Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **42** (1), 99-110 (2022).

Oscillatory integrals with variable Calderón-Zygmund kernel on generalized weighted Morrey spaces

Ismail Ekincioglu ⋆**, Salaudin Umarkhadzhiev**

Received: 18.09.2021 / Revised: 15.03.2022 / Accepted: 26.04.2022

Abstract. *In this paper, we investigate the boundedness of the oscillatory singular integrals with variable Calderón-Zygmund kernel on the generalized weighted Morrey spaces* $M^{p,\varphi}(w)$*. When* w *the weights are in the Muckenhoupt class* A_p*,* $1 < p < \infty$ *and* $(\varphi_1, \varphi_2, w)$ *satisfies some conditions, we show that the oscillatory singular integral operators* T_λ *and* T^*_λ *are bounded from* $M^{p,\varphi_1}(w)$ *to* $M^{p,\varphi_2}(w)$*. Meanwhile, the corresponding result for the oscillatory singular integrals with standard Calderón-Zygmund kernel are established.*

Keywords. Generalized weighted Morrey space, oscillatory integral, variable Calderón-Zygmund kernels.

Mathematics Subject Classification (2010): Primary 42B20, 42B25, 42B35.

1 Introduction and main results

The classical Morrey spaces were introduced by Morrey [21] to study the local behavior of solutions to second-order elliptic partial differential equations. Moreover, various Morrey spaces are defined in the process of study. Guliyev, Mizuhara and Nakai [4,20,22] introduced generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ (see, also [5,6,24]); Komori and Shirai [18] defined weighted Morrey spaces $L^{p,\kappa}(w)$; Guliyev [8] gave a concept of the generalized weighted Morrey spaces $M^{p,\varphi}(w)$ which could be viewed as extension of both $M^{p,\varphi}(\mathbb{R}^n)$ and $L^{p,\kappa}(w)$. In [8], the boundedness of the classical operators and their commutators in spaces $M^{p,\varphi}_w$ was also studied, see also [1,3,10–12,14,16,17].

The spaces $M^{p,\varphi}(w)$ defined by the norm

$$\|f\|_{M^{p,\varphi}_w} \equiv \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, w(B(x,r))^{-1/p}\, \|f\|_{L^p(B(x,r),w)},$$

⋆ Corresponding author

I. Ekincioglu
Dumlupinar University, Department of Mathematics, Kutahya, Turkey
Istanbul Medeniyet University, Department of Mathematics, Istanbul, Turkey
E-mail: ismail.ekincioglu@dpu.edu.tr

S. Umarkhadzhiev
Kh. Ibragimov Complex Institute of Russian Academy of Science, Grosny, Russia
Academy of Sciences of Chechen Republic, Grosny, Russia
E-mail: umsalaudin@gmail.com

where the function φ is a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and w is a non-negative measurable function on $\mathbb{R}^n$. Here and everywhere in the sequel $B(x,r)$ is the ball in $\mathbb{R}^n$ of radius r centered at x and $w(B(x,r)) = \int_{B(x,r)} w(y)dy$.

Suppose that K is the standard Calderón-Zygmund kernel. That is, $K \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is homogeneous of degree $-n$, and $\int_{S^{n-1}} K(x)d\sigma(x) = 0$, where $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. The oscillatory integral operator T_λ is defined by

$$T_\lambda f(x) = p.v. \int_{\mathbb{R}^n} e^{i\lambda\Phi(x,y)} K(x-y)\varphi(x,y)f(y)dy, \tag{1.1}$$

where $\lambda \in \mathbb{R}, \varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, the space of infinitely differentiable functions on $\mathbb{R}^n \times \mathbb{R}^n$ with compact supports, and Φ is a real-analytic function or a real-$C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ function satisfying that for any $(x_0,y_0) \in \operatorname{supp}\varphi$, there exists $(j_0,k_0), 1 \le j_0,k_0 \le n$, such that $\partial^2\Phi(x_0,y_0)/\partial x_{j_0}\partial y_{k_0}$ does not vanish up to infinite order. These operators have arisen in the study of singular integrals supported on lower dimensional varieties, and the singular Radon transform. In [23], Y. B. Pan proved that T_λ are uniformly in λ bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

Let $K(x,y)$ be a variable Calderón-Zygmund kernel. That means, for a. e. $x \in \mathbb{R}^n$, $K(x,\cdot)$ is a standard Calderón-Zygmund kernel and

$$\max_{|j| \le 2n, j \in \mathbb{N}_0^n} \left\| \frac{\partial^{|j|} k}{\partial y^j} \right\|_{L^\infty(\mathbb{R}^n \times S^{n-1})} = A < \infty. \tag{1.2}$$

Define the oscillatory integral operator with variable Calderón-Zygmund kernel T_λ^* by

$$T_\lambda^* f(x) = p.v \int_{\mathbb{R}^n} e^{i\lambda\Phi(x,y)} K(x, x-y)\varphi(x,y)f(y)dy, \tag{1.3}$$

where λ, φ and Φ satisfy the same assumptions as those in the operator defined by (1.1).

S. Z. Lu and D. C. Yang etc. [19] investigated the L^p boundedness about this class of oscillatory integral operators. The boundedness of some operators on these spaces can be see ([2,4,6,7,20–22,25,26]). Recently, A. Eroglu [15] obtained the boundedness of a class of oscillatory integral with Calderón-Zygmund kernel and polynomial phase on generalized Morrey spaces. In [13] Guliyev etc. proved the generalized Morrey spaces $M^{p,\varphi}$ boundedness of T_λ defined by (1.1).

The purpose of this paper is to generalize the results above to the case with real - $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ or analytic phase functions. Our main results in this paper are formulated as follows.

Theorem 1.1 *Let $\lambda \in \mathbb{R}, \varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and Φ is a real-$C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ function satisfying that for any $(x_0,y_0) \in \operatorname{supp}\varphi$, there exists $(j_0,k_0), 1 \le j_0,k_0 \le n$, such that $\partial^2\Phi(x_0,y_0)/\partial x_{j_0}\partial x_{k_0}$ does not vanish up to infinite order. Assume K is a standard Calderón- Zygmund kernel and T_λ is defined as in* (1.1). *Then for any $1 \le p < \infty$, and (φ_1,φ_2) satisfies the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau) w(B(x,\tau))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{p}}} \frac{dt}{t} \le C\,\varphi_2(x,r), \tag{1.4}$$

where C does not depend on x and r, the operator T_λ is bounded from $M^{p,\varphi_1}(w)$ to $M^{p,\varphi_2}(w)$ for $p > 1$ and from $M^{p,\varphi_1}(w)$ to $WM^{p,\varphi_2}(w)$ for $p \ge 1$.

Theorem 1.2 *Let $\lambda \in \mathbb{R}, \varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and Φ is a real-$C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ function satisfying that for any $(x_0, y_0) \in \operatorname{supp}\varphi$, there exists $(j_0, k_0), 1 \leq j_0, k_0 \leq n$, such that $\partial^2\Phi(x_0, y_0)/\partial x_{j_0}\partial x_{k_0}$ does not vanish up to infinite order. Assume K is a variable Calderón- Zygmund kernel and T_λ^* is defined as in* (1.3). *Then for any $1 \leq p < \infty$, (φ_1, φ_2) satisfies the condition* (1.4)*, the operator T_λ^* is bounded from $M^{p,\varphi_1}(w)$ to $M^{p,\varphi_2}(w)$ for $p > 1$ and from $M^{p,\varphi_1}(w)$ to $WM^{p,\varphi_2}(w)$ for $p \geq 1$.*

Note that for $\varphi_1(x, r) = \varphi_1(x, r) \equiv w(B(x, r))^{\frac{\kappa-1}{p}}$, from Theorems 1.1 and 1.2 we get the following results, which proved in [23].

Corollary 1.1 *Let $\lambda \in \mathbb{R}, \varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and Φ is a real-$C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ function satisfying that for any $(x_0, y_0) \in \operatorname{supp}\varphi$, there exists $(j_0, k_0), 1 \leq j_0, k_0 \leq n$, such that $\partial^2\Phi(x_0, y_0)/\partial x_{j_0}\partial x_{k_0}$ does not vanish up to infinite order. Assume K is a standard Calderón- Zygmund kernel and T_λ is defined as in* (1.1)*. Then for any $1 \leq p < \infty$, and $0 < \kappa < 1$, the operator T_λ is bounded on $L^{p,\kappa}(w)$ for $p > 1$ and from $L^{p,\kappa}(w)$ to $WL^{p,\kappa}(w)$ for $p \geq 1$.*

Corollary 1.2 *Let $\lambda \in \mathbb{R}, \varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and Φ is a real-$C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ function satisfying that for any $(x_0, y_0) \in \operatorname{supp}\varphi$, there exists $(j_0, k_0), 1 \leq j_0, k_0 \leq n$, such that $\partial^2\Phi(x_0, y_0)/\partial x_{j_0}\partial x_{k_0}$ does not vanish up to infinite order. Assume K is a standard Calderón- Zygmund kernel and T_λ^* is defined as in* (1.3)*. Then for any $1 \leq p < \infty$, and $0 < \kappa < 1$, the operator T_λ^* is bounded on $L^{p,\kappa}(w)$ for $p > 1$ and from $L^{p,\kappa}(w)$ to $WL^{p,\kappa}(w)$ for $p \geq 1$.*

2 Notations and preliminary Lemmas

Let $B = B(x_0, r)$ be the ball with the center x_0 and radius r. Given a ball B and $\lambda > 0$, λB denotes the ball with the same center as B whose radius is λ times that of B.

If w is a weight function, we denote by $L^p(w) \equiv L^p(\mathbb{R}^n, w)$ the weighted Lebesgue space defined by the norm

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)dx\right)^{\frac{1}{p}} < \infty, \quad \text{when } 1 \leq p < \infty$$

and by $\|f\|_{L^\infty(w)} = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} |f(x)|w(x)$ when $p = \infty$.

We recall that a weight function w is in the Muckenhoupt class A_p, $1 < p < \infty$, if

$$\begin{aligned}
[w]_{A_p} :&= \sup_B [w]_{A_p(B)} \\
&= \sup_B \left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w(x)^{1-p'}dx\right)^{p-1} < \infty,
\end{aligned}$$

where the sup is taken with respect to all the balls B and $\frac{1}{p} + \frac{1}{p'} = 1$. Note that, for all ball B by Hölder's inequality

$$[w]_{A_p(B)}^{1/p} = |B|^{-1}\|w\|_{L^1(B)}^{1/p}\,\|w^{-1/p}\|_{L^{p'}(B)} \geq 1. \tag{2.1}$$

For $p = 1$, the class A_1 is defined by the condition $Mw(x) \leq Cw(x)$ with $[w]_{A_1} = \sup\limits_{x\in\mathbb{R}^n} \frac{Mw(x)}{w(x)}$, and for $p = \infty$ we define $A_\infty = \bigcup_{1\leq p<\infty} A_p$.

Our argument based heavily on the following results.

Lemma 2.1 *[19] Assume T_λ is defined as in* (1.1). *Then for any $1 < p < \infty$ and $w \in A_p$, we have*

$$\|T_\lambda f\|_{L^p(w)} \leq C(n, p, \Phi, \varphi, C_{p,w})\, C_1\, \|f\|_{L^p(w)},$$

where $C(n, p, \Phi, \varphi, C_{p,w})$ is independent of λ, K, f and $C_1 = \|k\|_{C^1(S^{n-1})}$.

Lemma 2.2 *[19] Assume T^*_λ is defined as in* (1.3). *Then for any $1 < p < \infty$ and $w \in A_p$, we have*

$$\|T^*_\lambda f\|_{L^p(w)} \leq C(n, p, \Phi, \varphi, C_{p,w})\, A\, \|f\|_{L^p(w)},$$

where $C(n, p, \Phi, \varphi, C_{p,w})$ is independent of λ, K, f and A is defined in (1.2).

The generalized weighed Morrey spaces introduced by Guliyev in [8] are defined as follows.

Definition 2.1 *Let $1 \leq p < \infty$, φ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and w be non-negative measurable function on $\mathbb{R}^n$. We denote by $M^{p,\varphi}(w)$ the generalized weighted Morrey space, the space of all functions $f \in L^p_{\rm loc}(w)$ with finite norm*

$$\|f\|_{M^{p,\varphi}_w} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x, r)^{-1}\, w(B(x, r))^{-\frac{1}{p}}\, \|f\|_{L^p(B(x,r),w)},$$

where

$$\|f\|_{L^p(B(x,r),w)} = \left(\int_{B(x,r)} |f(y)|^p w(y) dy \right)^{\frac{1}{p}}.$$

Furthermore, $WM^{p,\varphi}(w)$ is the weak generalized weighted Morrey space of all functions $f \in WL^p_{\rm loc}(w)$ for which

$$\|f\|_{WM^{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x, r)^{-1}\, w(B(x, r))^{-\frac{1}{p}}\, \|f\|_{WL^p(B(x,r),w)} < \infty,$$

where $WL^p(B(x, r), w)$ denotes the weak $L^p(w)$-space of measurable functions f for which

$$\|f\|_{WL^p(B(x,r),w)} \equiv \|f\chi_{B(x,r)}\|_{WL^p(w)} = \sup_{t>0} t \left(\int_{\{y\in B(x,r):\, |f(y)|>t\}} w(y) dy \right)^{\frac{1}{p}}.$$

Remark 2.1 (1) If $w \equiv 1$, then $M^{p,\varphi}(1) = M^{p,\varphi}$ is the generalized Morrey space.

(2) If $\varphi(x, r) \equiv w(B(x, r))^{\frac{\kappa-1}{p}}$, then $M^{p,\varphi}(w) = L^{p,\kappa}(w)$ is the weighted Morrey space.

(3) If $\varphi(x, r) \equiv v(B(x, r))^{\frac{\kappa}{p}} w(B(x, r))^{-\frac{1}{p}}$, then $M^{p,\varphi}(w) = L^{p,\kappa}(v, w)$ is the two weighted Morrey space.

(4) If $w \equiv 1$ and $\varphi(x, r) = r^{\frac{\lambda-n}{p}}$ with $0 < \lambda < n$, then $M^{p,\varphi}(w) = L^{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey space and $WM^{p,\varphi}(w) = WL^{p,\lambda}(\mathbb{R}^n)$ is the weak Morrey space.

(5) If $\varphi(x, r) \equiv w(B(x, r))^{-\frac{1}{p}}$, then $M^{p,\varphi}(w) = L^p(\mathbb{R}^n, w)$ is the weighted Lebesgue space.

The Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|dy, \quad f \in L^1_{\rm loc}(\mathbb{R}^n).$$

A distribution kernel K is called a standard Calderòn-Zygmund kernel (SCZK) if it satisfies the following hypotheses:

$$|K(x,y)| \leq \frac{C}{|x-y|^n}, \forall x \neq y,$$

$$|\nabla_x K(x,y)| + |\nabla_y K(x,y)| \leq \frac{C}{|x-y|^{n+1}}, \forall x \neq y,$$

where C does not depend on x and y. The corresponding Calderòn-Zygmund integral operator S and oscillatory integral operator R are defined by

$$Sf(x) = p.v. \int_{\mathbb{R}^n} K(x,y) f(y) dy.$$

and

$$Rf(x) = p.v. \int_{\mathbb{R}^n} e^{iP(x,y)} K(x,y) f(y) dy,$$

where $P(x,y)$ is a real valued polynomial defined on $\mathbb{R}^n \times \mathbb{R}^n$.

Theorem 2.1 *[8] Let* $1 \leq p < \infty$ *and* (φ_1, φ_2) *satisfy the condition* (1.4). *Then the maximal operator* M *and the singular integral operator* T *are bounded from* $M^{p,\varphi_1}(w)$ *to* $M^{p,\varphi_2}(w)$ *for* $p > 1$ *and from* $M^{p,\varphi_1}(w)$ *to* $WM^{p,\varphi_2}(w)$ *for* $p \geq 1$.

Corollary 2.1 *Let* $1 \leq p < \infty$ *and* (φ_1, φ_2) *satisfy the condition* (1.4). *If* S *is of type* (L^2, L^2), *then for any real polynomial* $P(x,y)$, *the operator* R *is bounded from* $M^{p,\varphi_1}(w)$ *to* $M^{p,\varphi_2}(w)$ *for* $p > 1$ *and from* $M^{p,\varphi_1}(w)$ *to* $WM^{p,\varphi_2}(w)$ *for* $p \geq 1$.

Remark 2.2 Note that, in the case $w \equiv 1$ Corollary 2.1 were proved in [15].

Lemma 2.3 *[27] Denote by* $\mathcal{H}_m$ *the spaces of spherical harmonic functions of degree* m. *Then*

(a) $L^2(S^{n-1}) = \oplus_{m=0}^{\infty} \mathcal{H}_m$, *and* $g_m = dim \mathcal{H}_m \leq C(n) m^{n-2}$ *for any* $m \in \mathbb{N}$,

(b) for any $m = 0, 1, 2, \ldots$, *there exists an orthogonal system* $\{Y_{jm}\}_{j=1}^{g_m}$ *of* $\mathcal{H}_m$ *such that* $\|Y_{jm}\|_{L^\infty(S^{n-1})} \leq C(n) m^{n/2-1}, Y_{jm} = (-m)^{-n}(m+n-2)^{-n} \varLambda^n Y_{jm}, j = 1, \ldots, g_m$, *and* $\varLambda$ *is the Beltrami-Laplace operator on* S^{n-1}.

In the following the letter C will denote a constant which may vary at each occurrence.

3 Proof of Theorems 1.1 and 1.2

We will use the following results on the boundedness of the weighted Hardy operator

$$H_w^* g(t) := \int_t^\infty g(s) w(s) ds, \quad 0 < t < \infty,$$

where w is a weight.

The following theorem was proved in [9].

Theorem 3.1 *[9] Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t) H_w^* g(t) \le C \sup_{t>0} v_1(t) g(t) \tag{3.1}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$D := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty.$$

Moreover, the value $C = D$ is the best constant for (3.1).

The following lemma is valid.

Lemma 3.1 *Let $1 \le p < \infty$, $w \in A_p$ and T_λ is defined as in* (1.1).
Then, for $1 < p < \infty$ the inequality

$$\|T_\lambda f\|_{L^p(B,w)} \lesssim w(B)^{\frac{1}{p}} \int_{2r}^\infty \|f\|_{L^p(B(x_0,t),w)} \, w(B(x_0,t))^{-1/p} \, \frac{dt}{t}$$

holds for any ball $B = B(x_0,r)$ and for all $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$.
Moreover, for $p = 1$ the inequality

$$\|T_\lambda f\|_{WL^1(B,w)} \lesssim w(B) \int_{2r}^\infty \|f\|_{L^1(B(x_0,t),w)} \, w(B(x_0,t))^{-1} \, \frac{dt}{t} \tag{3.2}$$

holds for any ball $B = B(x_0,r)$ and for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n, w)$.

Proof. Let $p \in (1,\infty)$ and $w \in A_p$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0,r)$ for the ball centered at x_0 and radius r, $2B = B(x_0,2r)$. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{\complement_{(2B)}}(y)$$

and have

$$\|T_\lambda f\|_{L^p(B,w)} \le \|T_\lambda f_1\|_{L^p(B,w)} + \|T_\lambda f_2\|_{L^p(B,w)}.$$

It is known that (see Lemma 2.1) the operator T_λ is bounded on $L^p(w)$. Since $f_1 \in L^p(w)$, $T_\lambda f_1 \in L^p(w)$ and boundedness of T_λ in $L^p(w)$ (see [19]) it follows that

$$\|T_\lambda f_1\|_{L^p(B,w)} \le \|T_\lambda f_1\|_{L^p(w)} \le C\|f_1\|_{L^p(w)} = C\|f\|_{L^p(2B,w)},$$

where the constant $C > 0$ is independent of f.

We now estimate $T_\lambda f_2$. We can write

$$\Big|T_\lambda f_2(x)\Big| = \Big| \int_{\complement_{(2B)}} e^{i\lambda\Phi(x,y)} K(x-y)\varphi(x,y) f(y) dy \Big|.$$

Now by an argument similar to the proof of Lemma 6 in [19], we choose $\phi_1 \in C_0^\infty(\mathbb{R}^n)$ such that $\phi_1(x) \equiv 1$ when $|x| \le 1$, and $\phi_1(x) \equiv 0$ when $|x| > 2$. Let $\phi_2 = 1 - \phi_1$ and $N \in \mathbb{N}$ which is large enough and will be determined later. Write

$$K(x) = K_\lambda^1(x) + K_\lambda^2(x),$$

where

$$K_\lambda^j(x) = K(x)\phi_j(\lambda^{1/N}x), \ j = 1,2.$$

Then

$$T_\lambda f_2(x) = p.v. \int_{\complement_{(2B)}} e^{i\lambda\Phi(x,y)} K_\lambda^1(x-y)\varphi(x,y)f(y)dy$$
$$+ p.v. \int_{\complement_{(2B)}} e^{i\lambda\Phi(x,y)} K_\lambda^2(x-y)\varphi(x,y)f(y)dy := T_\lambda^1 f_2(x) + T_\lambda^2 f_2(x).$$

Let us first estimate $T_\lambda^1 f_2(x)$. To do so, using Taylor's expansion and the compactness of $\operatorname{supp}\varphi$, we write

$$\Phi(x,y) = \Phi(x,x) + P(x,y) + r_N(x,y)$$

for $(x,y) \in \operatorname{supp}\varphi$, where $P(x,y)$ is a polynomial with deg $P < N$ and $|r_N(x,y)| \leq C|x-y|^N$ with C in dependent of x and y. Define

$$Rf(x) = p.v. \int_{\complement_{(2B)}} e^{i\lambda P(x,y)} K_\lambda^1(x-y)\varphi(x,y)f(y)dy.$$

Therefore

$$\begin{aligned}
&e^{-i\lambda\Phi(x,x)} T_\lambda^1 f_2(x) - Rf(x) \\
&= \int_{B(x,2\lambda^{-1/N})} e^{i\lambda P(x,y)}[e^{i\lambda r_N(x,y)} - 1] K_\lambda^1(x-y)\varphi(x,y)f(y)dy \\
&= \sum_{j=0}^{\infty} \int_{B(x,2^{-j+1}\lambda^{-1/N})\setminus B(x,2^{-j}\lambda^{-1/N})} e^{i\lambda P(x,y)}[e^{i\lambda r_N(x,y)} - 1] K_\lambda^1(x-y)\varphi(x,y)f(y)dy \\
&\equiv \sum_{j=0}^{\infty} T_{\lambda j}^1 f_2(x).
\end{aligned}$$

On $T_{\lambda j}^1 f_2(x)$, by the properties of r_N and k, we have

$$|T_{\lambda j}^1 f_2(x)| \leq C 2^{-jN} Mf(x).$$

So we have

$$|T_\lambda^1 f_2(x)| \leq C \sum_{j=0}^{\infty} 2^{-jN} Mf(x) + C|Rf(x)| \leq CMf(x) + C|Rf(x)|.$$

By Theorem 3.1 in [8], we have

$$\begin{aligned}
\|T_\lambda^1 f_2\|_{L^p(B(x_0,r),w)} &\lesssim \|Mf\|_{L^p(B(x_0,r),w)} + \|Rf\|_{L^p(B(x_0,r),w)} \\
&\lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t),w)}\, w(B(x_0,t))^{-1/p} \frac{dt}{t}.
\end{aligned}$$

Now, let us turn to estimate $T_\lambda^2 f_2(x)$. We consider the following two cases.

Case 1. $\lambda \leq 1$. Similar to that estimate of T_λ^2 in Lemma 6 in [19], we have

$$|T_\lambda^2 f_2(x)| \leq CMf(x),$$

where the constant $C > 0$ is independent of f. By Theorem 3.1 in [8] we have

$$\|T_\lambda^2 f_2\|_{L^p(B(x_0,r),w)} \lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t),w)}\, w(B(x_0,t))^{-1/p} \frac{dt}{t}.$$

Case 2. $\lambda > 1$. We choose $\varphi_0 \in C_0^\infty(\mathbb{R}^n)$ such that

$$\operatorname{supp} \varphi_0 \subseteq \{x \in \mathbb{R}^n : 1 < |x| \leq 2\},$$

and

$$\phi_2(x) = \sum_{j=0}^{\infty} \varphi_0(2^{-j}x).$$

Let

$$K_{\lambda,j}^2(x) = K(x)\,\varphi_0(2^{-j}\lambda^{1/N}x).$$

Then

$$\begin{aligned} T_\lambda^2 f_2(x) &= \int_{\complement_{(2B)}} e^{i\lambda\Phi(x,y)} K_\lambda^2(x-y)\varphi(x,y)f(y)dy \\ &= \sum_{j=0}^{\infty} \int_{\complement_{(2B)}} e^{i\lambda\Phi(x,y)} K_{\lambda,j}^2(x-y)\varphi(x,y)f(y)dy \\ &\equiv \sum_{j=0}^{\infty} T_{\lambda,j}^2 f_2(x). \end{aligned}$$

For $T_{\lambda,j}^2$, by the definition of it, we can get

$$|T_\lambda^2 f_2(x)| \leq C \int_{B(x,2^{-j+1}\lambda^{-1/N})\setminus B(x,2^{-j}\lambda^{-1/N})} \frac{|f(y)|}{|x-y|^n}\,dy \leq CMf(x). \tag{3.3}$$

The inequality (3.3) also can be see in [19], we omit the detail here.
By Theorem 3.1 in [8], we have

$$\|T_\lambda^2 f_2\|_{L^p(B(x_0,r),w)} \lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t),w)}\, w(B(x_0,t))^{-1/p}\, \frac{dt}{t}.$$

Therefore

$$\begin{aligned} \|T_\lambda f_2\|_{L^p(B(x_0,r),w)} &\leq \|T_\lambda^1 f_2\|_{L^p(B(x_0,r),w)} + \|T_\lambda^2 f_2\|_{L^p(B(x_0,r),w)} \\ &\lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t),w)}\, w(B(x_0,t))^{-1/p}\, \frac{dt}{t}. \end{aligned}$$

This finishes the proof of Lemma 3.1.

Proof of Theorem 1.1.

By Lemma 3.1 and Theorem 3.1 we get

$$\begin{aligned} \|T_\lambda f\|_{M^{p,\varphi_2}(w)} &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^{\infty} \|f\|_{L^p(B(x,t),w)}\, w(B(x,t))^{-1/p}\, \frac{dt}{t} \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1}\, w(B(x,r))^{-1/p}\, \|f\|_{L^p(B(x,r),w)} = \|f\|_{M^{p,\varphi_1}(w)}. \end{aligned}$$

This finishes the proof of Theorem 1.1.

The following lemma is valid.

Lemma 3.2 *Let $1 \le p < \infty$ and T^*_λ is defined as in* (1.3).

Then, for $1 < p < \infty$ the inequality

$$\|T^*_\lambda f\|_{L^p(B,w)} \lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t),w)}\, w(B(x_0,t))^{-1/p} \frac{dt}{t}$$

holds for any ball $B = B(x_0, r)$ and for all $f \in L^{\mathrm{loc}}_p(\mathbb{R}^n)$.

Moreover, for $p = 1$ the inequality

$$\|T^*_\lambda f\|_{WL^1(B,w)} \lesssim w(B) \int_{2r}^{\infty} \|f\|_{L^1(B(x_0,t),w)}\, w(B(x_0,t))^{-1} \frac{dt}{t}$$

holds for any ball $B = B(x_0, r)$ and for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n, w)$.

Proof. Let $p \in (1, \infty)$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and radius r, $2B = B(x_0, 2r)$. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{\complement_{(2B)}}(y)$$

and have

$$\|T^*_\lambda f\|_{L^p(B,w)} \le \|T^*_\lambda f_1\|_{L^p(B,w)} + \|T^*_\lambda f_2\|_{L^p(B,w)}.$$

It is known that (see Lemma 2.2) the operator T^*_λ is bounded on $L^p(w)$. Since $f_1 \in L^p(w)$, $T^*_\lambda f_1 \in L^p(w)$ and boundedness of T^*_λ in $L^p(w)$ (see [19]) it follows that

$$\|T^*_\lambda f_1\|_{L^p(B,w)} \le \|T^*_\lambda f_1\|_{L^p(w)} \le C\|f_1\|_{L^p(w)} = C\|f\|_{L^p(2B,w)},$$

where the constant $C > 0$ is independent of f.

We now estimate $T^*_\lambda f_2$. For each $m \in \mathbb{N}$ and $j = 1, \ldots, g_m$, we get

$$a_{jm}(x) = \int_{S^{n-1}} \Omega(x,z) Y_{jm}(z) d\sigma_z,$$

where $\Omega(x,z) = |z|^n K(x,z)$. Then for $a.e. x \in \mathbb{R}^n$,

$$\Omega(x,z) = \sum_{m=1}^{\infty} \sum_{j=1}^{g_m} a_{jm}(x) Y_{jm}(z'), \tag{3.4}$$

where $z' = z/|z|$ for any $z \in \mathbb{R}^n \backslash \{0\}$. By Lemma 2.3, we have that for any $x \in \mathbb{R}^n$,

$$\begin{aligned} |a_{jm}(x)| &= m^{-n}(m+n-2)^{-n} \left| \int_{S^{n-1}} \Omega(x,z) \Lambda^n Y_{jm}(z) d\sigma_z \right| \\ &= m^{-n}(m+n-2)^{-n} \left| \int_{S^{n-1}} \Lambda^n \Omega(x,z) Y_{jm}(z) d\sigma_z \right| \\ &\le C(n) A m^{-2n}. \end{aligned} \tag{3.5}$$

By Lemma 2.3 again, we can verify that for any $\epsilon > 0, N \in \mathbb{N}$, and a.e. $x \in \mathbb{R}^n$, if $|y - x| \ge \epsilon$, then

$$\left| \sum_{m=1}^{N} \sum_{j=1}^{g_m} e^{i\lambda\Phi(x,y)} \frac{a_{jm}(x) Y_{jm}((x-y)')}{|x-y|^n} \varphi(x,y) f_2(y) \right| \le C(\epsilon) A |f_2(y)|. \tag{3.6}$$

Therefore, from (3.4), (3.6) and the Lebesgue dominated convergence theorem, it follows that

$$T_\lambda^* f_2(x) = \lim_{\epsilon\to 0} \int_{\complement B(x,\varepsilon)} e^{i\lambda\Phi(x,y)} K(x, x-y)\varphi(x,y) f_2(y)dy$$
$$= \lim_{\epsilon\to 0} \sum_{m=1}^{\infty}\sum_{j=1}^{g_m} \int_{\complement B(x,\varepsilon)} e^{i\lambda\Phi(x,y)} \frac{a_{jm}(x)Y_{jm}((x-y)')}{|x-y|^n}\varphi(x,y) f_2(y)dy$$
$$= \lim_{\epsilon\to 0} \sum_{m=1}^{\infty}\sum_{j=1}^{g_m} a_{jm}(x) \int_{\complement B(x,\varepsilon)} e^{i\lambda\Phi(x,y)} \frac{Y_{jm}((x-y)')}{|x-y|^n}\varphi(x,y) f_2(y)dy.$$

We write

$$R_{jm} f_2(x) = \int_{\complement B(x,\varepsilon)} e^{i\lambda\Phi(x,y)} \frac{Y_{jm}((x-y)')}{|x-y|^n}\varphi(x,y) f_2(y)dy.$$

It is easy to see that $R_{jm} f_2(x)$ is the oscillatory integral operator defined by (1.1). By Theorem 1.1 we have R_{jm} bounded from $M^{p,\varphi_1}(w)$ to $M^{p,\varphi_2}(w)$. Therefore, by (3.5) and the above discussion we have

$$\|T_\lambda^* f_2\|_{L^p(B(x_0,r),w)} \lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L^p(B(x_0,t),w)}\, w(B(x_0,t))^{-1/p}\, \frac{dt}{t}.$$

This finishes the Lemma 3.2.

Proof of Theorem 1.2.

By Lemma 3.2 and Theorem 3.1 we get

$$\|T_\lambda^* f\|_{M^{p,\varphi_2}(w)} \lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^{\infty} \|f\|_{L^p(B(x,t),w)}\, w(B(x,t))^{-1/p}\, \frac{dt}{t}$$
$$\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1}\, w(B(x,r))^{-1/p}\, \|f\|_{L^p(B(x,r),w)} = \|f\|_{M^{p,\varphi_1}(w)}.$$

This finishes the proof of Theorem 1.2.

Acknowledgements

The authors thank the referee(s) for careful reading the paper and useful comments. The research of Ekincioglu was partially supported by the grant of Cooperation Program 2532 TUBITAK - RFBR (RUSSIAN foundation for basic research) (Agreement number no. 119N455). The research of S. Umarkhadzhiev was supported by RFBR and TUBITAK according to the research project No. 20-51-46003.

References

1. Ahmadli, A.A., Akbulut, A., Isayev, F.A., Serbetci, A.: *Multilinear commutators of parabolic Caldern-Zygmund operators on generalized weighted variable parabolic Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **41** (4), Mathematics, 3-16 (2021).
2. Akbulut, A., Guliyev, V.S., Mustafayev, R.: *On the boundedness of the maximal operator and singular integral operators in generalized Morrey spaces*, Math. Bohem. **137** (1), 27-43 (2012).
3. Azizov, J.V.: *Fractional maximal operator and its higher order commutators in generalized weighted Morrey spaces on Heisenberg group*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **40** (1), Mathematics, 6678 (2020).

4. Guliyev, V.S.: *Integral operators on function spaces on the homogeneous groups and on domains in* $\mathbb{R}^n$, (in Russian) Doctor of Sciences, Mat. Inst. Steklova, Moscow, 329 pp. (1994).
5. Guliyev, V.S.: Function spaces, integral operators and two weighted inequalities on homogeneous groups. Some applications, (in Russian) *Baku, Elm*, 1-332 (1999).
6. Guliyev, V.S.: *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces*, J. Inequal. Appl. Art. ID 503948 (2009).
7. Guliyev, V.S., Aliyev, S.S., Karaman, T., Shukurov, P.S.: *Boundedness of sublinear operators and commutators on generalized Morrey Space*, Integral Equations Operator Theory **71**, 327-355 (2011).
8. Guliyev, V.S.: *Generalized weighted Morrey spaces and higher order commutators of sublinear operators*, Eurasian Math. J. **3** (3), 33-61 (2012).
9. Guliyev, V.S.: *Generalized local Morrey spaces and fractional integral operators with rough kernel*, Problems in mathematical analysis. No. 71. J. Math. Sci. (N. Y.) **193** (2), 211-227 (2013).
10. Guliyev, V.S., Alizadeh F.Ch.: *Multilinear commutators of Calderón-Zygmund operator on generalized weighted Morrey spaces*, J. Funct. Spaces, vol. 2014, Article ID 710542, 9 pp. (2014).
11. Guliyev, V.S., Omarova, M.N.: *Multilinear singular and fractional integral operators on generalized weighted Morrey spaces*, Azerb. J. Math. **5** (1), 104-132 (2015).
12. Guliyev, V.S., Hamzayev V.H.: *Rough singular integral operators and its commutators on generalized weighted Morrey spaces*, Math. Ineq. Appl. **19** (3), 863-881 (2016).
13. Guliyev, V.S., Ahmadli, A., Ekincioglu, S.E.: *Oscillatory integrals with variable Calderón-Zygmund kernel on vanishing generalized Morrey spaces*, Tbilisi Math. J. **13** (1), 69-82 (2020).
14. Guliyev, V.S., Omarova, M.N.: *Estimates for operators on generalized weighted Orlicz-Morrey spaces and their applications to non-divergence elliptic equations*, Positivity **26** (2), 40 (2022).
15. Eroglu, A.: *Boundedness of fractional oscillatory integral operators and their commutators on generalized Morrey spaces*, Bound. Value Probl. 2013:70, 12 pp. (2013).
16. Hamzayev, V.H.: *Maximal operator with rough kernel and its commutators in generalized weighted Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **40** (1), Mathematics, 96110 (2020).
17. Ismayilova, A.F.: *Fractional maximal operator and its higher order commutators on generalized weighted Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **39** (4), Mathematics, 8495 (2019).
18. Komori, Y., Shirai, S.: *Weighted Morrey spaces and a singular integral operator*, Math. Nachr. **282** (2), 219-231 (2009).
19. Lu, S., Yang, D., Zhou, Z.: *On local oscillatory integrals with variable Calderón-Zygmund kernels*, Integral Equations Operator Theory **33** (4), 456-470 (1999).
20. Mizuhara, T.: *Boundedness of some classical operators on generalized Morrey spaces*, Harmonic Analysis (S. Igari, Editor), ICM 90 Satellite Proceedings, Springer - Verlag, Tokyo, 183-189 (1991).
21. Morrey, C.B.: *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43**, 126-166 (1938).
22. Nakai, E.: *Hardy–Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces*, Math. Nachr. **166**, 95-103 (1994).
23. Pan, Y.: *Uniform estimate for oscillatory integral operators*, Y. Funct. Anal. 130 (1991), 207-220.
24. Sawano, Y.: *A thought on generalized Morrey spaces*, J. Indones. Math. Soc. **25** (3), 210-281 (2019).

25. Sawano, Y., Sugano, S., Tanaka, H.: *A note on generalized fractional integral operators on generalized Morrey spaces*, Bound. Value Probl. Art. ID 835865, 18 pp. (2009).
26. Softova, L.: *Singular integrals and commutators in generalized Morrey spaces*, Acta Math. Sin. (Engl. Ser.) **22** (3), 757-766 (2006).
27. Stein, E.M., Weiss, G.: Introduction to Fourier Analysis on Euclidean Spaces, *Princeton University Press, Princeton, NJ, USA* (1971).