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Solvability of a boundary value problem with bounded operator boundary conditions for second order elliptic differential-operator equations with a complex parameter

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Abstract. *In a separable Hilbert space H, we studied questions of the solvability of a boundary value problem for second order elliptic differential-operator equations with a complex parameter in the case when a bounded linear operator is contaired in one of the boundary conditions. Abstract results obtained are applied to elliptic boundary value problems.*

Keywords. Hilbert space, differential-operator equation, interpolation spaces

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1 Introduction

Before turning to the topic of this work, we briefly outline some of the available results that are close to this work on the problem statement.

In [1], in particular, in a separable Hilbert space H, we study the solvability of the following boundary value problem for second order elliptic differential-operator equations:

$$L(\lambda)\, u := \lambda u(x) - u''(x) + Au(x) = f(x), \quad x \in (0,1), \tag{1.1}$$

$$\begin{array}{l} L_1 u := u'(1) + Bu(0) = f_1, \\ L_2 u := u'(0) = f_2, \end{array} \tag{1.2}$$

where λ is a complex parameter; A is a φ-positive operator in H (definition of φ- positive operator will be given below), B is a linear unbounded operator in H, which is subordinate to the operator $A^{\frac{1}{2}}$ in a certain sense. Under these conditions, it is proved that, for $\lambda \in \mathbb{C}$ with a sufficiently large norm $|\lambda|$, belonging to the angle $|\arg \lambda| \leq \varphi < \pi$, where $\varphi \in [0, \pi)$ is any fixed number, there is a theorem on the isomorphism between the solutions belonging

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to $W_p^2\left((0,1)\,;\,H(A),H\right)$ and the right side of the boundary value problem (1.1), (1.2) belonging to the direct sum

$$L_p\left((0,1)\,;H\right)\dot{+}\left(H(A),\ H\right)_{\frac{1}{2}+\frac{1}{2p},p}\dot{+}\left(H(A),\ H\right)_{\frac{1}{2}+\frac{1}{2p},p},\quad p\in(1,+\infty)\,.$$

A certain estimate is also established for the solution of problem (1.1), (1.2) (with respect to u and λ) in the space $L_p\left((0,1)\,;H\right)$. In this case, it is customary to say that the boundary value problem (1.1), (1.2) is coercively solvable in the space $L_p\left((0,1)\,;H\right)$, with respect to u. Moreover, the established estimate for the solution of problem (1.1), (1.2) is called a coercive estimate.

In [2], in particular, in a separable Hilbert space H, the solvability of the following boundary value problem is studied for equation (1.1) with the following boundary condition:

$$\begin{array}{l} L_1\left(\lambda\right)u := u'\left(1\right)+\lambda Bu\left(0\right)=f_1, \\ L_2u := u'\left(0\right)=f_2, \end{array}\tag{1.3}$$

where H and are B linear bounded operators acting in the spaces $H(A)$ and H.

Under these conditions, it is proved that due to the presence of a complex parameter λ in the boundary condition (1.3), the boundary value problem (1.1), (1.3), in contrast to the boundary value problem (1.1), (1.2), becomes non-coercively solvable with respect to u in the space $L_p\left((0,1)\,;H\right),\ p\in(1,+\infty)$.

As noted in [2], the non-coercivity of the boundary value problem (1.1), (1.3) manifests itself mainly in two respects.

1. For the existence of a solution to problem (1.1), (1.3), which belongs to the space $W_p^2\left((0,1)\,;\,H(A),H\right)$, the element f_2 cannot, as follows from the trace theorem, belong to the interpolation space $\left(H(A),\ H\right)_{\frac{1}{2}+\frac{1}{2p},p}$ but must belong to a narrower interpolation space $\left(H(A),\ H\right)_{\frac{1}{2p},p}$, although in this case the element f_1 can belong to the space $\left(H(A),\ H\right)_{\frac{1}{2}+\frac{1}{2p},p}$.

2. The fact that the element f_2 does not belong to the space $\left(H(A),\ H\right)_{\frac{1}{2}+\frac{1}{2p},p}$ implies its, in turn, that the vector-function f cannot belong to the space $L_p\left((0,1)\,;H\right)$; it must be chosen from a narrower space $L_p\Big((0,1);H\Big(A^{\frac{1}{2}}\Big)\Big)$.

Further, the issues of solvability of boundary value problems for second order elliptic differential-operator equations with a complex parameter were studied in various aspects in [3]-[8] and others.

We note that the boundary value problems for equation (1.1) in the case when the coefficients in the boundary conditions are complex numbers and the boundary conditions are regular in the sense of Birkhoff-Tamarkin were first studied in the monograph ([9], Ch.5, Section 5.4), where the coercive solvability of the considered boundary value problems was proved in the space $L_p\left((0,1)\,;H\right),\ p\in(1,+\infty)$, with respect of u.

In this paper, in a separable Hilbert space H, we study the solvability of the following boundary value problem:

$$L\left(\lambda\right)u := \lambda u\left(x\right)-u''\left(x\right)+Au\left(x\right)=f\left(x\right),\quad x\in(0,1)\,,\tag{1.4}$$

$$\begin{array}{l} L_1u =: Bu'\left(1\right)+u\left(0\right)=f_1, \\ L_2u =: u\left(1\right)=f_2, \end{array}\tag{1.5}$$

where λ is a complex parameter; A is φ-positive operator in H; B is a linear bounded operator in the spaces $H(A)$ and H and is commutative with the operator A^{-1} in H.

Under these conditions, it is proved that for $f\in L_p\Big((0,1)\;;\;H(A^{\frac{1}{2}})\Big)$, $f_k\in(H(A^2)\,,$ $H)_{\frac{3}{4}-\frac{k}{4}+\frac{1}{4p},p}$, $k=1,2$ and for $\lambda\in\mathbb{C}$ with a sufficiently large norm $|\lambda|$, belonging to the angle $|\arg\lambda| \leq \varphi < \pi$, where $\varphi \in [0, \pi)$ is any fixed number, the boundary value problem (1.4), (1.5) has a unique solution from $W_p^2\left((0,1)\,; H\left(A\right),\, H\right)$ and some non-coercive estimate is valid for the solution. More precisely, the boundary value problem (1.4), (1.5) is non-coercively solvable with respect to u in the space $L_p((0,1); H), p \in (1, +\infty)$.

Since there is a linear bounded operator in the boundary conditions (1.5), the abstract results obtained allow us to study the non-coercive solvability of a new class of boundary value problems for second order elliptic partial differential equations with a complex parameter given on the square. In paragraph 4 of the work, one of such applications is given.

We introduce definitions and notions used in the paper.

Let E_1 and E_2 be Banach spaces. The set $E_1 + E_2$ of all pairs of the form (u, v), where $u \in E_1$, $v \in E_2$, with ordinary coordinate-linear operations and with the norm

$$\|(u,v)\|_{E_1 \dot{+} E_2} :\ = \|u\|_{E_1} + \|v\|_{E_2}$$

is a Banach space and is called the direct sum of the Banach spaces E_1 and E_2.

Let E_1 and E_2 be Banach spaces. Denote by $B(E_1, E)$ the Banach space of all linear bounded operators acting from E_1 to E with an ordinary operator norm. In the special case, $B(E) := B(E, E)$.

Definition 1.1 *A closed linear operator A in Hilbert space H is said to be φ-positive if its domain $D\left(A\right)$ is dense in H and, for some $\varphi \in [0, \pi)$ and for all points $\mu \in \mathbb{C}$ in the sector $|\arg\mu| \leq \varphi$ (including $\mu = 0$), there exist operators $\left(A + \mu I\right)^{-1}$ for which the following estimate holds with these μ:*

$$\left\|(A + \mu I)^{-1}\right\|_{B(H)} \leq C\left(1 + |\mu|\right)^{-1},$$

where I is the identity operator in H, $C = const > 0$.

A simple example of φ-positive operators is a self-adjoint, positive-definite operator acting in a Hilbert space.

Let A be a φ -positive operator in H. Since its inverse A^{-1} is bounded in H, we conclude that

$$H\left(A^n\right) := \left\{u : \quad u \in D\left(A^n\right),\ \ \|u\|_{H(A^n)} = \|A^n u\|_H\right\}, \quad n \in N$$

is a Hilbert space whose norm is equivalent to the graph norm of the operator A^n.

If the operator A is φ-positive operator in the space H, then it is well that the operator -A is the generator of a semigroup e^{-tA} analytic for $t > 0$, with this semigroup exponentially decaying; i.e. there exist such two numbers $C > 0$, $\delta_0 > 0$ that $\|e^{-tA}\| \leq Ce^{-\delta_0 t}$, $0 \leq t < +\infty$.

By Theorem 5.5. from ([10], ch.1, §5), the operator $A^{\frac{1}{2}}$ generates an analytic semigroup that descending to infinity for $t > 0$.

Definition 1.2 *(see. [11, Theorem 1.14.5]). Let A be a φ-positive operator in the space H. Then for $\theta \in (0, 1)$, $p > 1$, $n \in N$ the interpolation space $(H(A^n), H)_{\theta,p}$ of the Hilbert spaces $H(A^n)$ and H is defined by the relation*

$$(H(A^n),\, H)_{\theta,p} := \Big\{u : u \in H, \|u\|_{(H(A^n),H)_{\theta,p}} := \\ := \Big(\int_0^{+\infty} t^{-1+n\theta p} \|A^n e^{-tA} u\|_H^p dt\Big)^{\frac{1}{p}} < \infty\Big\}.$$

Here by definition $\left(H(A^n),\, H\right)_{0,p} := H(A^n)$, $\left(H(A^n),\, H\right)_{1,p} := H$.

By $L_p((0,1);H)$, $p \in (1,+\infty)$ we denote the Banach space (for $p=2$ the Hilbert space) strongly measurable p-integrable vector-functions $u(\cdot):[0,1] \to H$, with the norm

$$\|u\|_{L_p((0,1);H)} := \left(\int_0^1 \|u(x)\|_H^p \, dx \right)^{1/p} < \infty,$$

and by $W_p^{2n}((0,1);H(A^n),H) := \left\{ u : A^n u,\ u^{(2n)} \in L_p((0,1);H) \right\}$ a Banach space of vector functions with the norm

$$\|u\|_{W_p^{2n}((0,1);H(A^n),H)} := \|A^n u\|_{L_p((0,1);H)} + \left\| u^{(2n)} \right\|_{L_p((0,1);H)}.$$

It is well known that ([11], theorem 1.8.2) if $u \in W_p^{2n}((0,1);H(A^n),H)$, then for each $x_0 \in [0,1]$ one has the inclusion

$$u^{(j)}(x_0) \in (H(A^n),H)_{\frac{j+\frac{1}{p}}{2n},p}, \quad j = \overline{0,\ 2n-1}$$

is valid, and the estimate (the trace theorem)

$$\left\| u^{(j)}(x_0) \right\|_{(H(A^n),H)_{\frac{j+\frac{1}{p}}{2n},p}} \le C \|u\|_{W_p^{2n}((0,1);H(A^n),H)}$$

holds.

In what follows, to simplify the statements, we adopt the following notation: for $\lambda \in \mathbb{C}$ and $\varphi \in [0,\pi)$ the notation $\lambda \in \mathcal{A}_\varphi$ means that λ belongs to the sector $|\arg \lambda| \le \varphi$ and the notation $\lambda \in \mathfrak{U}_\varphi$ implies that $\lambda \in \mathcal{A}_\varphi$ and the norm $|\lambda|$ is sufficiently large. In addition, in what follows, by C and ω we denote some positive constants whose values depend on the situation in hand but are independent of λ.

2 Homogeneous equation.

Consider first in space H the following boundary value problem:

$$L(\lambda)u := \lambda u(x) - u''(x) + Au(x) = 0, x \in (0,\ 1), \tag{2.1}$$

$$\begin{aligned} L_1 u &:= Bu'(1) + u(0) = f_1, \\ L_2 u &:= u(1) = f_2. \end{aligned} \tag{2.2}$$

Theorem 2.1 *Let the following conditions be satisfied:*
1) A is a φ-positive operator in H;
2) B is a linear bounded operator acting in H;
3) The operator B is commutative with the operator A^{-1} in H.

Then, for $f_k \in \left(H(A^2),H\right)_{\frac{3}{4}-\frac{k}{4}+\frac{1}{4p},p}$, and for $\lambda \in \mathfrak{U}_\varphi$ problem (2.1), (2.2) *has such a unique solution $u \in W_p^2((0,1)\ ;\ H(A),\ H)$ such that $u'(1) \in D(B)$, and the following non-coercive estimate for solving it is valid*

$$\begin{aligned} &|\lambda|\ \|u\|_{L_p((0,1);H)} + \|u''\|_{L_p((0,1);H)} + \|Au\|_{L_p((0,1);H)} \\ &\le C \sum_{k=1}^{2} \left(\|f_k\|_{(H(A^2),\ H)_{\frac{3}{4}-\frac{k}{4}+\frac{1}{4p},p}} + |\lambda|^{\frac{k+1}{2}-\frac{1}{2p}} \|f_k\|_H \right). \end{aligned} \tag{2.3}$$

Proof. By virtue of ([9], Lemma 5.4.2/6) for $\lambda \in \mathcal{A}_\varphi$, there exists a semigroup $e^{-x(A+\lambda I)^{\frac{1}{2}}}$ analytic for $x > 0$ and strongly continuous for $x \geq 0$. According to ([9], Lemma 5.3.2/1), for a function $u\,(x)$ to be a solution of equation (2.1) belonging to the space $W_p^2((0,1); H(A), H)$, it is necessary and sufficient that for $\lambda \in \mathcal{A}_\varphi$ satisfies the equality

$$u(x) = e^{-x(A+\lambda I)^{\frac{1}{2}}} g_1 + e^{-(1-x)(A+\lambda I)^{\frac{1}{2}}} g_2, \tag{2.4}$$

where $g_1, g_2 \in (H(A), H)_{\frac{1}{2p},p}$.

We require that a function of the form (2.4) satisfies the boundary conditions (2.2). Then we get the following system for the elements g_1 and g_2

$$\begin{aligned} &\Big(I - B(A+\lambda I)^{\frac{1}{2}} e^{-(A+\lambda I)^{\frac{1}{2}}}\Big) g_1 + \Big(B(A+\lambda I)^{\frac{1}{2}} + e^{-(A+\lambda I)^{\frac{1}{2}}}\Big) g_2 = f_1, \\ &e^{-(A+\lambda I)^{\frac{1}{2}}} g_1 + g_2 = f_2. \end{aligned} \tag{2.5}$$

From the second equation of system (2.5) we have

$$g_2 = f_2 - e^{-(A+\lambda I)^{\frac{1}{2}}} g_1. \tag{2.6}$$

Taking into account (2.6) in the first equation of system (2.5), we formally define the element g_1

$$g_1 = (I - R\,(\lambda))^{-1} f_1 - (I - R\,(\lambda))^{-1} \left(B(A+\lambda I)^{\frac{1}{2}} + e^{-(A+\lambda I)^{\frac{1}{2}}}\right) f_2, \tag{2.7}$$

where $R\,(\lambda) = 2B(A+\lambda I)^{\frac{1}{2}} e^{-(A+\lambda I)^{\frac{1}{2}}} + e^{-2(A+\lambda I)^{\frac{1}{2}}}$.

By virtue of ([9], lemma 5.4.2/6) and condition 2) of theorem 1 for $\lambda \in \mathfrak{U}_\varphi$ we have

$$\begin{aligned} \|R(\lambda)\|_{B(H)} \leq 2\|B\|_{B(H)} \cdot \left\|(A+\lambda I)^{\frac{1}{2}} e^{-(A+\lambda I)^{\frac{1}{2}}}\right\|_{B(H)} + \\ + \left\|e^{-2(A+\lambda I)^{\frac{1}{2}}}\right\|_{B(H)} \leq C e^{-\omega|\lambda|^{\frac{1}{2}}} < 1, \exists C, \omega > 0. \end{aligned} \tag{2.8}$$

Hence, by the Neumann identity for $\lambda \in \mathfrak{U}_\varphi$

$$(I - R\,(\lambda))^{-1} = I + \sum_{k=1}^{\infty} R(\lambda)^k, \tag{2.9}$$

where the series on the right side of (2.9) converges in the norm of the space of bounded operators in H. Denote

$$R_{11}\Big(\lambda\Big) := \sum_{k=1}^{\infty} R(\lambda)^k = \sum_{k=1}^{\infty} \left[2B(A+\lambda I)^{\frac{1}{2}} e^{-(A+\lambda I)^{\frac{1}{2}}} + e^{-2(A+\lambda I)^{\frac{1}{2}}}\right]^k. \tag{2.10}$$

Then by virtue of (2.8) from (2.7) for $\lambda \in \mathfrak{U}_\varphi$ we have

$$g_1 = (I + R_{11}\,(\lambda))\, f_1 - \Big(B(A+\lambda I)^{\frac{1}{2}} + R_{12}\,(\lambda)\Big) f_2, \tag{2.11}$$

where

$$R_{12}\,(\lambda) := e^{-(A+\lambda I)^{\frac{1}{2}}} + R_{11}\,(\lambda) \left(B(A+\lambda I)^{\frac{1}{2}} + e^{-(A+\lambda I)^{\frac{1}{2}}}\right). \tag{2.12}$$

Taking into account (2.11) in (2.6), we obtain representations of the element g_2

$$g_2 = R_{21}(\lambda) f_1 + (I + R_{22}(\lambda)) f_2 \,, \tag{2.13}$$

where

$$R_{21}(\lambda) := -e^{-(A+\lambda I)^{\frac{1}{2}}} (I + R_{11}(\lambda)) \,, \tag{2.14}$$

$$R_{22}(\lambda) := e^{-(A+\lambda I)^{\frac{1}{2}}} \left(B(A+\lambda I)^{\frac{1}{2}} - R_{12}(\lambda) \right) . \tag{2.15}$$

Substituting (2.11) and (2.13) into (2.4) we have

$$\begin{aligned} u(x) &= e^{-x(A+\lambda I)^{\frac{1}{2}}} \left[(I + R_{11}(\lambda)) f_1 - \left(B(A+\lambda I)^{\frac{1}{2}} + R_{12}(\lambda) \right) f_2 \right] + \\ &+ e^{-(1-x)(A+\lambda I)^{\frac{1}{2}}} \left[R_{21}(\lambda) f_1 + (I + R_{22}(\lambda)) f_2 \right] . \end{aligned} \tag{2.16}$$

Then, using the Minkowski inequality for $\lambda \in \mathfrak{U}_\varphi$ we have

$$\begin{aligned} &|\lambda| \|u\|_{L_p((0,1);H)} + \|u''\|_{L_p((0,1);H)} + \|Au\|_{L_p((0,1);H)} \\ &\le |\lambda| \Big[\Big(\int_0^1 \Big\| e^{-x(A+\lambda I)^{\frac{1}{2}}} f_1 \Big\|_H^p dx \Big)^{\frac{1}{p}} + \Big(\int_0^1 \Big\| e^{-x(A+\lambda I)^{\frac{1}{2}}} R_{11}(\lambda) f_1 \Big\|_H^p dx \Big)^{\frac{1}{p}} \\ &+ \Big(\int_0^1 \Big\| e^{-x(A+\lambda I)^{\frac{1}{2}}} B(A+\lambda I)^{\frac{1}{2}} f_2 \Big\|_H^p dx \Big)^{\frac{1}{p}} + \Big(\int_0^1 \Big\| e^{-x(A+\lambda I)^{\frac{1}{2}}} R_{12}(\lambda) f_2 \Big\|_H^p dx \Big)^{\frac{1}{p}} \\ &+ \Big(\int_0^1 \Big\| e^{-(1-x)(A+\lambda I)^{\frac{1}{2}}} R_{21}(\lambda) f_1 \Big\|_H^p dx \Big)^{\frac{1}{p}} + \Big(\int_0^1 \Big\| e^{-(1-x)(A+\lambda I)^{\frac{1}{2}}} f_2 \Big\|_H^p dx \Big)^{\frac{1}{p}} \\ &+ \Big(\int_0^1 \Big\| e^{-(1-x)(A+\lambda I)^{\frac{1}{2}}} R_{22}(\lambda) f_2 \Big\|_H^p dx \Big)^{\frac{1}{p}} \Big] + (1 + \|A(A+\lambda I)^{-1}\|) \\ &\times \Big[\Big(\int_0^1 \Big\| (A+\lambda I) e^{-x(A+\lambda I)^{\frac{1}{2}}} f_1 \Big\|_H^p dx \Big)^{\frac{1}{p}} \\ &+ \Big(\int_0^1 \Big\| (A+\lambda I) e^{-x(A+\lambda I)^{\frac{1}{2}}} R_{11}(\lambda) f_1 \Big\|_H^p dx \Big)^{\frac{1}{p}} \\ &+ \Big(\int_0^1 \Big\| (A+\lambda I) e^{-x(A+\lambda I)^{\frac{1}{2}}} B(A+\lambda I)^{\frac{1}{2}} f_2 \Big\|_H^p dx \Big)^{\frac{1}{p}} \\ &+ \Big(\int_0^1 \Big\| (A+\lambda I) e^{-x(A+\lambda I)^{\frac{1}{2}}} R_{12}(\lambda) f_2 \Big\|_H^p dx \Big)^{\frac{1}{p}} \end{aligned}$$

$$+\Big(\int_0^1 \Big\|(A+\lambda I)e^{-x(A+\lambda I)^{\frac12}}R_{21}(\lambda)f_1\Big\|_H^p dx\Big)^{\frac1p}$$

$$+\Big(\int_0^1 \Big\|(A+\lambda I)e^{-x(A+\lambda I)^{\frac12}}f_2\Big\|_H^p dx\Big)^{\frac1p}$$

$$+\Big(\int_0^1 \Big\|(A+\lambda I)e^{-x(A+\lambda I)^{\frac12}}R_{22}(\lambda)f_2\Big\|_H^p dx\Big)^{\frac1p}\Big]. \tag{2.17}$$

Let us estimate some integrals participating in the right side of inequality (2.17). We estimate the integral

$$I_1=\Big(\int_0^1 \Big\|(A+\lambda I)e^{-x(A+\lambda I)^{\frac12}}B(A+\lambda I)^{\frac12}f_2\Big\|_H^p dx\Big)^{\frac1p}.$$

From the commutativity of the operator B with the operator A^{-1} it follows that the operator B is commutative with the operator $e^{-x(A+\lambda I)^{\frac12}}$, $x\ge 0$. Then, by virtue of conditions 2), 3) and ([9], Theorem 5.4.2/1 and Lemma 5.4.2/6) for $\lambda\in\mathfrak{U}_\varphi$ we have

$$I_1=\Big(\int_0^1 \Big\|(A+\lambda I)Be^{-x(A+\lambda I)^{\frac12}}(A+\lambda I)^{\frac12}f_2\Big\|_H^p dx\Big)^{\frac1p}$$

$$\le\Big(\int_0^1 \Big\|AB(A+\lambda I)^{\frac12}e^{-x(A+\lambda I)^{\frac12}}f_2\Big\|_H^p dx\Big)^{\frac1p}$$

$$+|\lambda|\Big(\int_0^1 \Big\|B(A+\lambda I)^{\frac12}e^{-x(A+\lambda I)^{\frac12}}f_2\Big\|_H^p dx\Big)^{\frac1p}$$

$$=\Big(\int_0^1 \Big\|ABA^{-1}A(A+\lambda I)^{\frac12}e^{-x(A+\lambda I)^{\frac12}}f_2\Big\|_H^p dx\Big)^{\frac1p}$$

$$+|\lambda|\Big\|(A+\lambda I)^{-1}\Big\|\|B\|\Big(\int_0^1 \Big\|(A+\lambda I)^{\frac32}e^{-x(A+\lambda I)^{\frac12}}f_2\Big\|_H^p dx\Big)^{\frac1p}$$

$$\le\|ABA^{-1}\|\Big(\int_0^1 \Big\|A(A+\lambda I)^{\frac12}e^{-x(A+\lambda I)^{\frac12}}f_2\Big\|_H^p dx\Big)^{\frac1p}$$

$$+\|B\|\Big(\int_0^1 \Big\|(A+\lambda I)^{\frac32}e^{-x(A+\lambda I)^{\frac12}}f_2\Big\|_H^p dx\Big)^{\frac1p}$$

$$\leq C\Big(\int\limits_0^1 \Big\|A(A+\lambda I)^{\frac{1}{2}}e^{-x(A+\lambda I)^{\frac{1}{2}}}f_2\Big\|_H^p dx\Big)^{\frac{1}{p}}$$
$$+\,C\Big(\int\limits_0^1 \Big\|(A+\lambda I)^{\frac{3}{2}}e^{-x(A+\lambda I)^{\frac{1}{2}}}f_2\Big\|_H^p dx\Big)^{\frac{1}{p}}$$
$$\leq C\Big\|A(A+\lambda I)^{-1}\Big\|_{B(H)}\Big(\int\limits_0^1 \Big\|(A+\lambda I)^{\frac{3}{2}}e^{-x(A+\lambda I)^{\frac{1}{2}}}f_2\Big\|_H^p dx\Big)^{\frac{1}{p}}$$
$$+\,C\Big(\int\limits_0^1 \Big\|(A+\lambda I)^{\frac{3}{2}}e^{-x(A+\lambda I)^{\frac{1}{2}}}f_2\Big\|_H^p dx\Big)^{\frac{1}{p}}$$
$$\leq C\Big(\int\limits_0^1 \Big\|(A+\lambda I)^{\frac{3}{2}}e^{-x(A+\lambda I)^{\frac{1}{2}}}f_2\Big\|_H^p dx\Big)^{\frac{1}{p}}$$
$$\leq C\Big(\|f_2\|_{(H(A^2),H))_{\frac{1}{4}+\frac{1}{4p},p}}+|\lambda|^{\frac{3}{2}-\frac{1}{2p}}\|f_2\|_H\Big).$$

Thus for $\lambda\in\mathfrak{U}_\varphi$ we have

$$I_1\leq C\left(\|f_2\|_{(H(A^2)\,,\,H)_{\frac{1}{4}+\frac{1}{4p},\,p}}+|\lambda|^{\frac{3}{2}-\frac{1}{2p}}\;\|f_2\|_H\right).$$

Let us estimate the integral

$$I_2=\left(\int_0^1\left\|(A+\lambda I)e^{-(1-x)(A+\lambda I)^{\frac{1}{2}}}f_1\right\|_H^p dx\right)^{\frac{1}{p}}.$$

By virtue of the theorem ([11], Theorem 5.4.2/1) for $\lambda\in\mathfrak{U}_\varphi$ we have

$$I_2\leq C\left(\|f_1\|_{(\,H(A),\,H)_{\frac{1}{2p},p}}+|\lambda|^{1-\frac{1}{2p}}\;\|f_1\|_H\right).$$

Let us estimate the integral

$$I_3=\left(\int_0^1\left\|(A+\lambda I)e^{-x(A+\lambda I)^{\frac{1}{2}}}R_{11}(\lambda)f_1\right\|_H^p dx\right)^{\frac{1}{p}}.$$

From the representation of the operator $R_{11}(\lambda)$ determined by formula (2.10), by virtue of the estimate (2.8), it follows that the operator $R_{11}(\lambda)$ for $\lambda\in\mathfrak{U}_\varphi$ boundedly acts from H to H and the following estimate holds

$$\|R_{11}(\lambda)\|_{B(H)}\leq C\,e^{-\omega\,|\lambda|^{\frac{1}{2}}}. \tag{2.18}$$

By virtue of condition 3) of Theorem 1 and estimate (2.18), we have

$$\|R_{11}(\lambda)\|_{B(H(A))}=\left\|AR_{11}(\lambda)A^{-1}\right\|_{B(H)}=\|R_{11}(\lambda)\|_{B(H)}\leq C\,e^{-\omega\,|\lambda|^{\frac{1}{2}}}. \tag{2.19}$$

From estimates (2.18) and (2.19), by virtue of the interpolation theorem ([11], Theorem 1.3.3/(a)) (see also [11], section 1.7.), it follows that for $\lambda \in \mathfrak{U}_\varphi$ the operator $R_{11}(\lambda)$ boundedly acts from $(H(A),\ H)_{\theta,p}$ to $(H(A),\ H)_{\theta,p}$ for any $\theta \in (0,1)$, including for $\theta = \frac{1}{2p}$, and the following estimate holds

$$\|R_{11}(\lambda)\|_{B\left((H(A),\ H)_{\frac{1}{2p},p}\right)} \le Ce^{-\omega\,|\lambda|^{\frac{1}{2}}}. \tag{2.20}$$

Then in view of ([11], Theorem 5.4.2/1) and estimates (2.18), (2.21) for $\lambda \in \mathfrak{U}_\varphi$ we have

$$I_3 \le C\left(\|f_1\|_{(H(A),\ H)_{\frac{1}{2p},p}} + |\lambda|^{1-\frac{1}{2p}}\|f_1\|_H\right).$$

Let us now estimate the integral

$$I_4 = \left(\int_0^1 \left\|(A+\lambda I)e^{-x(A+\lambda I)^{\frac{1}{2}}}R_{12}(\lambda)f_2\right\|_H^p dx\right)^{\frac{1}{p}},$$

where $R_{12}(\lambda)$ is the operator defined by formula (2.12).

Let us show that the operator $R_{12}(\lambda)$ for $\lambda \in \mathfrak{U}_\varphi$ boundedly acts from H to H, and from $(H(A^2),\ H)_{\frac{1}{4}+\frac{1}{4p},p}$ to $(H(A),\ H)_{\frac{1}{2p},p}$ and we have the estimates

$$\|R_{12}(\lambda)\|_{B(H)} \le Ce^{-\omega|\lambda|^{\frac{1}{2}}} \tag{2.21}$$

$$\|R_{12}(\lambda)\|_{B\left((H(A^2),\ H),\ ((H(A),\ H))_{\frac{1}{2p},p}\right)} \le Ce^{-\omega|\lambda|^{\frac{1}{2}}} \tag{2.22}$$

We represent the integral I_4 in the form

$$I_4 = \left(\int_0^1 \left\|(A+\lambda I)^{\frac{3}{2}}e^{-x(A+\lambda I)^{\frac{1}{2}}}(A+\lambda I)^{-\frac{1}{2}}R_{12}(\lambda)f_2\right\|_H^p dx\right)^{\frac{1}{p}}.$$

Obviously, in order to prove the estimates (2.21), (2.22) it will be sufficient to prove that for $\lambda \in \mathfrak{U}_\varphi$ the following estimates hold:

$$\left\|(A+\lambda I)^{-\frac{1}{2}}R_{12}(\lambda)\right\|_{B(H)} \le C\,e^{-\omega|\lambda|^{\frac{1}{2}}}, \tag{2.23}$$

$$\left\|(A+\lambda I)^{-\frac{1}{2}}R_{12}(\lambda)\right\|_{B((H(A^2),\ H)_{\frac{1}{4}+\frac{1}{4p},p}} \le C\,e^{-\omega|\lambda|^{\frac{1}{2}}}. \tag{2.24}$$

Let us show estimate (2.23). It's obvious that

$$\begin{aligned}\left\|(A+\lambda I)^{-\frac{1}{2}}R_{12}(\lambda)\right\|_{B(H)} &\le \left\|(A+\lambda I)^{-\frac{1}{2}}e^{-(A+\lambda I)^{\frac{1}{2}}}\right\|_{B(H)}\\ &+ \left\|(A+\lambda I)^{-\frac{1}{2}}R_{11}(\lambda)B(A+\lambda I)^{\frac{1}{2}}\right\|_{B(H)}\\ &+ \left\|(A+\lambda I)^{-\frac{1}{2}}R_{11}(\lambda)e^{-(A+\lambda I)^{\frac{1}{2}}}\right\|_{B(H)}.\end{aligned} \tag{2.25}$$

Due to condition 3), condition 2, estimate (2.18) and ([9], Lemma 5.4.2/6) for $\lambda \in \mathfrak{U}_\varphi$ we have

$$\left\|(A+\lambda I)^{-\frac{1}{2}}R_{11}(\lambda)B(A+\lambda I)^{\frac{1}{2}}\right\|_{B(H)} = \|R_{11}(\lambda)B\|_{B(H)} \leq \|R_{11}(\lambda)\|_{B(H)}\|B\|_{B(H)}. \tag{2.26}$$

By virtue of ([11], Lemma 5.4.2/6), for $\lambda \in \mathfrak{U}_\varphi$ we have

$$\left\|(A+\lambda I)^{-1}e^{-(A+\lambda I)^{\frac{1}{2}}}\right\|_{B(H)} \leq \left\|(A+\lambda I)^{-1}\right\|\left\|e^{-(A+\lambda I)^{\frac{1}{2}}}\right\| \leq Ce^{-\omega|\lambda|^{\frac{1}{2}}}. \tag{2.27}$$

By virtue of estimate (2.18) and ([11], Lemma 5.4.2/6) for $\lambda \in \mathfrak{U}_\varphi$

$$\left\|(A+\lambda I)^{-1}R_{11}(\lambda)e^{-(A+\lambda I)^{\frac{1}{2}}}\right\|_{B(H)} \leq \left\|(A+\lambda I)^{-1}\right\| \times \|R_{11}(\lambda)\|\left\|e^{-(A+\lambda I)^{\frac{1}{2}}}\right\| \leq Ce^{-\omega|\lambda|^{\frac{1}{2}}}. \tag{2.28}$$

Taking into account the estimates (2.26)-(2.28) in (2.25), we obtain the estimate (2.23). Let us now show estimate (2.24). Due to condition 3) and estimate (2.23) for $\lambda \in \mathfrak{U}_\varphi$ we have

$$\left\|(A+\lambda I)^{-1}R_{12}(\lambda)\right\|_{B(H(A^2))} = \left\|A^2(A+\lambda I)^{-1}R_{12}(\lambda)A^{-2}\right\|_{B(H)} = \left\|(A+\lambda I)^{-1}R_{12}(\lambda)\right\|_{B(H)} \leq C\,e^{-\omega|\lambda|^{\frac{1}{2}}}. \tag{2.29}$$

From the estimates (2.28) and (2.29), by virtue of the interpolation theorem ([11], Theorem 1.3.3/(a)) (see also ([9], section 1.7.9), it follows that for $\lambda \in \mathfrak{U}_\varphi$ the operator $(A+\lambda I)^{-1}R_{12}(\lambda)$ boundedly acts from $\left(H(A^2),\ H\right)_{\theta,p}$to $\left(H(A^2),\ H\right)_{\theta,p}$ for any $\theta \in (0,1)$ and, in particular, for $\theta = \frac{1}{4} + \frac{1}{4p}$ and the estimates (2.24) hold. Then, in view of ([9], Theorem 5.4.2/1) and estimates (2.23), (2.24) for $\lambda \in \mathfrak{U}_\varphi$ the integral I_4, we have

$$\begin{aligned} I_4 &\leq C\Big(\|(A+\lambda I)^{-1}R_{12}(\lambda)f_2\|_{(H(A^2,H))_{\frac{1}{4}+\frac{1}{4p},p}} + |\lambda|^{\frac{3}{2}-\frac{1}{2p}}\|(A+\lambda I)^{-1}R_{12}(\lambda)f_2\|_H\Big) \\ &\leq Ce^{-\omega|\lambda|^{\frac{1}{2}}}\left(\|f_2\|_{H(A^2,H)_{\frac{1}{4}+\frac{1}{4p},p}} + |\lambda|^{\frac{3}{2}-\frac{1}{2p}}\|f_2\|_H\right) \\ &\leq C\Big(\|f_2\|_{H(A^2,H)_{\frac{1}{4}+\frac{1}{4p},p}} + |\lambda|^{\frac{3}{2}-\frac{1}{2p}}\|f_2\|_H\Big). \end{aligned}$$

The remaining terms on the right side of inequality (2.17) are estimated in a similar way. Moreover, it is easy to prove that for $\lambda \in \mathfrak{U}_\varphi$ with a sufficiently large $|\lambda|$ from the angle $|\arg\lambda| \leq \varphi < \pi$, the following two statements hold: a) the operator $R_{21}(\lambda)$, defined by equality (2.14) boundedly acts from H to H, and from $(H(A),\ H)_{\frac{1}{2p},p}$ to $(H(A),\ H)_{\frac{1}{2p},p}$, and there are estimates

$$\|R_{21}(\lambda)\|_{B(H)} \leq Ce^{-\omega|\lambda|^{\frac{1}{2}}},$$

$$\|R_{21}(\lambda)\|_{B\left((H(A),\ H)_{\frac{1}{2p},p}\right)} \leq Ce^{-\omega|\lambda|^{\frac{1}{2}}}.$$

b) the operator $R_{22}(\lambda)$ defined by equality (2.15) boundedly acts from H to H, and from $(H(A^2),\ H)_{\frac{1}{4p},p}$ to $(H(A),\ H)_{\frac{1}{2p},p}$, and there are the estimates

$$\|R_{22}(\lambda)\|_{B(H)} \leq Ce^{-\omega|\lambda|^{\frac{1}{2}}},$$

$$\|R_{22}(\lambda)\|_{B\left((H(A^2),\ H)_{\frac{1}{4p},p},\ (H(A),\ H)_{\frac{1}{2p},p}\right)} \leq Ce^{-\omega|\lambda|^{\frac{1}{2}}}.$$

Theorem 1 is proved.

Remark. By virtue of the theorem ([11], Theorem 1.3.3/(c), Theorem 1.15.2 (formula 1.15.2./(4))), to within the equivalence of the norm

$$(H(A), H)_{\frac{1}{2p},p} = (H(A^2), H)_{\frac{1}{2}+\frac{1}{4p},p}.$$

Therefore, when formulating Theorem 1, the place of the interpolation space $(H(A),\ H)_{\frac{1}{2p},p}$, for the sake of notation symmetry, was taken $(H(A^2),\ H)_{\frac{1}{2}+\frac{1}{4p},p}$.

3 Nonhomogeneous equation.

Let us now consider a boundary value problem for an inhomogeneous equation with a complex parameter in H, i.e. the problem

$$L(\lambda)\,u := \lambda u(x) - u''(x) + Au(x) = f(x), \quad x \in (0,1), \tag{3.1}$$

$$\begin{aligned} L_1 u &:= Bu'(1) + u(0) = f_1, \\ L_2 u &:= u(1) = f_2\,. \end{aligned} \tag{3.2}$$

Theorem 3.1 *Let all the conditions of Theorem 1 be satisfied. Then, for* $f \in L_p((0,1);\ H(A^{\frac{1}{2}}))$, $f_k \in (H(A^2), H)_{\frac{3}{4}-\frac{k}{4}+\frac{1}{4p},p}$, $k = 1, 2$, $p \in (1, +\infty)$ *and for* $\lambda \in \mathfrak{U}_\varphi$ *problem* (3.1), (3.2) *has a unique solution* $u \in W_p^2((0,1);\ H(A), H)$, *and the solution satisfies the following non-coercive estimate*

$$\begin{aligned} &|\lambda|\|u\|_{L_p((0,1);H)} + \|u''\|_{L_p((0,1);H)} \\ &+ \|Au\|_{L_p((0,1);H)} \leq C\Big[|\lambda|^{\frac{1}{2}}\|f\|_{L_p((0,1);H(A^{\frac{1}{2}}))} \\ &+ \sum_{k=1}^{2}(\|f_k\|_{(H(A^2),H)_{\frac{3}{4}-\frac{k}{4}+\frac{1}{4p},p}} + |\lambda|^{\frac{k+1}{2}-\frac{1}{2p}}\|f_k\|_H)\Big]. \end{aligned} \tag{3.3}$$

Proof. Uniqueness follows from Theorem 1. Let us represent the solution (3.1), (3.2) belonging to $W_p^2\left((0,1); H(A), H\right)$ in the form of the sum $u(x) = u_1(x) + u_2(x)$, where $u_1(x)$ the restriction on the segment $[0,1]$ of the solution of the equation

$$L(\lambda)\,\tilde{u}_1 := \lambda\,\tilde{u}_1(x) - \tilde{u}_1''(x) + A\tilde{u}_1(x) = \tilde{f}(x), x \in R = (-\infty, +\infty), \tag{3.4}$$

where $\tilde{f}(x) := f(x)$, if $x \in [0,1]$ and $\tilde{f}(x) = 0$, if $x \notin [0,1]$, and $u_2(x)$ is the solution to the problem

$$\begin{aligned} &L(\lambda)\,u_2 = 0, \quad x \in (0,1)\,, \\ &L_1 u_2 = f_1 - L_1 u_1, \quad L_2 u_2 = f_2 - L_2 u_1\,. \end{aligned} \tag{3.5}$$

As shown in the proof of ([9], Theorem 5.4.4), the solution to equation (3.4) is given by the formula

$$\tilde{u}_1(x) = \frac{1}{\sqrt{2\pi}} \int_R e^{i\mu x} L(\lambda, i\mu)^{-1} F \tilde{f}(\mu)\, d\mu = \left(F^{-1} L(\lambda, i\mu)^{-1} F\tilde{f}(\mu)\right)(x), \tag{3.6}$$

where $F\tilde{f}$ is the Fourier transform of the function $\tilde{f}(x)$, F^{-1} is inverse Fourier transform; $L(\lambda, \sigma)$ is a characteristic operator pencil of the equation (3.4), i.e.,

$$L(\lambda, \sigma) = -\sigma^2 I + A + \lambda I, \lambda \in A_\varphi.$$

From (3.6) it follows that

$$\begin{aligned}
&\|\tilde{u}_1\|_{W_p^2\left(R; H(A^{\frac{3}{2}}), H(A^{\frac{1}{2}})\right)} = \|\tilde{u}_1\|_{L_p\left(R; H(A^{\frac{3}{2}})\right)} + \|\tilde{u}_1''\|_{L_p\left(R; H(A^{\frac{1}{2}})\right)} \\
&= \left\| \left(F^{-1} L(\lambda, i\mu)^{-1} F\widetilde{f}(\mu)\right)(\cdot) \right\|_{L_p\left(R; H(A^{\frac{3}{2}})\right)} \\
&\quad + \left\| \left(F^{-1} (i\mu)^2 L(\lambda, i\mu)^{-1} F\widetilde{f}(\mu)\right)(\cdot) \right\|_{L_p\left(R; H(A^{\frac{1}{2}})\right)} \\
&\quad \le \left\| \left(F^{-1} A L(\lambda, i\mu)^{-1} F\widetilde{f}(\mu)\right)(\cdot) \right\|_{L_p\left(R; H(A^{\frac{1}{2}})\right)} \\
&\quad + \left\| \left(F^{-1} (i\mu)^2 L(\lambda, i\mu)^{-1} F\widetilde{f}(\mu)\right)(\cdot) \right\|_{L_p\left(R; H(A^{\frac{1}{2}})\right)}.
\end{aligned} \tag{3.7}$$

Let us show that the operator-functions (with respect to μ)

$$T_{k+1}(\lambda, \mu) := (i\mu)^{2k} A^{1-k} L(\lambda, i\mu)^{-1}, \ k = 0, 1, \tag{3.8}$$

is the Fourier multiplication in the space $L_p\left((0,1); H(A^{1/2})\right)$. To do this, it suffices to check the conditions of the Dunford-Schwartz theorem ([12], Ch. XI, §28, 29) (see also [9], Theorem 1.37/1) on Fourier multiplications for an operator-function $\mu \to T_{k+1}(\lambda, \mu) : R \to B\left(H(A^{\frac{1}{2}})\right)$, $k = 0, 1$. Obviously, for $\lambda \in \mathcal{A}_\varphi$ and $\mu \in R$, $\left|\arg\left(\lambda + \mu^2\right)\right| \le \varphi < \pi$. Since the operator $A - \varphi$ is positive in H, then by definition, for $\lambda \in \mathcal{A}_\varphi$ and $\mu \in R$ we have the estimates

$$\begin{aligned}
&\left\| L(\lambda, i\mu)^{-1} \right\|_{B(H)} = \left\| \left[A + \left(\lambda + \mu^2\right) I\right]^{-1} \right\|_{B(H)} \\
&\le \frac{C}{1 + |\lambda + \mu^2|} \le \frac{C}{|\mu|^2}, \ \ \mu \neq 0
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
&\left\| A L(\lambda, i\mu)^{-1} \right\|_{B(H)} = \left\| A \left[A + \left(\lambda + \mu^2\right) I\right]^{-1} \right\|_{B(H)} \\
&= \left\| I - \left(\lambda + \mu^2\right) \left[A + \left(\lambda + \mu^2\right) I\right]^{-1} \right\|_{B(H)} \\
&\le 1 + \left|\lambda + \mu^2\right| \tfrac{C}{1 + |\lambda + \mu^2|} \le C,
\end{aligned} \tag{3.10}$$

uniformly by λ in the corner $\mathcal{A}_\varphi$. From estimates (3.9), (3.10) for $\lambda \in \mathcal{A}_\varphi$ and $\mu \in R$, we have

$$\|T_1(\lambda,\mu)\|_{B(H(A^{\frac{1}{2}}))} = \left\|AL(\lambda,i\mu)^{-1}\right\|_{B(H(A^{\frac{1}{2}}))} = \left\|AL(\lambda,i\mu)^{-1}\right\|_{B(H)} \le C, \quad (3.11)$$

$$\begin{aligned} &\left\|T_2(\lambda,\mu)\right\|_{B(H(A^{\frac{1}{2}}))} = \left\|(i\mu)^2 L(\lambda,i\mu)^{-1}\right\|_{B(H(A^{\frac{1}{2}}))} \\ &= \left\|(i\mu)^2 L(\lambda,i\mu)^{-1}\right\|_{B(H)} = |\mu^2|\left\|L(\lambda,i\mu)^{-1}\right\|_{B(H)} \le C \end{aligned} \quad (3.12)$$

uniformly by λ in the corner $\mathcal{A}_\varphi$. Since

$$\frac{\partial}{\partial\mu}T_1(\lambda,\mu) = A\frac{\partial}{\partial\mu}L(\lambda,i\mu)^{-1} = -AL(\lambda,i\mu)^{-1}\cdot 2\mu\, L(\lambda,i\mu)^{-1};$$

$$\begin{aligned} &\frac{\partial}{\partial\mu}T_2(\lambda,\mu) = \frac{\partial}{\partial\mu}(-\mu^2 L(\lambda,i\mu)^{-1}) \\ &= -2\mu L(\lambda,i\mu)^{-1} + \mu^2 L(\lambda,i\mu)^{-1} 2\mu L(\lambda,i\mu)^{-1} \end{aligned}$$

by virtue of (3.9) -(3.10) we have

$$\begin{aligned} &\left\|\tfrac{\partial}{\partial\mu}T_1(\lambda,\mu)\right\|_{B(H(A^{\frac{1}{2}}))} = \left\|AL(\lambda,i\mu)^{-1} 2\mu\, L(\lambda,i\mu)^{-1}\right\|_{B(H(A^{\frac{1}{2}}))} \\ &= \left\|AL(\lambda,i\mu)^{-1} 2\mu\, L(\lambda,i\mu)^{-1}\right\|_{B(H)} \le C2|\mu|\cdot\tfrac{1}{|\mu|^2} \le \tfrac{C}{|\mu|}, \quad \mu \ne 0 \end{aligned} \quad (3.13)$$

$$\begin{aligned} &\left\|\frac{\partial}{\partial\mu}T_2(\lambda,\mu)\right\|_{B(H(A^{\frac{1}{2}}))} \le 2|\mu|\left\|L(\lambda,i\mu)^{-1}\right\|_{B(H(A^{\frac{1}{2}}))} \\ &+ |\mu|^2\left\|L(\lambda,i\mu)^{-1}\right\|_{B(H(A^{\frac{1}{2}}))} 2|\mu|\left\|L(\lambda,i\mu)^{-1}\right\|_{B(H(A^{\frac{1}{2}}))} \\ &= 2|\mu|\left\|L(\lambda,i\mu)^{-1}\right\|_{B(H)} + 2|\mu|^3\left\|L(\lambda,i\mu)^{-1}\right\|_{B(H)} \\ &\times\left\|L(\lambda,i\mu)^{-1}\right\|_{B(H)} \le \frac{C}{|\mu|}, \mu \ne 0 \end{aligned} \quad (3.14)$$

By virtue of the Dunford-Schwartz theorem, from estimates (3.11)-(3.14) imply that the operator-functions $\mu \to T_{k+1}(\lambda,\mu)$, $k = 0,1$ defined by equalities (3.8) is a Fourier multiplication in the space $L_p\left(R; H(\frac{1}{2})\right)$. Then it follows from (3.7) that the following estimate holds

$$\|\tilde{u}_1\|_{W_p^2\left(R;H(A^{\frac{3}{2}}),H(A^{\frac{1}{2}})\right)} \le C\left\|\tilde{f}\right\|_{L_p\left(R;H(A^{\frac{1}{2}})\right)}, \quad (3.15)$$

uniformly by λ. It follows from (3.15) that $u_1 \in W_p^2\left((0,1); H\left(A^2\right), H(A)\right)$ and we have the estimate

$$\|u_1\|_{W_p^2\left((0,1);H(A^{\frac{3}{2}}),H(A^{\frac{1}{2}})\right)} \le C\|f\|_{L_p\left((0,1);H(A^{\frac{1}{2}})\right)}. \quad (3.16)$$

From (3.16), by continuity of the embedding

$$W_p^2\left((0,1); H\left(A^{\frac{3}{2}}\right), H(A^{\frac{1}{2}})\right) \subset W_p^2\left((0,1); H(A), H\right)$$

we have

$$\|u_1\|_{W_p^2((0,1);H(A),H)} \leq C\,\|f\|_{L_p\left((0,1);H(A^{\frac{1}{2}})\right)}. \tag{3.17}$$

From (3.4) for $u_1(x)$ we have

$$\lambda u_1(x) = f(x) + u_1''(x) - Au_1(x),\;\; x \in (0,1)\,.$$

Hence, in view of (3.17)

$$\begin{aligned} &|\lambda|\,\|u_1\|_{L_p((0,1);H)} \leq \|f\|_{L_p((0,1);H)} + \|u_1''\|_{L_p((0,1);H)} + \|Au_1\|_{L_p((0,1);H)} \\ &\leq \|f\|_{\;L_p((0,1);H)} + C\,\|f\|_{\;L_p\left((0,1);H\left(A^{\frac{1}{2}}\right)\right)} \leq C\,\|f\|_{\;L_p\left((0,1);H\left(A^{\frac{1}{2}}\right)\right)}. \end{aligned} \tag{3.18}$$

From inequalities (3.17) and (3.18) for $\lambda \in \mathcal{A}_\varphi$ we have

$$\begin{aligned} &|\lambda|\,\|u_1\|_{L_p((0,1);H)} + \left\|u_1''\right\|_{L_p((0,1);H)} \\ &+ \|Au_1\|_{L_p((0,1);H)} \leq C\,\|f\|_{\;L_p\left((0,1);H\left(A^{\frac{1}{2}}\right)\right)} \end{aligned} \tag{3.19}$$

By virtue of ([9], Theorem 1.7.7/1), it follows from estimate (3.16) that for any fixed $x_0 \in [0,1]$ and $k = 0,1$,

$$u_1^{(k)}(x_0) \in \left(H\left(A^{\frac{3}{2}}\right), H(A^{\frac{1}{2}})\right)_{\frac{k}{2}+\frac{1}{2p},p}.$$

By virtue of ([9], Lemmas 1.7.3/1, 1.7.3./6 and 1.7.3./5) for $k = 0,1$, we have

$$\left(H(A^{\frac{3}{2}}), H(A^{\frac{1}{2}})\right)_{\frac{1+kp}{2p},p} = \left(H\left(A^2\right), H\right)_{\frac{1}{4}+\frac{1+kp}{4p},p}.$$

Therefore, for any fixed $x_0 \in [0,1]$,

$$u_1(x_0) \in \left(H\left(A^2\right), H\right)_{\frac{1}{4}+\frac{1}{4p},p},\;\; u_1'(x_0) \in \left(H\left(A^2\right), H\right)_{\frac{1}{2}+\frac{1}{4p},p}.$$

Let us show that the operator B boundedly acts from the interpolation space $\left(H\left(A^2\right), H\right)_{\frac{1}{2}+\frac{1}{4p},p}$ to $\left(H\left(A^2\right), H\right)_{\frac{1}{2}+\frac{1}{4p},p}$ and we have the estimate

$$\|B\|_{\;B\left((H(A^2),H)_{\frac{1}{2}+\frac{1}{4p},p}\right)} \leq C. \tag{3.20}$$

Indeed, since the operator B boundedly acts from H to H and from $H(A)$ to $H(A)$, then, by virtue of the interpolation theorem, the operator B boundedly acts from $(H(A),\; H)_{\theta,p}$ to $(H(A), H)_{\theta,p}$ for any $\theta \in (0,1)$, including for $\theta = \frac{1}{2p}$. Since $(H(A), H)_{\frac{1}{2},p} = \left(H\left(A^2\right), H\right)_{\frac{1}{2}+\frac{1}{4p},p}$, the estimate (3.20) holds.

Thus, $L_1 u_1 \in \left(H\left(A^2\right), H\right)_{\frac{1}{2}+\frac{1}{4p},p}$; $L_2 u_1 \in \left(H\left(A^2\right), H\right)_{\frac{1}{4}+\frac{1}{4p},p}$.

Then, by virtue of Theorem 1, for $\lambda \in \mathfrak{U}_\varphi$, for the solution of problem (3.5) we have the inequalities

$$\begin{aligned}
&|\lambda|\|u_2\|_{L_p((0,1);H)} + \|u_2''\|_{L_p((0,1);H)} + \|Au_2\|_{L_p((0,1);H)} \\
&\le C\Big(\|f_1 - L_1u_1\|_{(H(A^2),H)_{\frac{1}{2}+\frac{1}{4p},p}} + \|f_2 - L_2u_2\|_{(H(A^2),H)_{\frac{1}{4}+\frac{1}{4p},p}} \\
&+ |\lambda|^{1-\frac{1}{2p}}\|f_1 - L_1u_1\|_H + |\lambda|^{\frac{3}{2}-\frac{1}{2p}}\|f_2 - L_2u_1\|_H\Big) \\
&\le C\Big(\|f_1\|_{(H(A^2),H)_{\frac{1}{2}+\frac{1}{4p},p}} + \|f_2\|_{(H(A^2),H)_{\frac{1}{4}+\frac{1}{4p},p}} \\
&+ |\lambda|^{1-\frac{1}{2p}}\|f_1\|_H + |\lambda|^{\frac{3}{2}-\frac{1}{2p}}\|f_2\|_H + \|Bu_1'(1)\|_{(H(A^2),H)_{\frac{1}{2}+\frac{1}{4p},p}} \\
&+ \|u_1(0)\|_{(H(A^2),H)_{\frac{1}{2}+\frac{1}{4p},p}} + \|u_1\|_{(H(A^2),H)_{\frac{1}{4}+\frac{1}{4p},p}} \\
&+ |\lambda|^{1-\frac{1}{2p}}(\|Bu_1'(1)\| + \|u_1(0)\|_H) + |\lambda|^{\frac{3}{2}-\frac{1}{2p}}\|u_1(1)\|_H\Big).
\end{aligned} \tag{3.21}$$

We estimate the norm $\|Bu_1'(1)\|_{(H(A^2),H)_{\frac{1}{2}+\frac{1}{4p},p}}$. By virtue of ([11], Theorem 1.8.2), and estimates (3.20), (3.16), we have

$$\begin{aligned}
&\|Bu_1'(1)\|_{(H(A^2),H)_{\frac{1}{2}+\frac{1}{4p},p}} \le C\,\|u_1'(1)\|_{(H(A^2),H)_{\frac{1}{2}+\frac{1}{4p},p}} \\
&\le C\,\|u_1\|_{W_p^2\left((0,1);H(A^{\frac{3}{2}}),H(A^{\frac{1}{2}})\right)} \le C\,\|f\|_{L_p((0,1);H(A^{\frac{1}{2}}))}\,.
\end{aligned} \tag{3.22}$$

We estimate the norm $\|u_1(0)\|_{(H(A^2),H)_{\frac{1}{2}+\frac{1}{4p},p}}$. Taking into account the continuity of the embedding $\left(H\left(A^2\right),H\right)_{\frac{1}{4}+\frac{1}{4p},p} \subset \left(H\left(A^2\right),H\right)_{\frac{1}{2}+\frac{1}{4p},p}$, by virtue of ([11], Theorem 1.8.2) (see also [9], Theorem 1.7.7/1) and estimate (3.16), we have

$$\begin{aligned}
&\|u_1(0)\|_{(H(A^2),H)_{\frac{1}{2}+\frac{1}{4p},p}} \le C\,\|u_1(0)\|_{(H(A^2),H)_{\frac{1}{4}+\frac{1}{4p},p}} \\
&\le C\,\|u_1\|_{W_p^2\left((0,1);H(A^{\frac{3}{2}}),H(A^{\frac{1}{2}})\right)} \le C\,\|f\|_{L_p((0,1);H(A^{\frac{1}{2}}))}\,.
\end{aligned} \tag{3.23}$$

By virtue of ([11], Theorem 1.8.2) (see also ([9], Theorem 1.7.7/1)) and estimate (3.16), we have

$$\begin{aligned}
&\|u_1(1)\|_{(H(A^2),H)_{\frac{1}{4}+\frac{1}{4p},p}} \le C\|u\|_{W_p^2((0,1);H(A^{\frac{3}{2}}),H(A^{\frac{1}{2}}))} \\
&\le C\|f\|_{L_p((0,1);H(A^{\frac{1}{2}}))}.
\end{aligned} \tag{3.24}$$

By virtue of ([9], Theorem 1.7.7/2) for any $\mu \in \mathbb{C}$, $\ |\mu| \to \infty$ and for any $\nu \in W_p^2\left((0,1)\,;H\right)$ the following inequality is true

$$|\mu|^{2-s}\left\|\nu^{(s)}\left(x_0\right)\right\|_H \le C\left(|\mu|^{\frac{1}{p}}\,\|\nu\|_{W_p^2((0,1);H)} + |\mu|^{2+\frac{1}{p}}\,\|\nu\|_{L_p((0,1);H)}\right), \tag{3.25}$$

where $x_0 \in [0,1]$, $s = \{0,1\}$, $p \in (1,+\infty)$.

We divide (3.25) by $|\mu|^{\frac{1}{p}}$ and denote $\lambda = \mu^2$. Then for $\lambda \in \mathbb{C}$, $\ |\lambda| \to \infty$ and for any $\nu \in W_p^2\left((0,1)\,;H\right)$, we have

$$|\lambda|^{1-\frac{s}{2}-\frac{1}{2p}}\left\|\nu^{(s)}\left(x_0\right)\right\|_H \le C\left(\|\nu\|_{W_p^2((0,1);H)} + |\lambda|\,\|\nu\|_{L_p((0,1);H)}\right). \tag{3.26}$$

In inequality (3.26) we take the function $\nu(x)$ instead of the $u_1(x)$. Then due to estimate (3.19) for $\lambda \in u_\varphi$, we have

$$\begin{aligned} &|\lambda|^{1-\frac{s}{2}-\frac{1}{2p}} \left\| u_1^{(s)}(x_0) \right\|_H \le C\left(\|u_1\|_{W_p^2((0,1);H(A),H)} + |\lambda| \, \|u_1\|_{L_p((0,1);H)} \right) \\ &\le C \, \|f\|_{L_p\left((0,1);H(A^{\frac{1}{2}})\right)} . \end{aligned} \tag{3.27}$$

Using inequality (3.27), we estimate the norms $|\lambda|^{1-\frac{1}{2p}} \left(\|Bu_1'(1)\|_H + \|u_1(0)\|_H \right)$ and $|\lambda|^{\frac{3}{2}-\frac{1}{2p}} \|u_1(1)\|_H$.

In inequality (3.27) we take $s=0, \ x_0=0$. Then for $\lambda \in \mathfrak{U}_\varphi$

$$|\lambda|^{1-\frac{1}{2p}} \|u_1(0)\|_H \le C \, \|f\|_{L_p\left((0,1);H(A^{\frac{1}{2}})\right)} . \tag{3.28}$$

In inequality (3.27) we take $s=0, \ x_0=1$. Then for $\lambda \in \mathfrak{U}_\varphi$

$$|\lambda|^{\frac{3}{2}-\frac{1}{2p}} \|u_1(1)\|_H = |\lambda|^{\frac{1}{2}} \cdot |\lambda|^{1-\frac{1}{2p}} \|u_1(1)\|_H \le C \, |\lambda|^{\frac{1}{2}} \, \|f\|_{L_p\left((0,1);H(A^{\frac{1}{2}})\right)} . \tag{3.29}$$

By virtue of condition 2) of Theorem 1, from inequality (3.27) for $s=1, \ x_0=1$, we have

$$\begin{aligned} &|\lambda|^{\frac{1}{2}-\frac{1}{2p}} \|Bu_1'(1)\|_H \le |\lambda|^{\frac{1}{2}} |\lambda|^{\frac{1}{2}-\frac{1}{2p}} \|B\|_{B(H)} \|u_1'(1)\|_H \\ &\le C|\lambda|^{\frac{1}{2}} |\lambda|^{\frac{1}{2}-\frac{1}{2p}} \|u_1'(1)\|_H \le C|\lambda|^{\frac{1}{2}} \|f\|_{L_p\left((0,1);H(A^{\frac{1}{2}})\right)} \end{aligned}$$

From estimate (3.28), (3.30) we have

$$|\lambda|^{1-\frac{1}{2p}} \left(\|Bu_1'(1)\|_H + \|u_1(0)\|_H \right) \le C|\lambda|^{\frac{1}{2}} \|f\|_{L_p\left((0,1);H(A^{\frac{1}{2}})\right)} . \tag{3.30}$$

Taking into account estimates (3.22), (3.23), (3.24), (3.29), (3.30) in (3.21), for $\lambda \in \mathfrak{U}_\varphi$ we obtain

$$\begin{aligned} &|\lambda| \|u_2\|_{L_p((0,1);H)} + \|u_2''\|_{L_p((0,1);H)} + \|Au_2\|_{L_p((0,1);H)} \\ &\le C\Big[|\lambda|^{\frac{1}{2}} \|f\|_{L_p((0,1);H(A^{\frac{1}{2}}))} + \sum_{k=1}^{2} \Big(\|f_k\|_{(H(A^2),H)_{\frac{3}{4}-\frac{k}{4}+\frac{1}{4p},p}} \\ &+ |\lambda|^{\frac{k+1}{2}-\frac{1}{2p}} \|f_k\|_H \Big) \Big]. \end{aligned} \tag{3.31}$$

Estimates (3.31) and (3.19) imply estimate (3.3). Theorem 3.2. is proved.

4 Application of abstract results to elliptic partial differential equations

In the square $\Omega = [0,1] \times [0,1]$ we consider the following boundary value problem for an elliptic differential equation with a complex parameter.

$$\begin{aligned} &Lu := \lambda u(x,y) - D_x^2 u(x,y) - D_y \left(a(y) D_y(x,y) \right) \\ &= f(x,y), \qquad (x,y) \in (0,1) \times (0,1), \end{aligned} \tag{4.1}$$

$$L_1 u := \int_0^1 b(t,y) \frac{\partial u(1,t)}{\partial x} dt + u(0,y) = f_1(y), \tag{4.2}$$

$$L_2u := u\,(1,y) = f_2(y), \quad y \in [0,1]\,, \\ L_3u := u\,(x,0) = 0, \;\; L_4u := u\,(x,1) = 0, \;\; x \in [0,1]\,, \tag{4.3}$$

where λ is a complex parameter; $a\,(y)\,,\ b\,(x,y)$ are some continuous functions;

$$D_x := \frac{\partial}{\partial x},\ D_y := \frac{\partial}{\partial y}.$$

Let $S_0,\ S_1$ be non-negative integers, $0 < \theta < 1$, $1 < p, q < \infty$ and $S = (1-\theta)\,S_0 + \theta S_1$.

Denote by $B^s_{p,q}(0,1) := (W^{S_0}_p(0,1),\ \ W^{S_1}_p(0,1))_{\theta,q}$ the interpolation space of Sobolev spaces.

In particular, $W^s_p(0,1) := B^s_p(0,1) := \Big(W^{S_0}_p(0,1), W^{S_1}_p(0,1)\Big)_{\theta,p}$, if the positive number s is different from an integer. We also denote $W^{\ell,s}_{p,q}(\Omega); = W^{\ell}_p\Big((0,1); W^s_q(0,1), L_q(0,1)\Big)$, where $0 \le \ell, s$ are non-negative integers, $1 < p < \infty$,$1 < q < \infty$. If $p = q$ and $\ell = s$, then $W^{\ell,s}_{p,q}\,(\Omega) := W^{\ell}_p\,(\Omega)$. And, finally, denote $L_{p,q}\,(\Omega) := W^{0,0}_{p,q}\,(\Omega) = L_p\,((0,1)\,;\ L_q\,(0,1))$. We have that $L_{p,q}\,(\Omega)$ is a Banach space of measurable functions $u\,(x,y)$ on $(0,1)\times(0,1)$, for which the following norm is finite

$$\|u\|_{L_{p,q}(\Omega)} = \Big(\int\limits_0^1 \Big(\int\limits_0^1 |u(x,y)|^q dy\Big)^{\frac{p}{q}} dx\Big)^{\frac{1}{p}},$$

and $W^{\ell,s}_{p,q}\,(\Omega)$ is a Banach space of functions $u(x,y)$ having on $(0,1)\times(0,1)$ generalized derivatives $\frac{\partial^\ell u(x,y)}{\partial x^\ell}$, $\frac{\partial^s u(x,y)}{\partial y^s}$, and the finite norm

$$\|u\|_{W^{\ell,s}_{p,q}(\Omega)} = \|u\|_{L_{p,q}(\Omega)} + \Big\|D^{\ell}_x u\Big\|_{L^{(\Omega)}_{p,q}} + \big\|D^s_y u\big\|_{L^{(\Omega)}_{p,q}}\,.$$

Theorem 4.1 *Let $a(y) \in C^3\,[0,1]\,,\ a(y) > 0$ for $y \in [0,1]$ and $a''(0) = a''(1) = 0$; $b(x,y) \in C^{0,2}(\Omega)$, where $C^{0,2}(\Omega)$ is the space of continuous functions in the square Ω, which have continuous derivatives in this square $\frac{\partial b(x,y)}{\partial y}$, $\frac{\partial^2 b(x,y)}{\partial y^2}$ and $b(x,0) = b(x,1) = 0$ for $x \in [0,1]$.*

Then for $f\,(x,y) \in L_p\,\big((0,1)\,;\,\big(W^1_2\,(0,1)\,;\,L_v f = 0,\ v = 3,4\big)\,\big)$, $p \in (1,\infty)$, $f_k\,(y) \in B^{k+1-\frac{1}{p}}_{2,p,*}\ (0,1)$ (the definition for $B^{k+1-\frac{1}{p}}_{2,p,*}$ will be given in the proof of Theorem 3) and for $\lambda \in u_p$ problem (4.1)-(4.3) *has a unique solution from $W^2_p((0,1);\ W^2_2((0,1);\ u(0) = u(1) = 0),\ L_2(0,1))$ and the following non-coercive estimate is valid for it*

$$|\lambda|\,\|u\|_{\mathrm{L_p}\left((0,1);\mathrm{L_2}(0,1)\right)} + \big\|D^2_x u(x,y)\big\|_{\mathrm{L_p}\left((0,1);\mathrm{L_2}(0,1)\right)} + \big\|D^2_y u(x,y)\big\|_{\mathrm{L_p}\left((0,1);\mathrm{L_2}(0,1)\right)} \le$$
$$\le C\left[\ |\lambda|^{1/2}\,\|f(x,y)\|_{\mathrm{L_p}\left((0,1);\mathrm{W^1_2}(0,1)\right)} + \textstyle\sum_{k=1}^{2}\left(\ \|f_k\|_{B^{k+1-\frac{1}{p}}_{2,p}(0,1)} + \right.\right.$$
$$\left.\left. + |\lambda|^{\frac{k+1}{2}-\frac{1}{2p}}\,\|f_k\|_{\mathrm{L_2}(0,1)}\ \right)\right]\ .$$

Proof. In space $H := L_2(0,1)$ consider the operator A defined by the equalities

$$D(A) := W^2_2\,((0,1)\,;\ u(0) = u(1) = 0)\ , Au := -\left(a\,(y)\ \ u'(y)\right)'\ , \tag{4.4}$$

$$D(B) := L_2\,(0,1)\,, \quad Bu := \int_0^1 b\,(t,y)\,u(t)\,dt. \tag{4.5}$$

Then we write problem (4.1)–(4.3) in the operator form

$$\lambda u\left(x\right)-u''\left(x\right)+Au(x)=f(x),\;\; x\in\left(0,1\right), \tag{4.6}$$

$$\begin{aligned} &Bu'\left(1\right)+u\left(0\right)=f_1,\\ &u(1)\;=f_2 \end{aligned} \tag{4.7}$$

where $u\left(x\right):=u\left(x,\;\cdot\right),\;\; f(x)\;=f(x,\,\cdot)$ are vector-valued functions with the values from Hilbert space $L_2\left(0,1\right)$ and $f_k:=f_k(\,\cdot)$. It is obvious that the proof of Theorem 3 reduces to checking the condition of Theorem 2 for the boundary value problems (4.6)-(4.7) . It is known that the operator A, defined by equalities (4.4) is self-adjoint and positively definite in the space $L_2\left(0,1\right)$, i.e. condition 1) of Theorem 2 for problem (4.6), (4.7) is satisfied. As shown in [1], the operator B defined by equalities (4.5) boundly acts in the spaces $H=L_2\left(0,1\right)$ and $H(A)$. Consequently, conditions 2) of Theorem 2 are also satisfied. It is known that the operator A defined by equalities (4.4) has a bounded inverse operator A^{-1} and this operator commutes with the operator B in $L_2\left(0,1\right)$, i.e., conditions 3 of Theorem 2 are also satisfied. It can be easily verified that

$$A^2u=\left(\left(a(y)(a(y)u'(y))''\right)\right)',D\left(A^2\right)=W_2^4\left((0,1)\,;L_vu=0,\;\;L_vAu=0,v=3,4\right).$$

Obviously, the boundary conditions $L_vAu\,=\,0,v\,=\,3,4$ coincide with the boundary conditions $u''(0)=u''(1)=0$. Since the order of the boundary conditions $L_vu=0,\;\;v=3,4,$ is equal to zero, and the order of the boundary conditions $L_vAu=0,\;\;v=3,4$ is equal to two, then due to ([11], Theorem 4.3.3) we have

$$\left(H\left(A^2\right),H\right)_{\frac{1}{2}+\frac{1}{4p},p}=B_{2,p,*}^{2-\frac{1}{p}}\left(0,1\right)=$$
$$=\begin{cases} B_{2,p}^{2-\frac{1}{p}}\left((0,1)\,;\;\;u(0)=u(1)=0\right),\;\;1<p<2\,,\\ W_2^{\frac{3}{2}}\left((0,1)\,;\;\;u(0)=u(1)=0,\;\;\int_0^1\left(\min\left\{x,1-x\right\}\right)^{-1}\left|u''\left(x\right)\right|^2dx<\infty\right),\;\;p=2,\\ B_{2,p}^{2-\frac{1}{p}}\left((0,1)\,;\;\;u^{(j)}(0)=\;u^{(j)}(1)=0,\;\;j=0,2\right),\;\;p>2; \end{cases};$$

$$\left(H\left(A^2\right),H\right)_{\frac{1}{4}+\frac{1}{4p},p}=B_{2,p,*}^{3-\frac{1}{p}}\left(0,1\right)=$$
$$=\begin{cases} B_{2,p}^{3-\frac{1}{p}}\left((0,1)\,;\;\;u(0)=u(1)=0\right),\;\;1<p<2\,,\\ W_2^{\frac{5}{2}}\left((0,1)\,;\;\;u(0)=u(1)=0,\;\;\int_0^1\left(\min\left\{x,1-x\right\}\right)^{-1}\left|u''\left(x\right)\right|^2dx<\infty\right),\;\;p=2,\\ B_{2,p}^{3-\frac{1}{p}}\left((0,1)\,;\;\;u^{(j)}(0)=\;u^{(j)}(1)=0,\;\;j=0,2\right),\;\;p>2. \end{cases}$$

References

1. Aliev, B.A., Yakubov, Ya.: *Second order elliptic differential-operator equations with unbounded operator boundary conditions in UMD Banach spaces*, Inregral Equations Operator Theory **69**, 269-300 (2011).
2. Aliev, B.A., Kurbanova, N.K., Yakubov, Ya.: *Solvability of a boundary value problem for second-order elliptic differential-operator equations with spectral parameter in the equation and in the boundary contions*, Differ. Equ. **54**(1), 67-85 (2018).
3. Denche, M.: *Abstract differential equation with a spectral parameter in the boundary conditions*, Result. Math. **35**, 216-227 (1999).

4. Aibeche, A., Favini, A., Mezoued, Ch.: *Deficient coerciveness estimate for an abstract differential equation with a parameter dependent boundary conditions*, Boll Unione Mat. Ital. Sez. B Artic. Ric. Mat. **10**(3), 535-547 (2007).
5. Hammou H., Labbas R., Maingot S., Medeghri A.: *Nonlocal general boundary value problems of elliptic type in L^p cases*, Mediterr. J. Math. **13**(4), 1669-1683 (2016).
6. Aliev, B.A., Kerimov, V.Z., Yakubov, Ya.S.: *Solvability of a boundary value problem for a second order elliptic differential operator equation with a complex parameter*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **47** (2), 309-326 (2021).
7. Aliev, B.A., Kerimov, V.Z., Yakubov, Ya.: *Solvability of a boundary value problem for elliptic diferential-operator equations of the second order with qvadratic spectral parameter*, Differ. Equ. **56**(10), 1306-1317 (2020).
8. Aliev, B.A., Kerimov, V.Z.: *On solvability of a boundary value problem for a second order elliptic diferential-operator equations with a complex linear parameter*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics, **41**(4), 17-30 (2021).
9. Yakubov, S., Yakubov, Ya.: Differential-Operator Equations. Ordinary and Partial Differential Equations, *Chapman and Hall / CRC, Boca Raton*, (2000).
10. Krein, S.G.: Linear Differential Equations in Banach Spaces, *(in Russian), Nauka, Moscow* (1967).
11. Tribel, H.: Interpolation Theory. Functions spaces. Differential Operators, *North-Holland, Amsterdam*, (1978).
12. Dunford, N., Schwartz, T.J.: Linear Operators, *Part II, Spectral Theory, Wiley-Interscinece, New York*, (1963).