Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **42** (1), 127-131 (2022).

On a generalized Norden-Walker 4-manifold

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Received: 07.05.2021 / Revised: 03.11.2021 / Accepted: 22.12.2021

Abstract. *The aim of this paper is to express geometric properties of a generalized almost complex structure on 4-dimensional Walker manifolds. We study the integrability and Kähler (holomorphic) conditions of a generalized Norden-Walker structure by using of the vanishing of Nijenhuis tensor and the Tachibana operator applied to the Walker metric.*

Keywords. Almost complex structure, Holomorphic metrics, Norden metric, Walker 4-manifolds.

Mathematics Subject Classification (2010): 53C15, 53C25, 53B30

1 Introduction

The investigation of some classes of four-dimensional Norden-Walker manifolds is important in the context of mainstream of modern differential geometry. Walker obtained a local canonical form for the pseudo-Riemannian metric of a $C^{\infty}-$manifold [10, Theorem 3.1]. Moreover, he proved that the Walker metric of dimension 4 is depending on three smooth functions [10, p. 76].

Let (M_{2n}, g) be a Riemannian manifold, with a neutral metric, i.e., with a pseudo-Riemannian metric g of signature (n, n). $\Im_q^p(M_{2n})$ is a set of all tensor fields of type (p, q) on M_{2n}. Manifolds and tensor fields are belonged to the class C^{∞} .

Next let (M_{2n}, φ, g) be an almost complex manifold, i.e. we assume that φ is an almost complex structure satisfying $\varphi^2 = -I$. An almost complex structure φ is said to be integrable if φ is reduced to the constant form in a collection of holonomic (natural) coordinates on M_{2n} [3]. Also, an almost complex structure φ is integrable if and only if the Nijenhuis tensor $N_{\varphi} \in \Im_2^1(M_{2n})$ vanishes [6, p. 124]. The triple (M_{2n}, φ, g) is called complex manifold if φ is integrable.

We say that a neutral metric g is a Norden metric [9] if

$$g(\varphi X, \varphi Y) = -g(X, Y)$$

or equivalently

$$g(\varphi X, Y) = g(X, \varphi Y)$$

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where $X, Y \in \Im_0^1(M_{2n})$. An almost Norden manifold is a triple (M_{2n}, φ, g) with the Norden metric g. The triple is called Norden manifold if φ is integrable.

We say that a Norden metric g on a Norden manifold (M_{2n}, φ, g) is holomorphic if

$$(\varPhi_\varphi g)(X, Y, Z) = 0$$

for any vector fields X, Y, Z on M_{2n}, where $\varPhi_\varphi g$ is the Tachibana operator [11]:

$$(\varPhi_\varphi g)(X, Y, Z) = (\varphi X)(g(Y, Z)) - Xg(\varphi Y, Z) + g((L_Y\varphi)X, Z) + g(Y, (L_Z\varphi)X). \tag{1.1}$$

By assigning natural vector fields instead of vector fields X, Y, Z in the equation (1.1), we can write this equation in coordinates such as

$$(\varPhi_\varphi g)_{kij} = \varphi_k^m \partial_m g_{ij} - \varphi_i^m \partial_k g_{mj} + g_{mj}(\partial_i \varphi_k^m - \partial_k \varphi_i^m) + g_{im}\partial_j \varphi_k^m.$$

A triple (M_{2n}, φ, g) is holomorphic Norden manifold if g is the holomorphic Norden metric. In some literatures, holomorphic Norden manifolds and Kähler manifolds are identical [2, p. 73], [4].

2 Walker metric

Let M_4 be a $4-$dimensional $C^\infty-$manifold. A neutral metric g on a manifold M_4 is said to be Walker metric if there is a totally isotropic parallel $2-$dimensional null distribution D on M_4. By a result of Walker theorem [10, p. 76], for every Walker metric g on a $4-$manifold M_4, there exist a system of coordinates which the matrix of $g = (g_{ij})$ in these coordinates has following form:

$$g = (g_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix}, \tag{2.1}$$

where $a,\ b,\ c$ are differentiable functions depending on the coordinates (x, y, z, t). The parallel $2-$dimensional null distribution D is spanned locally by $\{\partial_x, \partial_y\}$, where $\partial_x = \frac{\partial}{\partial x}$, $\partial_y = \frac{\partial}{\partial y}$.

Let be an almost complex structure on a Walker $4-$manifold M_4, which satisfies

1 $\varphi^2 = -I$,
2 $g(\varphi X, Y) = g(X, \varphi Y)$ (Nordenian property)
3 $\varphi\partial_x = \partial_y,\ \varphi\partial_y = -\partial_x$ (φ induces a positive $\frac{\pi}{2}$ rotation on the degenerate parallel field D).

The almost complex structure φ is completely determined by the metric as follows:

$$\begin{cases} \varphi\partial_x = \partial_y \\ \varphi\partial_y = -\partial_x \\ \varphi\partial_z = d\partial_x + \frac{1}{2}(a+b)\partial_y - \partial_t \\ \varphi\partial_t = -\frac{1}{2}(a+b)\partial_x + d\partial_y + \partial_z \end{cases}$$

and φ has the local components with respect to the natural frame $\{\partial_x, \partial_y, \partial_z, \partial_t\}$

$$\varphi = \left(\varphi_j^i\right) = \begin{pmatrix} 0 & -1 & d & -\frac{1}{2}(a+b) \\ 1 & 0 & \frac{1}{2}(a+b) & d \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{2.2}$$

where $d = d(x, y, z, t)$ is an arbitrary function.

The triple (M_4, φ, g) is called generalized almost Norden-Walker manifold. In some literature [1], [7], [8] φ with $d = c$ is called the proper almost complex structure. Our purpose here is to investigate integrability and holomorphic (Kähler) conditions of a generalized almost complex structure φ.

3 Norden-Walker manifold

An almost complex structure φ is integrable if the Nijenhuis tensor N_φ with the coordinates

$$(N_\varphi)_{jk}^i = \varphi_j^m \partial_m \varphi_k^i - \varphi_k^m \partial_m \varphi_j^i - \varphi_m^i \partial_j \varphi_k^m + \varphi_m^i \partial_k \varphi_j^m = 0$$

vanishes.

From (2.1) and (2.2), we have

$$(N_\varphi)_{13}^1 = (N_\varphi)_{24}^1 = (N_\varphi)_{31}^1 = (N_\varphi)_{42}^1 = (N_\varphi)_{14}^2 = (N_\varphi)_{23}^2 = (N_\varphi)_{32}^2 = (N_\varphi)_{41}^2 = a_x + b_x + 2d_y = 0,$$

$$(N_\varphi)_{14}^1 = (N_\varphi)_{23}^1 = (N_\varphi)_{32}^1 = (N_\varphi)_{41}^1 = (N_\varphi)_{13}^2 = (N_\varphi)_{24}^2 = (N_\varphi)_{31}^2 = (N_\varphi)_{42}^2 = a_y + b_y - 2d_x = 0,$$

$$(N_\varphi)_{34}^1 = -(N_\varphi)_{43}^1 = -\frac{1}{2} d(a_x + b_x + 2d_y) - \frac{1}{4}(a+b)(a_y + b_y - 2d_x) = 0,$$

$$(N_\varphi)_{34}^2 = -(N_\varphi)_{43}^2 = -\frac{1}{2} d(a_y + b_y - 2d_x) + \frac{1}{4}(a+b)(a_x + b_x + 2d_y) = 0.$$

So we have obtained the following theorem:

Theorem 3.1 *An almost complex structure φ on a generalized almost Norden-Walker manifold is integrable if and only if*

$$\begin{cases} a_x + b_x + 2d_y = 0, \\ a_y + b_y - 2d_x = 0. \end{cases}$$

From here we have the following identities:

$$\begin{cases} a_{xy} + b_{xy} + 2d_{yy} = 0, \\ a_{yx} + b_{yx} + 2d_{xx} = 0. \end{cases} \Rightarrow d_{xx} + d_{yy} = 0,$$

i.e., if an almost complex structure φ is integrable, then the function d is harmonic with respect to the arguments x and y. Thus we have

Theorem 3.2 *If the triple (M_4, φ, g) is a generalized Norden-Walker manifold, then d is harmonic with respect to the arguments x and y.*

4 Holomorphic Norden-Walker manifold

Now let (M_4, φ, g) be a generalized almost Norden-Walker manifold. First, we note that if

$$(\Phi_\varphi g)_{kij} = \varphi_k^m \partial_m g_{ij} - \varphi_i^m \partial_k g_{mj} + g_{mj}\left(\partial_i \varphi_k^m - \partial_k \varphi_i^m\right) + g_{im} \partial_j \varphi_k^m = 0,$$

then φ is integrable and the manifold (M_4, φ, g) is called a holomorphic Norden-Walker or a Kähler-Norden-Walker manifold [3].

After some straightforward calculations, we have

$$(\Phi_\varphi g)_{xzz} = a_y + c_x - d_x,$$

$$(\Phi_\varphi g)_{xzt} = (\Phi_\varphi g)_{xtz} = \tfrac{1}{2}(b_x - a_x) + c_y,$$

$$(\Phi_\varphi g)_{xtt} = b_y - c_x - d_x, \quad (\Phi_\varphi g)_{yzz} = -a_x + c_y - d_y,$$

$$(\Phi_\varphi g)_{yzt} = (\Phi_\varphi g)_{ytz} = -c_x + \tfrac{1}{2}(b_y - a_y),$$

$$(\Phi_\varphi g)_{ytt} = -b_x - c_y - d_y,$$

$$(\Phi_\varphi g)_{zxz} = (\Phi_\varphi g)_{zzx} = (\Phi_\varphi g)_{txt} = (\Phi_\varphi g)_{ttx} = d_x,$$

$$(\Phi_\varphi g)_{zxt} = (\Phi_\varphi g)_{ztx} = -(\Phi_\varphi g)_{txz} = -(\Phi_\varphi g)_{tzx} = \tfrac{1}{2}(a_x + b_x),$$

$$(\Phi_\varphi g)_{zyz} = (\Phi_\varphi g)_{zzy} = (\Phi_\varphi g)_{tyt} = (\Phi_\varphi g)_{tty} = d_y,$$

$$(\Phi_\varphi g)_{zyt} = (\Phi_\varphi g)_{zty} = -(\Phi_\varphi g)_{tyz} = -(\Phi_\varphi g)_{tzy} = \tfrac{1}{2}(a_y + b_y),$$

$$(\Phi_\varphi g)_{zzz} = da_x - a_t + c_z + d_z + \tfrac{1}{2}(a+b)\,a_y,$$

$$(\Phi_\varphi g)_{zzt} = (\Phi_\varphi g)_{ztz} = dc_x - c_t + b_z + d_t + \tfrac{1}{2}(a+b)\,c_y,$$

$$(\Phi_\varphi g)_{ztt} = db_x - c_z + d_z + a_t + \tfrac{1}{2}(a+b)\,b_y,$$

$$(\Phi_\varphi g)_{tzz} = da_y - b_z + c_t - d_t - \tfrac{1}{2}(a+b)\,a_x,$$

$$(\Phi_\varphi g)_{tzt} = (\Phi_\varphi g)_{ttz} = dc_y + c_z - a_t + d_z - \tfrac{1}{2}(a+b)\,c_x,$$

$$(\Phi_\varphi g)_{ttt} = db_y + b_z - c_t + d_t - \tfrac{1}{2}(a+b)\,b_x.$$

So, we have the following theorem:

Theorem 4.1 *The generalized Norden-Walker manifold (M_4, φ, g) is holomorphic if and only if the following PDEs hold:*

$$\begin{array}{l} a_x = -b_x = c_y, \ \ a_y = -b_y = -c_x, \quad d_x = d_y = 0, \\ da_x - a_t + c_z + d_z + \tfrac{1}{2}(a+b)\,a_y = 0, \\ da_y - b_z + c_t - d_t - \tfrac{1}{2}(a+b)\,a_x = 0. \end{array}$$

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