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Extreme Restrained Geodesic Graphs

A.P. Santhakumaran · P. Titus · K. Ganesamoorthy*

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Abstract. For a connected graph $G = (V, E)$ of order at least two, a *geodetic set* of G is a set S of vertices such that every vertex of G lies on a geodesic joining some pair of vertices in S. The *geodetic number* of G is the minimum cardinality of its geodetic sets and is denoted by $g(G)$. A geodetic set $S \subseteq V$ of a graph G is a *restrained geodetic set* if either $S = V$ or the subgraph $G[V - S]$ induced by $V - S$ has no isolated vertex. The minimum cardinality of a restrained geodetic set of G is the *restrained geodetic number* of G and is denoted by $g_r(G)$. The number of extreme vertices in G is its *extreme order* $ex(G)$. A graph G is an *extreme restrained geodesic graph* if $g_r(G) = ex(G)$. It is shown that every pair a, b of integers with $b \geq 3$ and $0 \leq a \leq b$ is realized as the extreme order and geodetic number, respectively, of some graph. For positive integers r, d and $k \geq 2$ with $r < d \leq 2r$, it is shown that there exists an extreme restrained geodesic graph G of radius r, diameter d and restrained geodetic number k.

Keywords. geodetic number · restrained geodetic number · extreme order · extreme restrained geodesic graph.

1 Introduction

By a *graph* $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by p and q, respectively. For basic graph theoretic terminology we refer to [10]. For vertices u and v in a connected graph G, the *distance* $d(u, v)$ is the length of a shortest $u - v$ path in G. It is known that the distance is a *metric* on the vertex set of G. A $u - v$ path of

* Corresponding author

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A.P. Santhakumaran
Former Professor, Department of Mathematics, Hindustan Institute of Technology and Science, Chennai - 603 103, India
E-mail: apskumar1953@gmail.com

P. Titus
Department of Mathematics, University College of Engineering Nagercoil, Nagercoil-629 004, India
E-mail: titusvino@yahoo.com

K. Ganesamoorthy
Department of Mathematics, Coimbatore Institute of Technology, Coimbatore-641 014, India
E-mail: kvgm_2005@yahoo.co.in

length $d(u,v)$ is called a $u-v$ *geodesic*. For any vertex u of G, the *eccentricity* of u is $e(u) = max\{d(u,v) : v \in V\}$. The *radius* $rad(G)$ and *diameter* $diam(G)$ are defined by $rad(G) = min\{e(v) : v \in V\}$ and $diam(G) = max\{e(v) : v \in V\}$, respectively [2]. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v is an *extreme vertex* of G if the subgraph induced by its neighbors is complete. The number of extreme vertices in G is its *extreme order* $ex(G)$.

A *geodetic set* of G is a set S of vertices such that every vertex of G lies on a geodesic path joining some pair of vertices in S. The *geodetic number* of G is the minimum cardinality of its geodetic sets and is denoted by $g(G)$. A geodetic set of cardinality $g(G)$ is called a *g-set*. The geodetic number of a graph was introduced in [11] and further studied in [3,4,6–9,12]. A geodetic set $S \subseteq V$ of a graph G is a *restrained geodetic set* if either $S = V$ or the subgraph $G[V-S]$ induced by $V-S$ has no isolated vertex. The minimum cardinality of a restrained geodetic set of G is the *restrained geodetic number* of G and is denoted by $g_r(G)$. The restrained geodetic number of a graph was introduced and studied in [1]. A graph G is an *extreme geodesic graph* if $g(G) = ex(G)$. Extreme geodesic graphs were introduced and studied in [5]. The following theorems will be used in the sequel.

Theorem 1 [1] Each extreme vertex of a connected graph G belongs to every restrained geodetic set of G.

Theorem 2 [1] If T is a tree of order p with k endvertices and $p-k \geq 2$, then $g_r(T) = k$.

2 Main Results

Definition 1 A graph G is said to be an *extreme restrained geodesic graph* if $g_r(G) = ex(G)$.

For the graph G given in Figure 2.1, v_1 and v_3 are the only two extreme vertices so that $ex(G) = 2$. The set $S = \{v_1, v_3\}$ is the unique minimum restrained geodetic set of G and so $g_r(G) = ex(G) = 2$. Therefore, G is an extreme restrained geodesic graph.

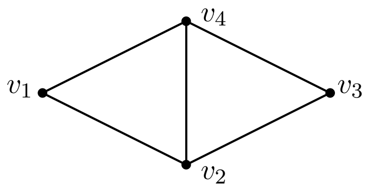

Figure 2.1: G

For any nontrivial tree G of order p with k endvertices and $p-k \geq 2$, $ex(G) = k$ and by Theorem 2, $g_r(G) = k$. Thus any nontrivial tree with atleast two internal vertices is an extreme restrained geodesic graph. The cycle $C_p (p \geq 4)$ and the complete bipartite graph $K_{r,s} (2 \leq r \leq s)$ are not an extreme restrained geodesic graphs. By Theorem 1, we see that for any connected graph G of order p, $0 \leq ex(G) \leq g_r(G) \leq p$. It is an easy consequence of Theorem 1 that a connected graph G of order $p \geq 2$ is an extreme restrained geodesic graph with extreme restrained geodetic number p if and only if $G = K_p$.

Theorem 3 If $G = K_2 + \bigcup m_i K_j$, where each m_i is a positive integer such that $\sum m_i \geq 2$ and $j \geq 1$, then G is an extreme restrained geodesic graph with $g_r(G) = p-2$.

Proof. Let the vertex set of K_2 be $\{x, y\}$. It is observed that every vertex of G except x and y is an extreme vertex so that $ex(G) = p - 2$. It is clear that the set S of all extreme vertices of G is a minimum geodetic set of G and the subgraph induced by $V - S$ has no isolated vertex. Hence S is the unique restrained geodetic set of G and so $g_r(G) = p - 2 = ex(G)$. Thus G is an extreme restrained geodesic graph with $g_r(G) = p - 2$.

Remark 1 The converse of Theorem 3 need not be true. For the graph G in Figure 2.2, $ex(G) = g_r(G) = 4 = p - 2$. Thus G is an extreme restrained geodesic graph, and it is not in the form $G = K_2 + \bigcup m_i K_j$.

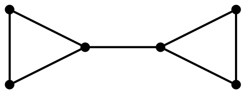

Figure 2.2: G

Theorem 4 There does not exist an extreme restrained geodesic graph G of order p with $ex(G) = p - 1$.

Proof. Suppose that there exists an extreme restrained geodesic graph G of order p with $ex(G) = p - 1$. Then every vertex of G is an extreme vertex except one, say x. Let S be the set of all extreme vertices of G. Then S is a geodetic set of G and the subgraph induced by $V - S$ has the isolated vertex x. Thus S is not a restrained geodetic set of G. Hence $V(G)$ is the unique minimum restrained geodetic set of G and so $g_r(G) = p \neq ex(G)$, which is a contradiction to G is an extreme restrained geodesic graph. Therefore, there does not exist an extreme restrained geodesic graph G of order p with $ex(G) = p - 1$.

Theorem 5 For any integer k such that $2 \leq k \leq p$ and $k \neq p - 1$, there is an extreme restrained geodesic graph G of order p such that $g_r(G) = k$.

Proof. For $k = p$, the result follows from Theorem 1 by taking $G = K_p$. For $2 \leq k \leq p - 2$, the tree T given in Figure 2.3 has p vertices with $ex(G) = k$ and it follows from Theorem 2 that $g_r(G) = k$.

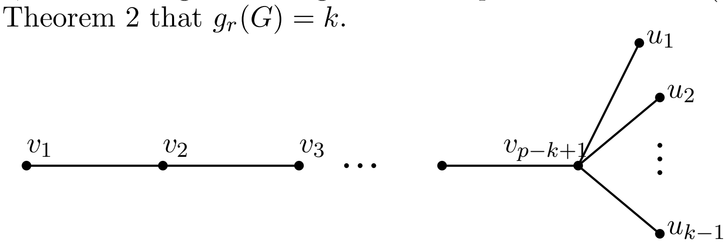

Figure 2.3: G

For any connected graph G, we have $0 \leq ex(G) \leq g_r(G)$ and $2 \leq g_r(G) \leq p$, $g_r(G) \neq p - 1$. In view of this, we have the following realization result.

Theorem 6 For every pair a, b of integers with $b \geq 3$ and $0 \leq a \leq b$, there exists a connected graph G with $ex(G) = a$ and $g_r(G) = b$.

Proof. We consider two cases, according to whether $a = 0$ or $a \geq 1$.

Case (i) $a = 0$. Let G be the graph obtained from the cycle $C_4 : v_1, v_2, v_3, v_4, v_1$ of length 4 by adding $b - 2$ new vertices $u_1, u_2, \cdots, u_{b-2}$ and joining each $u_i (1 \leq i \leq b-2)$ to the vertices v_1 and v_3; also joining the vertices v_2 and v_4. The graph G is shown in Figure 2.4. Clearly, no vertex of G is an extreme vertex and so $ex(G) = 0$. Let $S = \{v_1, v_3\}$. It is easily verified that S is a minimum geodetic set of G. Since the subgraph induced by $V - S$ has the isolated vertices $u_1, u_2, \cdots, u_{b-2}$, S is not a restrained geodetic set of G. Let $S' = S \cup \{u_1, u_2, \cdots, u_{b-2}\}$. It is easily observed that S' is a minimum restrained geodetic set of G and so $g_r(G) = b$.

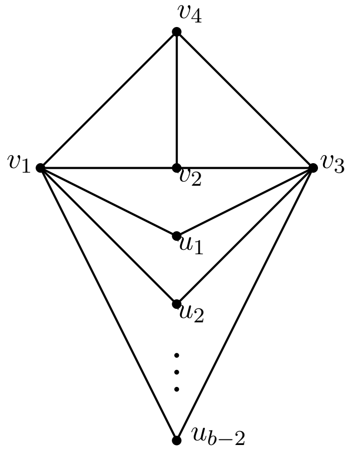

Figure 2.4: G

Case (ii) $a \geq 1$. If $a = b$, then the complete graph $G = K_a$ has the desired properties.

If $a < b$ and $b = a + 1$, let G be the graph obtained from the cycle C_5 : $v_1, v_2, v_3, v_4, v_5, v_1$ by adding 'a' new vertices $u_1, u_2, \cdots, u_a$ and joining each $u_i (1 \leq i \leq a)$ to the vertices v_1 and v_2, thereby producing the graph G which is shown in Figure 2.5. Since $S = \{u_1, u_2, \cdots, u_a\}$ is the set of all extreme vertices of G, $ex(G) = a$. By Theorem 1, every restrained geodetic set of G contains S. Clearly S is not a restrained geodetic set of G. It is easily verified that $S \cup \{v_4\}$ is a minimum restrained geodetic set of G and so $g_r(G) = a + 1 = b$.

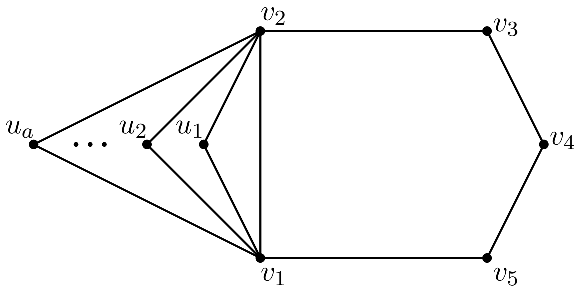

Figure 2.5: G

Now, if $a < b$ and $b = a + 2$. Let G be the graph obtained from the graph in Figure 2.5 by adding a new vertex x and joining x with v_3 and v_5. Then as above $ex(G) = a$. By Theorem 1, every restrained geodetic set of G contains S. Clearly S is not a restrained geodetic set of G. Also for any vertex $u \in V - S$, $S \cup \{u\}$ is not a restrained geodetic set of G. It is clear that $S \cup \{v_4, x\}$ is a minimum restrained geodetic set of G and so $g_r(G) = a + 2 = b$.

If $a < b$ and $b - a \geq 3$, let H be the graph obtained from the cycle C_5 : $v_1, v_2, v_3, v_4, v_5, v_1$ by adding $b - a - 2$ new vertices $u_1, u_2, \cdots, u_{b-a-2}$ and joining each $u_i (1 \leq i \leq b - a - 2)$ to the vertices v_1 and v_3; and also joining the vertex v_2 with both v_4 and v_5. Now, add 'a' new vertices $w_1, w_2, \cdots, w_a$ to H and join each $w_j (1 \leq j \leq a)$ to the vertices v_4 and v_5, thereby producing the graph G which is shown in Figure 2.6. Since $S = \{w_1, w_2, \cdots, w_a\}$ is the set of all extreme vertices of G, $ex(G) = a$. By Theorem 1, every restrained geodetic set of G contains S. It is easily verified that S is not a restrained geodetic set of G. Also for any vertex $x \in V - S$, $S \cup \{x\}$ is not a restrained geodetic set of G. It is clear that $S_1 = S \cup \{v_1, v_3\}$ is a minimum geodetic set of G and the subgraph induced by $V - S_1$ has the isolated vertices $u_1, u_2, \cdots, u_{b-a-2}$. Hence S_1 is not a restrained geodetic set of G. Let $S_2 = S_1 \cup \{u_1, u_2, \cdots, u_{b-a-2}\}$. It is easily verified that S_2 is a minimum restrained geodetic set of G and so $g_r(G) = b$.

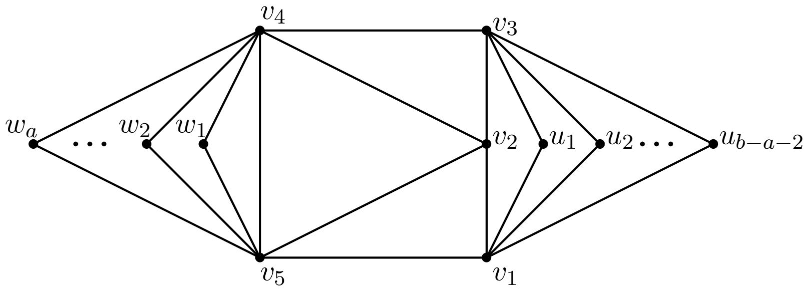

Figure 2.6: G

For any connected graph G, $rad(G) \leq diam(G) \leq 2\,rad(G)$. Ostrand [13] showed that every two positive integers r and d are realizable as the radius and diameter, respectively, of some connected graph. Ostrand's theorem can be extended to extreme restrained geodesic graphs so that the restrained geodetic number can also be prescribed.

Theorem 7 For positive integers r, d and $k \geq 2$ with $r < d \leq 2r$, there exists an extreme restrained geodesic graph G with $rad(G) = r$, $diam(G) = d$ and $g_r(G) = k$.

Proof. If $r = 1$ and $d = 2$. Let G be the graph obtained from the cycle C_4 : u, v, w, x, u of length 4 by adding $k-2$ new vertices $u_1, u_2, \cdots, u_{k-2}$ and joining each $u_i (1 \leq i \leq k-2)$ to the vertex v; and also joining the vertices v and x, thereby producing the graph G with radius 1 and diameter 2. The graph G is shown in Figure 2.7. Since $S = \{u_1, u_2, \cdots, u_{k-2}, u, w\}$ is the set of all extreme vertices of G, $ex(G) = k$.

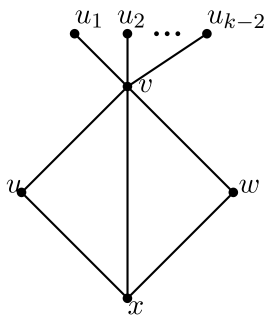

Figure 2.7: G

By Theorem 1, every restrained geodetic set of G contains S. It is easily verified that S is the unique minimum restrained geodetic set of G and so $g_r(G) = k = ex(G)$. Thus G is an extreme restrained geodesic graph.

Now, let $r \geq 2$ and $r < d$. Let $C_{2r} : u_1, u_2, \ldots, u_{2r}, u_1$ be a cycle of order $2r$ and let $P_{d-r} : v_0, v_1, \ldots, v_{d-r}$ be a path of length $d-r$. Let H be the graph obtained from C_{2r} and P_{d-r} by identifying v_0 of P_{d-r} and u_1 of C_{2r}. Now, add $k-2$ new vertices $w_1, w_2, \ldots, w_{k-2}$ to the graph H and join each vertex $w_i (1 \leq i \leq k-2)$ to both u_r and u_{r+2}, thereby producing the graph G which is shown in Figure 2.8. It is easy to verify that $r \leq e(x) \leq d$ for any vertex x in G and $e(u_1) = r$ and $e(v_{d-r}) = d = e(u_{r+1}) = e(w_i)(1 \leq i \leq k-2)$. Then $rad(G) = r$ and $diam(G) = d$. Since $S = \{w_1, w_2, \ldots, w_{k-2}, u_{r+1}, v_{d-r}\}$ is the set of all extreme vertices of G, $ex(G) = k$. It is easily verified that S is the unique minimum restrained geodetic set of G and so $g_r(G) = k = ex(G)$. Thus G is an extreme restrained geodesic graph.

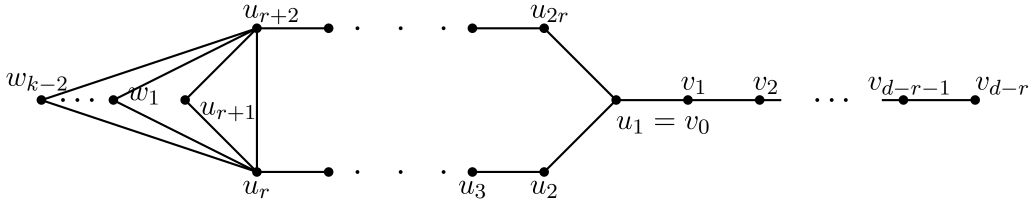

Figure 2.8: G

We leave the following problem as an open question.

Problem 1 For any three positive integers r, d and $k \geq 2$ with $r = d$, does there exist an extreme restrained geodesic graph G with $rad(G) = r$, $diam(G) = d$ and $g_r(G) = k$?

Theorem 8 For any three positive integers d, k and p with $3 \leq d < p$ and $2 \leq k < p$ and $p - d - k \geq 0$, there exists an extreme restrained geodesic graph G of order p with diameter d and $g_r(G) = k$.

Proof. Let $P_{d+1} : u_1, u_2, \ldots, u_{d+1}$ be a path of length d. Add $p-d-1$ new vertices $w_1, w_2, \ldots, w_{p-d-k+1}, v_1, v_2, \ldots, v_{k-2}$ to P_{d+1} and join each $w_i (1 \leq i \leq p-d-k+1)$ to the vertices u_1, u_2 and u_3; and join each $v_j (1 \leq j \leq k-2)$ to u_d; and also join each $w_j (1 \leq j \leq p-d-k)$ to $w_i (j+1 \leq i \leq p-d-k+1)$, thereby producing the graph G of order p with diameter d, which is shown in Figure 2.9. Since $S = \{u_1, u_{d+1}, v_1, v_2, \ldots, v_{k-2}\}$ is the set of all extreme vertices of G, $ex(G) = k$.

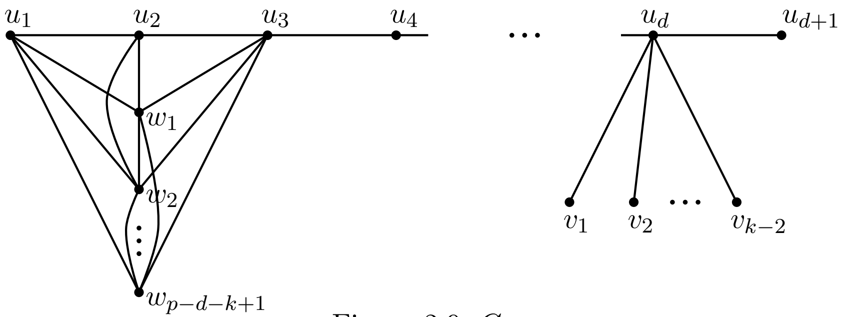

Figure 2.9: G

By Theorem 1, every restrained geodetic set of G contains S. It is clear that S is a geodetic set of G and also the subgraph induced by $V - S$ has no isolated vertex. Hence S is the unique minimum restrained geodetic set of G and so $g_r(G) = k = ex(G)$. Thus G is an extreme restrained geodesic graph G of order p with diameter d and $g_r(G) = k$.

In the following theorem we construct a non-extreme restrained geodesic graph of G of order p with diameter d and $g_r(G) = k$.

Theorem 9 For any three positive integers d, k and p with $3 \leq d < p$, $2 \leq k < p$ and $p - d - k \geq 0$, there exists a non-extreme restrained geodesic graph G of order p with diameter d and $g_r(G) = k$.

Proof. Let $P_{d+1} : u_1, u_2, \ldots, u_{d+1}$ be a path of length d. Add $p-d-1$ new vertices $w_1, w_2, \ldots, w_{p-d-k+1}, v_1, v_2, \ldots, v_{k-2}$ to P_{d+1} and join each $w_i (2 \leq i \leq p-d-k+1)$ to the vertices u_1, u_2 and u_3; and join each $v_j (1 \leq j \leq k-2)$ to u_d; and join each $w_j (2 \leq j \leq p-d-k)$ to $w_i (j+1 \leq i \leq p-d-k+1)$; and also join the vertex w_1 to the vertices u_1 and u_3, thereby producing the graph G of order p with diameter d which is shown in Figure 2.10. Since $S = \{u_{d+1}, v_1, v_2, \ldots, v_{k-2}\}$ is the set of all extreme vertices of G, $ex(G) = k - 1$. By Theorem 1, every restrained geodetic set of G contains S. It is clear that S is not a geodetic set of G. It is easily proved that $S_1 = S \cup \{u_1\}$ is a geodetic set of G. Since the subgraph induced by $V - S_1$ has no isolated vertex and hence S_1 is a restrained geodetic set of G so that $g_r(G) = k$. Thus $ex(G) = k - 1 \neq g_r(G)$. Hence G is a non-extreme restrained geodesic graph of order p with diameter d and $g_r(G) = k$.

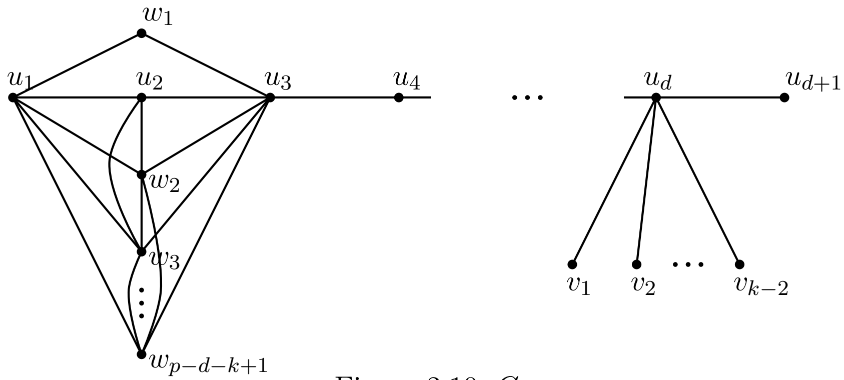

Figure 2.10: G

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