Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **42** (1), 162-171 (2022).

Location of eigenvalues and structures of root subspaces of some spectral problem for the equation of a vibrating rod

Faiq M. Namazov

Received: 07.11.2021 / Revised: 27.04.2022/ Accepted: 09.05.2022

Abstract. *In this paper we consider a eigenvalue problem for ordinary differential equations of fourth order with a spectral parameter contained in two of boundary conditions. This problem describes the bending vibrations of a homogeneous rod, in cross-sections of which the longitudinal force acts, at both ends of which elastically fixed loads are concentrated. We investigate the location of eigenvalues on the complex plane (the real axis) and study the structures of root subspaces of this problem.*

Keywords. eigenvalue problem, spectral parameter in boundary conditions, eigenfunction, system of root functions, oscillatory properties of eigenfunctions

Mathematics Subject Classification (2010): 34B05, 34B08, 34B09, 34L10, 34L15, 47A75, 47B50, 74H45

1 Introduction

Consider the homogeneous Euler-Bernoulli beam of length L, density ρ and cross-sectional area F, at both ends of which loads with masses of m_1 and m_2 are concentrated. These loads are elastically fixed to springs with stiffnesses κ_1 and κ_2, respectively, preventing the vertical displacement of the beam.

The free bending vibrations of a homogeneous rod of constant rigidity, in cross sections of which the longitudinal force acts, are described by the equation [13, Ch. 8, § 5, formula (84)]

$$EJ\frac{\partial^4 U(X,t)}{\partial X^4}-\frac{\partial}{\partial X}\left(\overline{Q}(X)\frac{\partial U(X,t)}{\partial X}\right)+\rho F\frac{\partial^2 U(X,t)}{\partial t^2}=0,$$

where $U(X,t)$ is a flexure of the current point of axis of the rod, EJ is the flexural rigidity of the rod, $\overline{Q}(X)$ is longitudinal force.

If both ends are fixed elastically and there are concentrated loads on these ends, then the boundary conditions can be written in the following form [13, Ch. 8, § 5, p. 154]:

$$EJ\frac{\partial^2 U(0,t)}{\partial X^2}=0,$$

F.M. Namazov
Baku State University, Baku AZ1148, Azerbaijan
E-mail: faig-namazov@mail.ru

$$EJ\frac{\partial^3 U(0,t)}{\partial X^3} - \overline{Q}(0)\frac{\partial U(0,t)}{\partial X} - c_1 U(0,t) = m_1\frac{\partial^2 U(0,t)}{\partial t^2},$$
$$EJ\frac{\partial^2 U(L,t)}{\partial X^2} = 0,$$
$$EJ\frac{\partial^3 U(L,t)}{\partial X^3} - \overline{Q}(L)\frac{\partial U(L,t)}{\partial X} + c_2 U(L,t) = -m_2\frac{\partial^2 U(L,t)}{\partial t^2}.$$

Introducing the notation $x = \frac{X}{L}$, $u = \frac{U}{L}$ we write these equations and the boundary conditions in the following form

$$\frac{\partial^4 u(x,t)}{\partial x^4} - \frac{\partial}{\partial x}\left(Q(x)\,\frac{\partial u(x,t)}{\partial x}\right) + \frac{\rho F L^4}{EJ}\frac{\partial^2 u(x,t)}{\partial t^2} = 0,$$
$$\frac{\partial^2 u(0,t)}{\partial x^2} = 0,$$
$$\frac{\partial^3 u(0,t)}{\partial x^3} - Q(0)\,\frac{\partial u(0,t)}{\partial x} - \frac{k_1 L^3}{EJ}u(0,t) = \frac{m_1 L^3}{EJ}\frac{\partial^2 u(0,t)}{\partial t^2},$$
$$\frac{\partial^2 u(1,t)}{\partial x^2} = 0,$$
$$\frac{\partial^3 u(1,t)}{\partial x^3} - Q(1)\,\frac{\partial u(1,t)}{\partial x} + \frac{k_2 L^3}{EJ}u(1,t) = -\frac{m_2 L^3}{EJ}\frac{\partial^2 u(1,t)}{\partial t^2},$$

where $Q(x) = \frac{L^2}{EJ}\,\overline{Q}(Lx)$.

By λ we denote $\rho F L^4\omega^2/EJ$. Then, by the change of variables $u(x,t) = y(x)\,\cos\omega t$ [13, Ch. 11, § 2, formula (12)]), this problem reduces to the following spectral problem:

$$y^{(4)}(x) - (q(x)y'(x))' = \lambda y(x),\ x \in (0,1), \tag{1.1}$$
$$y''(0) = 0,\ y''(1) = 0, \tag{1.2}$$
$$Ty(0) - (a\lambda + b)\,y(0) = 0, \tag{1.3}$$
$$Ty(1) - (c\lambda - d)\,y(1) = 0, \tag{1.4}$$

where $q(x) \equiv Q(x)$, $Ty \equiv y''' - qy'$, $a = -\frac{m_1}{\rho FL}$, $b = \frac{\kappa_1 L^3}{EJ}$, $c = \frac{m_2}{\rho FL}$ and $d = \frac{\kappa_2 L^3}{EJ}$. Hence it follows that $q(x) > 0$, $x \in [0,1]$, $a < 0$, $b > 0$, $c > 0$ and $d > 0$. In addition, we suppose that $q(x)$ is an absolutely continuous function on $[0,1]$.

Problem (1.1)-(1.4) in the case $b = d = 0$ was considered in [8]. Should be noted that in this case problem (1.1)-(1.4) describing bending vibrations of a homogeneous rod, in cross-sections of which the longitudinal force acts, at both ends of which only the masses are concentrated. In [7] we study the general characteristic of location of eigenvalues on the real axis, find the multiplicities of all eigenvalues, investigate the oscillatory properties of eigenfunctions to problem (1.1)-(1.4) with $b = d = 0$. Moreover, we establishe sufficient conditions for the subsystems of root functions of this problem to form a basis in $L_p(0,1)$, $1 < p < \infty$. Similar results for equation (1.1) so for these, also under various boundary conditions, were obtained in the papers [1, 3-8, 21]. In the recent papers [3], the uniform convergence of the expansions of continuous functions in Fourier series in the system of root functions of the spectral problem for equation (1.1) with boundary conditions depending on the spectral parameter (these conditions given at the point $x = 1$) was also studied. Similar results for the Sturm-Liouville problems with a spectral parameter in the boundary conditions were demonstrated in [2, 11, 12, 14-19, 22, 23].

Should be noted that to study the basis properties in the space L_p, $1 < p < \infty$ of the system of root functions of problem (1.1)-(1.4), it is necessary to study the arrangement of eigenvalues on the complex plane (on the real axis) and the structure of root subspaces. The present paper is devoted precisely to the investigation of these spectral properties of problem (1.1)-(1.4) for $d > 0$ and some $b > 0$.

2 Operator interpretation of problem (1.1)-(1.4)

As is known the eigenvalue problem (1.1)-(1.4) is reduce to a spectral problem for the linear operator L in a Hilbert space $H = L_2(0,1) \oplus \mathbb{C}^2$ with the inner product

$$(\hat{u}, \hat{v}) = (\{y, m, n\}, \{v, s, t\}) = (y, v)_{L_2} + \ |a|^{-1} m\bar{s} \ + \ |\, c|^{-1} n\bar{t},$$

where $(y, v)_{L_2} = \int\limits_0^1 y(x)\, \overline{v(x)}\, dx$, and L defined by

$$L\hat{y} = L\{y, m, n\} = \{(Ty(x))',\, Ty(0) - b\, y(0),\, Ty(1) + d\, y(1)\}$$

on

$$D(L) = \{\{y\ (x),\, m,\, n\} \ : \ y \in W_2^4(0,1),\ (Ty(x))' \in L_2(0,1),$$

$$y''(0) = y''(1) = 0,\ m = ay(0),\ n = cy(1)\}$$

It is obvious that problem (1.1)-(1.4) is equivalent to the problem

$$L\hat{y} = \lambda \hat{y},\ \hat{y} \in D(L), \tag{2.1}$$

i.e., the eigenvalues λ_k, $k \in \mathbb{N}$, of problem (1.1)-(1.4) and (2.1) coincide together with their multiplicities, and there exists one-to-one correspondence between the root functions of these problems,

$$y_k(x) \leftrightarrow \ \{y_k(x),\, m_k,\, n_k\},\ \ m_k = ay_k(0),\ n_k = cy_k\,(1).$$

By conditions $a < 0$ and $c > 0$ the operator L is closed (nonself-adjoint) in H with compact resolvent. In this case we define an operator $J : H \to H$ as follows:

$$J\{y, m; n\} = \{y, -m, -n\}.$$

It is easy to show that J is a unitary, symmetric operator on H and its spectrum consists of two eigenvalues: $-\,1$ with multiplicity two and $+\,1$ with infinite multiplicity. Consequently, this operator generates the Pontryagin space $\Pi_2 = L_2(0,1) \oplus \mathbb{C}^2$ equipped with inner product (J-metric)[9]

$$[\hat{y}, \hat{v}] = (\hat{y}, \hat{v})_{\Pi_2} = (\{y, m, n\}, \{v, s, t\})_{\Pi_2} = (y, v)_{L_2} + a^{-1} m\bar{s} \ - c^{-1} n\bar{t},$$

Theorem 2.1 (see [8, Theorem 1]) *L is $J-$self-adjoint operator in Π_2.*

Let λ be an eigenvalue of L with algebraic multiplicity ν, and let $\varrho(\lambda)$ to be equal to ν if $\Im(\lambda) \neq 0$, and to the integer part of $\frac{\nu}{2}$ if $\Im(\lambda) = 0$.

Theorem 2.2 (see [24, Theorem 3]) *The eigenvalues of the operator L are arranged symmetrically around the real axis, and*

$$\sum_{k=1}^{n} \varrho(\lambda_k) \leq 2$$

for any system $\{\lambda_k\}_{k=1}^n$, $n \leq +\infty$, of eigenvalues with nonnegative imaginary parts.

By virtue of Theorem 2.2 the spectral problem (1.1)-(1.4) may have real multiple eigenvalues the sum of algebraic multiplicities of which does not exceed five, or non-real eigenvalues the sum of algebraic multiplicities whose imaginary parts are positive does not exceed two. But below in § 5 we will prove that in the case $b < b_0$ (see § 3) problem (1.1)-(1.4) have either one real multiple eigenvalue whose multiplicity does not exceed three or one pair of complex conjugate non-real eigenvalues.

3 Some auxiliary facts and assertions

Alongside the spectral problem (1.1)-(1.4) we will consider the following eigenvalue problems :

$$\begin{cases} y^{(4)}(x) - (q(x)y'(x))' = \lambda y(x),\ x \in (0,1), \\ y''(0) = 0,\ y(0)\cos\beta + Ty(0)\sin\beta = 0, \\ y''(1) = 0,\ y(1)\cos\delta - Ty(1)\sin\delta = 0. \end{cases} \tag{3.1}$$

and

$$\begin{cases} y^{(4)}(x) - (q(x)y'(x))' = \lambda y(x),\ x \in (0,1), \\ y''(0) = 0,\ Ty(0) - (a\lambda + b)y(0) = 0, \\ y''(1) = 0,\ y(1)\cos\delta - Ty(1)\sin\delta = 0. \end{cases} \tag{3.2}$$

It follows [10, Theorem 5.4] and [20, Remark 3] that the eigenvalues of the spectral problem (3.1) are real, simple and form an unboundedly increasing sequence $\{\lambda_k(\beta,\delta)\}_{k=1}^{\infty}$ such that $\lambda_1(\beta,\delta) > 0$ for $\beta + \delta < \pi$ and $\lambda_1\left(\frac{\pi}{2},\frac{\pi}{2}\right) = 0$; the eigenfunction $v_k^{(\beta,\delta)}(x)$, $k \in \mathbb{N}$, corresponding to the eigenvalue $\lambda_k(\beta,\delta)$, has exactly $k-1$ simple zeros in $(0,1)$. Moreover, by [10, Property 1] for each $k \in \mathbb{N}$ the eigenvalues $\lambda_k(\beta,\delta)$ is a continuous, strictly decreasing function of β and δ.

Let $D_k = (\lambda_k(0,\pi/2), \lambda_{k-1}(0,\pi/2))$, $k \in \mathbb{N}$, where $\lambda_0(0,\pi/2) = -\infty$.

By making the change of variables $t = 1 - x$, we transform the problem (3.2) into the eigenvalue problem

$$\begin{cases} v^{(4)}(t) - (\tilde{q}(t)v'(t))' = \lambda v(t),\ t \in (0,1), \\ v''(0) = 0,\ v(0)\cos\widetilde{\beta} + \tilde{T}v(0)\sin\widetilde{\beta} = 0, \\ v''(1) = 0,\ \tilde{T}v(1) - (\tilde{a}\lambda - \tilde{b})\, v(1) = 0, \end{cases} \tag{3.3}$$

where $v(t) = y(1-t),\ \widetilde{q}(t) = q(1-t)$, $(\widetilde{T}v)(t) = v'''(t) - \tilde{q}(t)v'(t),\ t \in [0,1]$, $\widetilde{\beta} = \delta$, $\widetilde{a} = -a > 0$ and $\widetilde{b} = b$. The problem (3.3) in a more general setting was considered in [1], where the spectral properties (except for the oscillatory properties of eigenfunctions) of this problem were studied in detail.

By virtue of [20, Lemma 2] for each fixed $\widetilde{\beta} \in \left[0, \frac{\pi}{2}\right]$ and each $\lambda \in \mathbb{C}$ there exits the unique (to within constant factor) nontrivial solution $v(x,\widetilde{\beta},\lambda)$ of problem

$$\begin{gathered} v^{(4)}(t) - (\tilde{q}(t)v'(t))' = \lambda v(t),\ t \in (0,1), \\ v''(0) = 0,\ v(0)\cos\widetilde{\beta} + \widetilde{T}v(0)\sin\widetilde{\beta} = 0,\ v''(1) = 0. \end{gathered} \tag{3.4}$$

Remark 3.1 Without loss of generality, we can assume that the solution $v(x,\delta,\lambda)$ of problem (3.4) is an entire function of λ for each $x \in [0,1]$ and each $\delta \in \left[0, \frac{\pi}{2}\right]$.

By virtue of [20, Remark 3] we have

$$\widetilde{b}_0^* = \frac{Tv(1,0,0)}{v(1,0,0)} < 0 \text{ and } \frac{Tv(1,0,0)}{v\,(1,\pi/2,0)} = 0.$$

Then it follows from [1, Theorem 4.1] that the following result holds.

Theorem 3.1 *The eigenvalues of problem* (3.2) *with* $b < b_0$, *where* $b_0 = -\widetilde{b}_0^*$, *and* $\delta = 0$ *are real, simple and form an unboundedly increasing sequence* $\{\lambda_k(0)\}_{k=1}^{\infty}$ *such that* $\lambda_1(0) < 0$ *and* $\lambda_k(0) > 0$ *for* $k \geq 2$. *The eigenvalues of problem* (3.2) *with* $b < b_0$ *and* $\delta = \frac{\pi}{2}$ *form an infinite sequence* $\{\lambda_k\left(\frac{\pi}{2}\right)\}_{k=1}^{\infty}$, *accumulating only at* $+\infty$, *and only following cases are possible:*

(i) all eigenvalues are real; in this case, the interval D_1 contains algebraically two eigenvalues (either two simple eigenvalues or one double eigenvalue), and the interval D_k for $k \geq 2$ contains one simple eigenvalue;

(ii) all eigenvalues are real; in this case, the interval D_1 contains no eigenvalues, and there exists a natural number $m_0 \geq 2$ such that D_{m_0} contains algebraically three eigenvalues (either three simple eigenvalues, or one double eigenvalue and one simple eigenvalue, or one triple eigenvalue), and the interval D_k for $k \geq 2$, $k \neq m_0$, contains one simple eigenvalue;

(iii) this problem has one pair of non-real complex conjugate eigenvalues; in this case, the interval D_1 contains no eigenvalues, and the interval D_k for $k \geq 2$ contains one simple eigenvalue.

But in below in § 5 we show that the statement (ii) of Theorem 3.1 does not hold.

Theorem 3.2 *The negative eigenvalues of spectral problem* (3.2) *is a continuous strictly increasing function of δ, and the positive eigenvalues of the same problem are continuous strictly decreasing functions of δ.*

To prove this theorem, first the problem (3.2) is reduced to a spectral problem in the corresponding space with an indefinite metric, and then the max-min properties of eigenvalues of [25, § 2] is applied.

Remark 3.2 By virtue of Theorems 3.1 and 3.2 we have the following relations:

$$\lambda_1(0) < \lambda_1(\pi/2) \leq \lambda_2(\pi/2) < \lambda_2(0) < \lambda_3(\pi/2) < \lambda_3(0) < \lambda_4(\pi/2) < \ldots,$$

if statement (i) of Theorem 3.1 holds (in this case either $\lambda_1(\pi/2), \lambda_2(\pi/2) < 0$ or $\lambda_1(\pi/2), \lambda_2(\pi/2) > 0$)),

$$\lambda_1(0) < 0 < \lambda_2(0) < \lambda_3(\pi/2) < \lambda_3(0) < \lambda_4(\pi/2) < \lambda_4(0) < \ldots,$$

if statement (iii) of Theorem 3.1 holds.

4 The main properties of the initial-boundary value problem (1.1)-(1.3)

Theorem 4.1 *Let $b < b_0$. Then for each fixed $\lambda \in \mathbb{C}$ there exists a nontrivial solution $y(x, \lambda)$ of the problem* (1.1)-(1.3) *which is unique up to a constant factor.*

Proof. Let $\varphi_k(x, \lambda), k = \overline{1, 4}$, be solutions of equation (1.1), normalized for $x = 0$ by the Cauchy conditions

$$\varphi_k^{(s-1)}(0, \lambda) = \delta_{ks},\ s = \overline{1, 3},\ T\varphi_k(0, \lambda) = \delta_{k4}, \tag{4.1}$$

where δ_{ks} is the Kronecker delta.

We will seek the function $y(x, \lambda)$ in the following form:

$$y(x, \lambda) = \sum_{k=1}^{4} C_k \varphi_k(x, \lambda), \tag{4.2}$$

where $C_k,\ k = 1,\ 2,\ 3,\ 4$, are constants.

It follows from (4.1), (4.2) and boundary conditions (1.2), (1.3) that $C_3 = 0$, $C_4 = (a\lambda + b)\, C_1$ and

$$C_1\{\varphi_1''(1,\lambda) + (a\lambda + b)\,\varphi_4''(1,\lambda)\} + C_2\,\varphi_2''(1,\lambda) = 0.$$

To complete the proof of this theorem it suffices to show that

$$|\,\varphi_1''(1,\lambda) + (a\lambda + b)\,\varphi_4''(1,\lambda)| + |\,\varphi_2''(1,\lambda)|\, > 0. \tag{4.3}$$

If $\lambda > 0$, then it follows from [10, Lemma 2.1] that $\varphi_k''(1,\lambda) > 0,\ k = 1,\,2,\,3,\,4$. Hence (4.3) holds for $\lambda > 0$.

Let $\lambda \in \mathbb{C}\backslash(0,+\infty)$. If (4.3) is fails for such λ, then the functions $\varphi_1(x,\lambda) + (a\lambda + b)\,\varphi_4(x,\lambda)$ and $\varphi_2(x,\lambda)$ solve the problem (1.1)-(1.3). We now define the functions $v(x,\lambda)$ and $w(x,\lambda)$ as follows:

$$\begin{aligned} v(x,\lambda) &= \varphi_2(1,\lambda)\,\{\varphi_1(x,\lambda) + (a\lambda + b)\,\varphi_4(x,\lambda)\} \\ &\quad - \{(\varphi_1(1,\lambda) + (a\lambda + b)\varphi_4(1,\lambda)\}\,\varphi_2(x,\lambda), \\ w(x,\lambda) &= T\varphi_2(1,\lambda)\,\{\varphi_1(x,\lambda) + (a\lambda + b)\,\varphi_4(x,\lambda)\} \\ &\quad - \{(T\varphi_1(1,\lambda) + (a\lambda + b)T\varphi_4(1,\lambda)\}\,\varphi_2(x,\lambda). \end{aligned}$$

Since $v(1,\lambda) = 0$ and $Tw(1,\lambda) = 0$, the functions $v(x,\lambda)$ and $w(x,\lambda)$ are eigenfunctions of the spectral problem (3.2) for $\delta = 0$ and $\delta = \pi/2$, respectively, corresponding to the same eigenvalue λ, which contradicts Remark 3.3. This contradiction proves (4.3). The proof of this theorem is complete.

Remark 4.1 By Theorem 4.1 we can represent the solution $y(x,\lambda)$ of problem (1.1)-(1.3) in the following form:

$$\begin{aligned} y(x,\lambda) &= \varphi_2''(1,\lambda)\,\{\varphi_1(x,\lambda) + (a\lambda + b)\,\varphi_4(x,\lambda)\} \\ &\quad - \left\{\varphi_1''(1,\lambda) + (a\lambda + b)\,\varphi_4''(1,\lambda)\right\}\,\varphi_2(x,\lambda). \end{aligned}$$

Then it follows from the general theory of linear differential equations that $y(x,\lambda)$ is an entire function of λ for each fixed $x \in [0,1]$.

Remark 4.2 Set $m(\lambda) = ay(0,\lambda),\ n(\lambda) = c\,y(1,\lambda)$. Note that, if λ is an eigenvalue of problem (1.1)-(1.4), then it follows from Remark 3.3 that $m(\lambda)\,n(\lambda) \neq 0$.

Let $A_k = (\lambda_{k-1}(0), \lambda_k(0)),\ n = 1, 2, \ldots$, where $\lambda_0(0) = -\infty$.

Obviously, the eigenvalues $\lambda_k(0)$ and $\lambda_k\left(\frac{\pi}{2}\right)$ of the spectral problem (3.2) for $\delta = 0$ and $\delta = \frac{\pi}{2}$ are zeros of entire functions $y(1,\lambda)$ and $Ty(1,\lambda)$, respectively. We note that the function $G(\lambda) = \frac{Ty(1,\lambda)}{y(1,\lambda)}$ is defined in

$$A \equiv (\mathbb{C}\backslash\mathbb{R}) \cup \left(\bigcup_{k=1}^{\infty} A_k\right),$$

and, $\lambda_k(0)$ and $\lambda_k\left(\frac{\pi}{2}\right)$, $k \in \mathbb{N}$, are the zeros and poles of this function, respectively.

Lemma 4.1 *One has the formula*

$$\frac{dG(\lambda)}{d\lambda} = \frac{1}{y^2(1,\lambda)}\left\{\int_0^l y^2(x,\lambda)\,dx + ay^2(0,\lambda)\right\},\ \lambda \in D. \tag{4.4}$$

The proof of this lemma is similar to that of [8, Lemma 3].

Lemma 4.2 *The following relation holds:*

$$\lim_{\lambda \to -\infty} G(\lambda) = -\infty. \tag{4.5}$$

The proof of this lemma is similar to that of [8, Lemma 4].

Lemma 4.3 *Let* $b \in (0, b_0)$. *Then we have the following relations:*
(i) $G(\lambda_0(0)+0) = -\infty$ *and* $G(\lambda_1(0)-0) = -\infty$*;*
(ii) $G(\lambda_1(0)+0) = +\infty$ *and* $G(\lambda_2(0)-0) = +\infty$*;*
(iii) $G(\lambda_k(0)+0) = -\infty$ *and* $G(\lambda_{k+1}(0)-0) = +\infty$ *for* $k \geq 2$*;*
(iv) $G(\lambda) < 0$ *in* $(\lambda_1(\pi/2), \lambda_2(\pi/2))$ *in the case* $\lambda_1(\pi/2), \lambda_2(\pi/2) \in \mathbb{R}$, *and* $\lambda_1(\pi/2) \neq \lambda_2(\pi/2)$*,*
$G(\lambda_1(\pi/2)) = 0$ *in the case* $\lambda_1(\pi/2) = \lambda_2(\pi/2) \in \mathbb{R}$*,*
$G(\lambda) > 0$ *in* $(\lambda_1(0), \lambda_2(0))$ *in the case* $\lambda_1(\pi/2), \lambda_2(\pi/2) \in \mathbb{C}\backslash\mathbb{R}$.

Proof. By virtue of (4.5) we have $G(\lambda) < 0$ in $(-\infty, \lambda_1(0)$. Since $\lambda_1(0)$ is a simple pole of the function $G(\lambda)$ it follows that this function changes sign when passing the point $\lambda_1(0)$. Then we get
$G(\lambda) > 0$ for $\lambda \in (\lambda_1(0), \lambda_2(0))$ in the case $\lambda_1(\pi/2), \lambda_2(\pi/2) \in \mathbb{C}\backslash\mathbb{R}$;
$G(\lambda) > 0$ for $\lambda \in (\lambda_1(0), \lambda_1(\pi/2)) \cup (\lambda_2(\pi/2), \lambda_2(0))$
and
$G(\lambda) < 0$ for $\lambda \in (\lambda_1(\pi/2), \lambda_2(\pi/2))$ in the case $\lambda_1(\pi/2), \lambda_2(\pi/2) \in \mathbb{R}$.
Consequently, we have the following relations

$$\lim_{\lambda \to \lambda_1(0)-0} G(\lambda) = -\infty, \quad \lim_{\lambda \to \lambda_1(0)+0} G(\lambda) = +\infty \text{ and } \lim_{\lambda \to \lambda_2(0)-0} G(\lambda) = +\infty.$$

Next, since $\lambda_k(0)$, $k \geq 2$, are simple poles of the function $G(\lambda)$ and by Remark 3.2 the interval A_k for $k \geq 3$, $k \neq m_0$, contain one simple zeros and A_{m_0} contain algebraically three zeros of this function it follows that

$$\lim_{\lambda \to \lambda_k(0)+0} G(\lambda) = -\infty, \quad \lim_{\lambda \to \lambda_{k+1}(0)-0} G(\lambda) = +\infty \text{ for } k \geq 2.$$

The proof of this lemma is complete.

Lemma 4.4 *One has the following representation*

$$G(\lambda) = \sum_{k=1}^{\infty} \frac{\lambda\, c_k}{\mu_k(\lambda - \mu_k)}, \tag{4.6}$$

where $c_k = \operatorname*{res}_{\lambda=\lambda_k(0)} G(\lambda)$, $k \in \mathbb{N}$, *and* $c_1 > 0$, $c_k < 0$, $k \geq 2$.

The proof of this lemma is similar to that of [3, Lemma 3.3] with the use of Theorems 3.1, 3.2, Lemmas 4.1-4.3 and Remarks 3.2, 4.2.

Remark 4.3 In view of (4.6) we have

$$G''(\lambda) = 2\sum_{k=1}^{\infty} \frac{c_k}{(\lambda - \lambda_k(0))^3}, \ \lambda \in A. \tag{4.7}$$

Since $c_1 > 0$ and $c_k < 0$ for $k \geq 2$ it follows from (4.7) that $G''(\lambda) > 0$ for $\lambda \in A_2 = (\lambda_1(0), \lambda_2(0))$, i.e. the function $G(\lambda)$ is convex on A_2.

5 The location of eigenvalues in real axis and structures of root subspaces of problem (1.1)-(1.4)

Below we need the following result.

Lemma 5.1 *Let $b < b_0$. Then the statement (ii) of Remark 3.2 does not hold.*

Proof. If this statement holds, then we have:

$G(\lambda) > 0$ in $(\lambda_1(0), \lambda_2(0))$;

$G(\lambda) < 0$ in $(\lambda_{m_0-1}(0), \lambda_{m_0-1}(\pi/2))$ and $G(\lambda) > 0$ in $\lambda \in (\lambda_{m_0+1}(\pi/2), \lambda_{m_0}(0))$ in the case of A_{m_0} contain three zeros of the function $G(\lambda)$ (two of which are different);

$G(\lambda) > 0$ in $(\lambda_{m_0-1}(\pi/2), \lambda_{m_0}(\pi/2))$ and $G(\lambda) < 0$ in $(\lambda_{m_0}(\pi/2), \lambda_{m_0+1}(\pi/2))$, in the case of $\lambda_{m_0-1}(\pi/2) < \lambda_{m_0}(\pi/2) < \lambda_{m_0+1}(\pi/2)$,

$G(\lambda) > 0$ in $(\lambda_{m_0-1}(\pi/2), \lambda_{m_0}(\pi/2))$ in the case of $\lambda_{m_0-1}(\pi/2) < \lambda_{m_0}(\pi/2) = \lambda_{m_0+1}(\pi/2)$;

$G(\lambda) < 0$ in $(\lambda_{m_0}(\pi/2), \lambda_{m_0+1}(\pi/2))$ in the case of $\lambda_{m_0-1}(\pi/2) = \lambda_{m_0}(\pi/2) < \lambda_{m_0+1}(\pi/2)$.

Suppose that A_{m_0} contain three different zeros of the function $G(\lambda)$. Then by virtue of above relations there exist the numbers $c_0 < 0$ and $d_0 < 0$ such that the line $c_0\lambda - d_0$ tangent to the graph of the function $G(\lambda)$ at some points $\lambda_0^* \in D_2$ and $\lambda_0^{**} \in D_{m_0}$. Then λ_0^* and λ^{**} are real double eigenvalues of the spectral problem (1.1)-(1.4) with $c = c_0$ and $d = d_0$. In other hand as is known that (see [7, Theorem 4.1]) problem (1.1)-(1.4) with $c = c_0$ and $d = d_0$ reduce to the eigenvalue problem for the linear self-adjoint operator in the Pontryagin space $\Pi_1 = L_2(0,1) \oplus \mathbb{C}$. Hence by [24, Theorem 3] this operator can have only one real multiple eigenvalue. The obtained contradiction shows that the case considered by us above is impossible.

The case when A_{m_0} contain two different zeros of the function $G(\lambda)$ is considered in a similar way.

Now let A_{m_0} contain one triple zero of the function $G(\lambda)$. Then for $\tilde{a} < a$ and $\tilde{c} > c$ sufficiently close to a and c, respectively, the function $\tilde{G}(\lambda)$ corresponding to the initial boundary-value problem (1.1)-(1.3) with (a, c) replaced by $(\tilde{a}, \tilde{c})$ in the interval A_{m_0} has three different zeros. Repeating now the above reasoning, we come to a contradiction, and consequently, the case considered by us above is also impossible. The proof of this lemma is complete.

The following theorem is the main result of this paper.

Theorem 5.1 *Let $b < b_0$. Then one of the following statements holds.*

(i) all eigenvalues of problem (1.1)-(1.4) *are real and simple; in this case either A_2 contains two eigenvalues, and A_k, $k = 1, 3, 4, \ldots$, contains one eigenvalue, or A_2 contains no eigenvalues, but there exists a positive integer $m_1 \geq 3$ such that A_{m_1} contains three eigenvalues, and A_k, $k = 1, 3, \ldots, k \neq m_1$, contains one eigenvalue;*

(ii) all eigenvalues of problem (1.1)-(1.4) *are simple and real, with the exception of one pair of non-real complex conjugated eigenvalues;*

(iii) all eigenvalues of problem (1.1)-(1.4) *are real; in this case either A_2 contains one double eigenvalue, and A_k, $k = 1, 3, 4, \ldots$, contains one eigenvalue, or A_2 contains no eigenvalues, and A_{m_1} contains algebraically three eigenvalues (either one double eigenvalue and one simple eigenvalue, or one triple eigenvalue), and A_k, $k = 1, 3, \ldots, k \neq m_1$, contains one eigenvalue.*

Proof. By Remark 4.2 it follows from (1.4) that the eigenvalues of problem (1.1)-(1.4) are the roots of the equation

$$G(\lambda) = c\lambda - d. \tag{5.1}$$

Since $c > 0$ and the function $G(\lambda)$ is convex in A_2 there exists $\widetilde{d_c} > 0$ such that the line $c\lambda + d_c$ tangent to the graph of the function $G(\lambda)$ at some point of the interval A_2. Then for small fixed $\tau_0 > 0$ the line $c\lambda + d_{c,0}$, where $d_{c,0} = d_c + \tau_0$, intersects the graph of the function $G(\lambda)$ at two points of the interval A_2. By following the arguments in Lemmas 4.1 and 4.2 of [1] we make sure that Eq. (5.1) with $d = -d_{c,0}$ does not have non-real roots and has a unique root in each interval A_k for $k = 1, 3, 4, \ldots$.

It follows from Lemmas 4.3 and 5.1 that for each $d > 0$ there exist $c_{d,1} > 0$, $c_{d,2} > 0$ such that $c_{d,1} < c_{d,2}$ and the line $c\lambda - d$ for $c \in (c_{d,1}, c_{d,2})$ intersects the graph of the function $G(\lambda)$ at three points of the interval A_{m_1}.

Let $r_k = \lambda_k\left(\frac{\pi}{2}\right) + \epsilon$, where ϵ is a sufficiently small positive number, and let k_0 ($k_0 > m_0 + 2$) be the sufficiently large natural number such that

$$c\,r_{k_0} - d > 0,\ |G(\lambda) - (c\lambda + d_c)| > |d + d_c|,\ \lambda \in \partial B_{r_{k_0}}, \tag{5.2}$$

where $B_r = \{\lambda \in \mathbb{C} : |\lambda| < r\}$ for $r > 0$.

Using (5.2) and following the corresponding arguments given in Theorem 4.1 of [1], we obtain

$$\sum_{\lambda_n \in B_{r_{k_0}}} \rho(\lambda_n) = k_0,$$

and consequently, we have

$$\sum_{\lambda_n \in B_{r_k}} \rho(\lambda_n) = k \text{ for } k \geq k_0. \tag{5.3}$$

Now all the statements of this theorem implies from relation (5.3) in view of Lemma 4.3. The proof of this theorem is complete.

References

1. Aliev, Z.S.: Basis properties in L_p of systems of root functions of a spectral problem with spectral parameter in a boundary condition, *Differ. Equ.* **47**(6), 766-777 (2011).
2. Aliev, Z.S., Dunyamalieva, A.A.: Basis properties of root functions of the Sturm-Liouville problem with a spectral parameter in the boundary conditions, *Dokl. Math.* **88**(1), 441-445 (2013).
3. Aliyev, Z.S., Kerimov, N.B., Mehrabov, V.A.: Convergence of eigenfunction expansions for a boundary value problem with spectral parameter in the boundary conditions, I, *Differ. Equ.* **56**(2), 143-157 (2020).
4. Aliyev, Z.S., Kerimov, N.B., Mehrabov, V.A.: Convergence of eigenfunction expansions for a boundary value problem with spectral parameter in the boundary conditions, II, *Differ. Equ.* **56**(3), 277289 (2020).
5. Aliyev, Z.S., Mamedova, G.T.: Some properties of eigenfunctions for the equation of vibrating beam with a spectral parameter in the boundary conditions, *J. Differential Equations* **269**(2), 1383-1400 (2020).
6. Aliyev, Z.S., Namazov, F.M.: Spectral properties of a fourth-order eigenvalue problem with spectral parameter in the boundary conditions, *Electron. J. Differential Equations* **2017**(307), 1-11 (2017).
7. Aliyev, Z.S., Namazov, F.M.: On the spectral problem arising in the mathematical model of bending vibrations of a homogeneous rod, *Complex Anal. Oper. Theory* **13**(8), 3675-3693 (2019).
8. Aliyev, Z.S., Namazov, F.M.: Spectral properties of the equation of a vibrating rod at both ends of which the masses are concentrated, *Banach J. Math. Anal.* **14**(2), 585-606 (2020).

9. Azizov, T.Ya., Iokhvidov, I.S.: Linear operators in spaces with an indefinite metric, John Vielcy, Chichester, UK, 1989
10. Banks, D.O., Kurowski, G.J.: A Prufer transformation for the equation of a vibrating beam subject to axial forces, *J. Differential Equations* **24**(1), 57-74 (1977)..
11. Binding, P.A., Browne, P.J., Seddici, K.: SturmL-iouville problems with eigenparameter dependent boundary conditions, *Proc. Edinburgh Math. Soc.* **37**(1), 57-72 (1993).
12. Binding, P.A., Browne, P.J.: Application of two parameter eigencurves to Sturm-Liouville problems with eigenparameter dependent boundary conditions, *Proc. Roy. Soc. Edinburgh, Sect. A* **125**, 1205-1218 (1995).
13. Bolotin, B.B.: Vibrations in technique: Handbook in 6 volumes, The vibrations of linear systems, I, Engineering Industry, Moscow, 1978.
14. Kapustin, N.Yu.: On a spectral problem arising in a mathematical model of torsional vibrations of a rod with pulleys at the ends, *Differ. Equ.* **41**(10), 1490-1492 (2005).
15. Kapustin, N.Yu.: On the uniform convergence in C^1 of Fourier series for a spectral problem with squared spectral parameter in a boundary condition. *Differ. Equ.* **47**(10), 14081413 (2011).
16. Kapustin, N.Yu., Moiseev, E.I.: Convergence of spectral expansions for functions of the Hölder class for two problems with a spectral parameter in the boundary condition, *Differ. Equ.* **36**(8), 1182-1188 (2000).
17. Kapustin, N.Yu., Moiseev, E.I.: On the basis property in the space L_p of systems of eigenfunctions corresponding to two problems with spectral parameter in the boundary condition, *Differ. Equ.* **36**(10) 14981501 (2000).
18. Kapustin, N.Yu., Moiseev, E.I.: On the problem of the convergence of spectral expansions for a classical problem with a spectral parameter in the boundary condition, *Differ. Equ.* **37**(2), 1677-1683 (2001).
19. Kerimov, N. B.: Basis properties in L_p of a Sturm-Liouville operator with spectral parameter in the boundary conditions. Differ. Equ. 55(2), 149-158 (2019).
20. Kerimov, N.B., Aliyev, Z.S.: On oscillation properties of the eigenfunctions of a fourth order differential operator, *Trans. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.* **25**(4), 63-76 (2005).
21. Kerimov, N.B., Aliyev, Z.S.: On the basis property of the system of eigenfunctions of a spectral problem with spectral parameter in a boundary condition, *Diff. Equ.* **43**(7), 905-915 (2007).
22. Kerimov, N.B., Goktas, S., Maris, E.A.: Uniform convergence of the spectral expansions in terms of root functions for a spectral problem. *Electron. J. Differential Equations* **2016**(80), 1-14 (2016).
23. Kerimov, N.B., Poladov, R.G.: Basis properties of the system of eigenfunctions in the Sturm-Liouville problem with a spectral parameter in the boundary conditions, *Dokl. Math.* **85**(1), 8-13 (2012).
24. Pontryagin, L.S.: Hermitian operators in a space with indefinite metric, *Izv. Akad. Nauk SSSR Ser. Mat.* **8**(6), 243-280 (1944).
25. Textorius, B.: Minimaxprinzipe zur Bestimmung der Eigenverte J-nichtnegativer Operatoren, *Math. Scand.* **35**(1) 105-114 (1974).