T_1, +\infty)$, by (3.13) we obtain

$$F'(t) + K_1 F^{\frac{p^-}{2}}(t) \le 0.$$

By (3.10) and (3.11), we obtain

$$F(t) \leq \left[F^{\frac{2-p^-}{2}}(0) - \frac{K_1\left(2-p^-\right)}{2}t\right]^{\frac{2}{2-p^-}}$$

for $t \in [T_1, T_2)$, and

$$F(t) \equiv 0$$

for $t \in [T_2, +\infty)$, where

$$T_2 = \frac{2F^{\frac{2-p^-}{2}}(0)}{K_1\left(2-p^-\right)}.$$

Thus the proof of Theorem 2.3 is complete.

Proof of Theorem 2.4. (i) In case $p > q^+ + 1$, $\lambda > 0$.

Set $\overline{u} = A\varphi(x)$, φ function is the solution of problem (2.4), $A > 0$ is a constant will be determined later. Then we have

$$-\text{div}\left(|\nabla\overline{u}|^{p-2}\nabla\overline{u}\right) - \lambda\overline{u}^{q(x)} \leq -A^{p-1} + \lambda M^{q^+}A^{q^+} \leq \overline{u}_t = 0,$$

where constant A satisfies that

$$A \geq \max\left\{\left(\lambda M^{q^+}\right)^{\frac{1}{p-q^+-1}}, \max_{x\in\Omega} u_0(x)\right\}.$$

(ii) In case $p < q^+ + 1, \lambda > 0$.

We can write

$$\text{div}\left(|\nabla\overline{u}|^{p-2}\nabla\overline{u}\right) + \lambda\overline{u}^{q(x)} \leq -A^{p-1} + \lambda M^{q^+}A^{q^+} \leq \overline{u}_t = 0,$$

with

$$\lambda M^{q^+}A^{q^+-p+1} \leq 1,$$

where

$$A = \max\left\{\max u_0\left(.\right), 1\right\}.$$

(iii) In case $p = q^+ + 1$, $\lambda > 0$ the following inequality is true:

$$\text{div}\left(|\nabla\overline{u}|^{p-2}\nabla\overline{u}\right) + \lambda\overline{u}^{q(x)} \leq -A^{q^+} + \lambda M^{q^+}A^{q^+} = A^{q^+}\left(\lambda M^{q^+} - 1\right) \leq \overline{u}_t = 0,$$

with $\lambda \leq M^{-q^+}$.

We know that, $\overline{u} \geq 0$ on $\partial\Omega\times\,(0,T)$ and $\overline{u}(x,0) \geq u_0(x)$ in Ω. By the comparison principle, $\overline{u}$ is a globally bounded supersolution of (1.1). Thus the proof of Theorem 2.4 is complete.

References

1. Acerbi, E., Mingione, G.: *Regularity results for stationary electro-rheological fluids*, Arch. Ration. Mech. Anal. **164,** 213-259 (2002).
2. Antontsev, S.N., Shmarev, S.I.: *Anisotropic parabolic equations with variable nonlinearity*, Publ. Mat. **53**(2), 355-399 (2009).
3. Antontsev, S.N., Shmarev, S.I.: *Blow-up of solutions to parabolic equations with non-standard growth conditions*, J. Comput. Math. Appl. **234**, 2633-2645 (2010).
4. Ayazoglu (Mashiyev), R., Akkoyunlu, E., Agırman Aydın, T.: *On global existence and bounds for theblow-up time in a semilinear heat equation involving parametric variable sources,* C.R. Mecanique **1**, 1-10 (2021), doi:10.5802/crmeca. 98.
5. Bokalo, M.M., Buhrii, O.M., Mashiyev, R.A.: *Unique solvablity of initial-boundary-value problems for anisotropic elliptic-parabolic equations with variable exponents of nanlinearity*, J. Nonl. Evol. Equ. Appl. **2013**(6), 67-87 (2014).
6. Buhrii, O.M., Mashiyev, R.A.: *Uniqueness of solutions of the parabolic variational inequality with variable exponent of nonlinearity*, Nonlinear Anal. **70**, 2325-2331 (2009).
7. Diening, L., Harjulehto, P., Hästö, P., Růžička, M.: Lebesgue and Sobolev spaces with variable exponents, *Springer-Verlag, Heidelberg* (2011).
8. Fan, X.L., Shen, J.S., Zhao, D.: *Sobolev embedding theorems for spaces* $W^{1,p(x)}\ (\Omega)$, J. Math. Anal. Appl. **262,** 749-760 (2001).
9. Giacomoni, J., Tiwari, S., Warnault, G.: *Quasilinear parabolic problem with* $p(x)$*-laplacian: existence, uniqueness of weak solutions and stabilization*, Nonlinear Differ. Equ. Appl. **23**(24), 1-30 (2016).
10. Guo, B., Li, Y., Gao, W.: *Singular phenomena of solutions for nonlinear diffusion equations involving* $p(x)$*-Laplace operator and nonlinear sources*, Z. Angew. Math. Phys. **66**(3), 989-1005 (2015).
11. Guo, B., Gao, W.J.: *Non-extinction of solutions to a fast diffusive* p*-Laplace equation with Neumann boundary conditions,* J. Math. Anal. Appl. **422,** 1527-1531 (2015).
12. Kholyavka, O., Buhrii, O., Bokalo, M., Ayazoglu (Mashiyev), R.: I*nitial-boundary-value problem for third order equations of Kirchhoff type with variable exponents of nonlinearity*, Advances in Math. Sciences and Appl. **23**(2), 509-528 (2013).
13. Kováčik, O., Răkosnik, J.: *On spaces* $L^{p(x)}$ *and* $W^{k,p(x)}$, Czechoslov. Math. J. **41**(116), 592-618 (1991).
14. Levine, S., Chen, Y.M., Stanich, J.: Image restoration via nonstandard diffusion, *Department of Mathematics and Computer Science, Duquesne University* (2004).
15. Liu, W., Wu, B.: *A note on extinction for fast diffusive* p*-Laplacian with sources*, Math. Methods Appl. Sci. **31**, 1383-1386 (2008).
16. Liu, B., Yang, J.: *Blow-up properties in the parabolic problems with anisotropic non-standard growth conditions*, Z. Angew. Math. Phys. **67**(13), 1-26 (2016).
17. Liu, B., Dong, M., Li, F.: *Singular solutions in nonlinear parabolic equations with anisotropic nonstandard growth conditions*, J. Math. Phys. **59**(12), 1-16 (2018).
18. Mashiyev, R.A., Buhrii, O.M.: *Existence of solutions of the parabolic variational inequality with variable exponent of nonlinearity*, J. Math. Anal. Appl. **377**, 450-463 (2011).
19. Rajagopal, K., Ružička, M.: *Mathematical modelling of electro-rheological fluids*, Contin. Mech. Thermodyn. **13**, 59-78 (2001).
20. Ružička, M.: Electrorheological Fluids: Modeling and Mathematical Theory. *Springer-Verlag, Berlin, Germany* (2002).
21. Tian, Y., Mu, C.: *Extinction and non-extinction for a* p*-Laplacian equation with nonlinear source*, Nonlinear Anal. **69**, 2422-2431 (2008).

22. Ya, T., Mu, C.: *Extinction and non-extinction for a p-Laplacian equation with nonlinear source*, Nonlinear Anal. **69**(8), 2422-2431 (2008).
23. Yin, J.X., Jin, C.H.: *Critical extinction and blow-up exponents for fast diffusive p-Laplacian with sources*, Math. Methods Appl. Sci. **30**, 1147-1167 (2007).
24. Zhou, J., Mu, C.L.: *Globle existence and blow up for a non-Newton polytropic filtration system with nonlocal source*, ANZIAM J. **50**, 13-29 (2008).