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# Multilinear Calderón-Zygmund operators with kernels of Dini's type and their commutators on generalized local Morrey spaces

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**Abstract.** Let $T$ be a multilinear Calderón-Zygmund operator of type $\omega$ with $\omega(t)$ being nondecreasing and satisfying a kind of Dini's type condition and $T_{\Pi\mathbf{b}}$ be the iterated commutators of the operator $T$ with $BMO^m$ functions. In this paper, we study the boundedness of the operators $T$ and $T_{\Pi\mathbf{b}}$ on generalized local Morrey spaces $M_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ and generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$. We find the sufficient conditions on the pair $(\varphi_1, \varphi_2)$ with $\mathbf{b} \in BMO^m$ which ensures the boundedness of the operators $T$ and $T_{\Pi\mathbf{b}}$ from $M_{p_1,\varphi_{11}}^{\{x_0\}}(\mathbb{R}^n) \times \cdots \times M_{p_m,\varphi_{1m}}^{\{x_0\}}(\mathbb{R}^n)$ to $M_{p,\varphi_2}^{\{x_0\}}(\mathbb{R}^n)$.

**Keywords.** Multilinear Calderón-Zygmund operator, generalized local Morrey spaces, commutator, $BMO$

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## 1 Introduction and Main Results

Let $T$ be a multilinear Calderon-Zygmund operator such that

$$T(f_1, \dots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \dots, y_m) f_1(y_1) \dots f_m(y_m) dy_1 \dots dy_m,$$

$x \notin \bigcap_{j=1}^{m} supp f_j$, where $K(x, y_1, \dots, y_m)$ is an $m$-linear Calderon-Zygmund kernel of type $\omega(t)$ and each $f_j \in C_c^\infty(\mathbb{R}^n)$, $j = 1, \dots, m$. Let $\mathbf{b} = (b_1, \dots, b_m) \in BMO^m$ such that

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$b_j \in BMO(\mathbb{R}^n)$, $j = 1, \ldots, m$. The $m$-linear commutator of $T$ with $\mathbf{b}$ is defined by

$$T_{\Sigma \mathbf{b}}(f_1, \ldots, f_m) = \sum_{j=1}^{m} T^j_{b_j}(\mathbf{f}),$$

where

$$T^j_{b_j}(\mathbf{f}) = b_j T(f_1, \ldots, f_j, \ldots, f_m) - T(f_1, \ldots, f_j, \ldots, f_m). \tag{1.1}$$

The iterated commutator of $T$ with $\mathbf{b}$ is defined by

$$T_{\Pi \mathbf{b}}(\mathbf{f})(x) = [b_1, [b_2, \ldots [b_{m-1}, [b_m, T]_m]_{m-1} \ldots]_2, ]_1(\mathbf{f})(x). \tag{1.2}$$

For an $m$-linear Calderón-Zygmund operator with associated kernel $K(x, \mathbf{y})$, the iterated commutator $T_{\Pi \mathbf{b}}$ is given formally by

$$T_{\Pi \mathbf{b}}(\mathbf{f})(x) = \int_{(\mathbb{R}^n)^m} \Big( \prod_{j=1}^{m} (b_j(x) - b_j(y_j)) \Big) K(x, \mathbf{y}) f_1(y_1) \ldots f_m(y_m) d\mathbf{y}.$$

Here and in what follows, $\mathbf{y} = (y_1, \ldots, y_m)$, $(x, \mathbf{y}) = (x, y_1, \ldots, y_m)$ and $d\mathbf{y} = dy_1 \ldots dy_m$.

The study of multilinear Calderón-Zygmund theory goes back to the pioneering works of Coifman and Meyer in 1970s, see e.g. [2,3]. This topic was then further investigated by many authors in the last few decades, see for example [6,15,18,23]. The classical Morrey spaces $M_{p,\lambda}$ were introduced by Morrey [21] to study the local behavior of solutions to second-order elliptic partial differential equations. Moreover, various Morrey spaces are defined in the process of study. The first author, Mizuhara and Nakai [7,20,22] introduced generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$ (see, also [8,12,24]). In [4], the boundedness of the multilinear commutator of Calderón-Zygmund operator $T_{\mathbf{b}}$ from one generalized variable exponent Morrey space $M_{p(\cdot),\varphi_1}$ to another $M_{p(\cdot),\varphi_2}$ for $\mathbf{b} = (b_1, \ldots, b_m) \in BMO^m$ was proved, see also [1,5,13,14].

The purpose of this paper is to prove the boundedness of multilinear Calderón-Zygmund operators $T$ of type $\omega$ and their iterated commutators $T_{\Pi \mathbf{b}}$ with $BMO^m$ functions from $M^{\{x_0\}}_{p_1,\varphi_{11}}(\mathbb{R}^n) \times \cdots \times M^{\{x_0\}}_{p_m,\varphi_{1m}}(\mathbb{R}^n)$ to $M^{\{x_0\}}_{p,\varphi_2}(\mathbb{R}^n)$.

Our main results formulated as follows:

**Theorem 1.1** *Let $m \geq 2$, $x_0 \in \mathbb{R}^n$, $T$ be an $m$-linear $\omega - CZO$ and $\omega \in Dini(1)$. Suppose that $p_1, \ldots, p_m \in [1, \infty)$, $p \in (0, \infty)$ with $1/p = \sum\limits_{i=1}^{m} 1/p_i$, $\boldsymbol{\varphi}_1 = (\varphi_{11}, \ldots, \varphi_{1m})$ and $(\boldsymbol{\varphi}_1, \varphi_2)$ satisfies the condition*

$$\int_r^{\infty} t^{-\frac{n}{p}} \operatorname*{ess\,inf}_{t<s<\infty} s^{\frac{n}{p}} \prod_{i=1}^{m} \varphi_{1i}(x_0, s) \, \frac{dt}{t} \leq C \, \varphi_2(x_0, r), \tag{1.3}$$

*where $C$ does not depend on $r$.*

(1) *If $1 < p_j < \infty$ for all $j = 1, \ldots, m$, then there exists a constant $C > 0$ such that*

$$\|T(\mathbf{f})\|_{M^{\{x_0\}}_{p,\varphi_2}} \leq C \prod_{j=1}^{m} \|f_j\|_{M^{\{x_0\}}_{p_i,\varphi_{1i}}}.$$

(2) *If $1 \leq p_j < \infty$ for all $j = 1, \ldots, m$, and at least one of the $p_j = 1$, then there exists a constant $C > 0$ such that*

$$\|T(\mathbf{f})\|_{WM^{\{x_0\}}_{p,\varphi_2}} \leq C \prod_{j=1}^{m} \|f_j\|_{M^{\{x_0\}}_{p_i,\varphi_{1i}}}.$$

**Corollary 1.1** *Let $T$ be an $m$-linear $\omega - CZO$, $m \geq 2$, and $\omega \in Dini(1)$. Suppose that $p_1, \dots, p_m \in [1, \infty)$, $p \in (0, \infty)$ with $1/p = \sum_{i=1}^{m} 1/p_i$, $\boldsymbol{\varphi}_1 = (\varphi_{11}, \dots, \varphi_{1m})$ and $(\boldsymbol{\varphi}_1, \varphi_2)$ satisfies the condition*

$$\int_r^\infty t^{-\frac{n}{p}} \operatorname*{ess\,inf}_{t<s<\infty} s^{\frac{n}{p}} \prod_{i=1}^{m} \varphi_{1i}(x, s) \frac{dt}{t} \leq C \varphi_2(x, r), \tag{1.4}$$

*where $C$ does not depend on $x$ and $r$.*

(1) *If $1 < p_j < \infty$ for all $j = 1, \dots, m$, then there exists a constant $C > 0$ such that*

$$\|T(\mathbf{f})\|_{M_{p,\varphi_2}} \leq C \prod_{j=1}^{m} \|f_j\|_{M_{p_i,\varphi_{1i}}}.$$

(2) *If $1 \leq p_j < \infty$ for all $j = 1, \dots, m$, and at least one of the $p_j = 1$, then there exists a constant $C > 0$ such that*

$$\|T(\mathbf{f})\|_{WM_{p,\varphi_2}} \leq C \prod_{j=1}^{m} \|f_j\|_{M_{p_i,\varphi_{1i}}}.$$

**Theorem 1.2** *Let $m \geq 2$, $x_0 \in \mathbb{R}^n$, $T$ be an $m$-linear $\omega - CZO$ and $\omega$ satisfies*

$$\int_0^1 \frac{\omega(t)}{t} \Big(1 + \log \frac{1}{t}\Big)^m dt < \infty. \tag{1.5}$$

*Suppose that $p_1, \dots, p_m \in (1, \infty)$, $p \in (0, \infty)$ with $1/p = \sum_{i=1}^{m} 1/p_i$, and $(\boldsymbol{\varphi}_1, \varphi_2)$ satisfies the condition*

$$\int_r^\infty \Big(1 + \ln \frac{t}{r}\Big)^m t^{-\frac{n}{p}} \operatorname*{ess\,inf}_{t<s<\infty} s^{\frac{n}{p}} \prod_{i=1}^{m} \varphi_{1i}(x_0, s) \frac{dt}{t} \leq C \varphi_2(x_0, r), \tag{1.6}$$

*where $C$ does not depend on $r$.*

*If $b_i \in BMO$, $\mathbf{b} = (b_1, \dots, b_m)$, then there exist constants $C > 0$ independent of $\mathbf{f} = (f_1, \dots, f_m)$ such that*

$$\|T_{\Pi\mathbf{b}}(\mathbf{f})\|_{M^{\{x_0\}}_{p,\varphi_2}} \leq C \|\mathbf{b}\|_* \prod_{i=1}^{m} \|f_i\|_{M^{\{x_0\}}_{p_i,\varphi_{1i}}}, \tag{1.7}$$

*where $\|\mathbf{b}\|_* = \prod_{i=1}^{m} \|b_i\|_*$.*

**Corollary 1.2** *Let $m \geq 2$, $T$ be an $m$-linear $\omega - CZO$ and $\omega$ satisfies (1.5). Suppose that $p_1, \dots, p_m \in (1, \infty)$, $p \in (0, \infty)$ with $1/p = \sum_{i=1}^{m} 1/p_i$, and $(\boldsymbol{\varphi}_1, \varphi_2)$ satisfies the condition*

$$\int_r^\infty \Big(1 + \ln \frac{t}{r}\Big)^m t^{-\frac{n}{p}} \operatorname*{ess\,inf}_{t<s<\infty} s^{\frac{n}{p}} \prod_{i=1}^{m} \varphi_{1i}(x, s) \frac{dt}{t} \leq C \varphi_2(x, r), \tag{1.8}$$

*where $C$ does not depend on $x$ and $r$.*

*If $b_i \in BMO$, $\mathbf{b} = (b_1, \ldots, b_m)$, then there exist constants $C > 0$ independent of $\mathbf{f} = (f_1, \ldots, f_m)$ such that*

$$\|T_{\Pi\mathbf{b}}(\mathbf{f})\|_{M_{p,\varphi_2}} \leq C\,\|\mathbf{b}\|_* \prod_{i=1}^m \|f_i\|_{M_{p_i,\varphi_{1i}}}. \tag{1.9}$$

**Theorem 1.3** *Let $m \geq 2$, $x_0 \in \mathbb{R}^n$, $T$ be an $m$-linear $\omega - CZO$ and $\omega$ satisfies* (1.5). *Suppose that $(\boldsymbol{\varphi}_1, \varphi_2)$ satisfies the condition*

$$\int_r^\infty \Big(1 + \ln\frac{t}{r}\Big)^m t^{-n} \operatorname*{ess\,inf}_{t<s<\infty} s^n \prod_{i=1}^m \varphi_{1i}(x_0, s)\,\frac{dt}{t} \leq C\,\varphi_2(x_0, r), \tag{1.10}$$

*where $C$ does not depend on $r$.*

*If $b_i \in BMO$, $\mathbf{b} = (b_1, \ldots, b_m)$, then there exist constants $C > 0$ independent of $\mathbf{f} = (f_, \ldots, f_m)$ such that*

$$\|T_{\Pi\mathbf{b}}(\mathbf{f})\|_{WM^{\{x_0\}}_{1,\varphi_2}} \leq C\,\|\mathbf{b}\|_* \prod_{i=1}^m \|f_i\|_{M^{\{x_0\}}_{\Phi^{(m)},\varphi_{1i}}}, \tag{1.11}$$

*where $\Phi(t) = t(1 + \log^+ t)$ and $\Phi^{(m)} = \underbrace{\Phi \circ \cdots \circ \Phi}_{m \text{ times}}$.*

**Corollary 1.3** *Let $m \geq 2$, $T$ be an $m$-linear $\omega - CZO$ and $\omega$ satisfies* (1.5). *Suppose that $(\boldsymbol{\varphi}_1, \varphi_2)$ satisfies the condition*

$$\int_r^\infty \Big(1 + \ln\frac{t}{r}\Big)^m t^{-n} \operatorname*{ess\,inf}_{t<s<\infty} s^n \prod_{i=1}^m \varphi_{1i}(x, s)\,\frac{dt}{t} \leq C\,\varphi_2(x, r),$$

*where $C$ does not depend on $x$ and $r$.*

*If $b_i \in BMO$, $\mathbf{b} = (b_1, \ldots, b_m)$, then there exist constants $C > 0$ independent of $\mathbf{f} = (f_, \ldots, f_m)$ such that*

$$\|T_{\Pi\mathbf{b}}(\mathbf{f})\|_{WM_{1,\varphi_2}} \leq C\,\|\mathbf{b}\|_* \prod_{i=1}^m \|f_i\|_{M_{\Phi^{(m)},\varphi_{1i}}},$$

This paper is arranged as follows. In Section 2, we recall some basic definitions and known results. Section 3 is devoted to proving our main theorems.

Throughout this article, we will use $C$ to denote a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence. Besides, we will denote the conjugate exponent of $p > 1$ by $p' = p/(p-1)$. By $A \lesssim B$, we mean that $A \leq CB$ for some constant $C > 0$, and $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

## 2 Notations and definitions

We now recall the definition of multilinear Calderón-Zygmund operators of type $\omega$.

**Definition 2.1** *Let $w(t) : [0, \infty) \to [0, \infty)$ be a nondecreasing function. A locally integrable function $K(x, y_1, \ldots, y_m)$, defined away from the diagonal $x = y_1 = \ldots = y_m$ in $(\mathbb{R}^n)^{m+1}$, is called an $m$-linear Calderón-Zygmund kernel of type $\omega$ if, for some constants $0 < \tau < 1$, there exists a constant $A > 0$ such that*

$$|K(x, y_1, \ldots, y_m)| \leq \frac{A}{(|x - y_1| + \cdots + |x - y_m|)^{mn}}$$

*for all* $(x, y_1, \ldots, y_m) \in (\mathbb{R}^n)^{m+1}$ *with* $x \neq y_j$ *for some* $j \in \{1, 2, \ldots, m\}$*, and*

$$|K(x, y_1, \ldots, y_m) - K(x', y_1, \ldots, y_m)| \leq \frac{A}{(|x-y_1| + \cdots + |x-y_m|)^{mn}} \omega\Big(\frac{|x-x'|}{|x-y_1| + \cdots + |x-y_m|}\Big),$$

*whenever* $|x - x'| \leq \tau \max\limits_{1 \leq j \leq m} |x - y_j|$*, and*

$$|K(x, y_1, \ldots, y_j, \ldots, y_m) - K(x', y_1, \ldots, y_j', \ldots, y_m)| \leq \frac{A}{(|x-y_1| + \cdots + |x-y_m|)^{mn}} \omega\Big(\frac{|y_j - y_j'|}{|x-y_1| + \cdots + |x-y_m|}\Big),$$

*whenever* $|y_j - y_j'| \leq \tau \max\limits_{1 \leq i \leq m} |x - y_i|$.

*We say* $T : \mathcal{S}(\mathbb{R}^n) \times \ldots \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ *is an* $m$*-linear operator with an* $m$*-linear Calderón-Zygmund kernel of type* $\omega$*,* $K(x, y_1, \ldots, y_m)$*, if*

$$T(f_1, \ldots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) f_1(y_1) \ldots f_m(y_m) dy_1 \ldots dy_m,$$

*whenever* $x \notin \bigcap\limits_{j=1}^{m} supp f_j$ *and each* $f_j \in C_c^\infty(\mathbb{R}^n)$*,* $j = 1, \ldots, m$*.*

If $T$ can be extended to a bounded multilinear operator from $L_{q_1}(\mathbb{R}^n) \times \cdots \times L_{q_m}(\mathbb{R}^n)$ to $L_{q,\infty}(\mathbb{R}^n)$ for some $1 \leq q_1, \ldots, q_m < \infty$, and $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, then $T$ is called an m-linear Calderón-Zygmund operator of type $w$, abbreviated to $m$-linear $w$-CZO.

When $\omega(t) = t^\varepsilon$ for some $\varepsilon > 0$, the $m$-linear $\omega$-CZO is exactly the multilinear Calderón-Zygmund operator studied in [6] and [16]. The linear $\omega$-CZO was studied by Yabuta [25].

**Definition 2.2** *Let* $\omega(t) : [0, \infty) \to [0, \infty)$ *be a nondecreasing function. For* $a > 0$*, we say that* $\omega$ *satisfies the* $Dini(a)$ *condition and write* $\omega \in Dini(a)$*, if*

$$|\omega|_{Dini(a)} := \int_0^1 \frac{\omega^a(t)}{t} dt < \infty.$$

We would like to note that Maldonado and Naibo [19] studied the bilinear $\omega$- CZOs when $\omega$ is a nondecreasing, concave function and belongs to $Dini(1/2)$. Recently, Lu and Zhang [18] improve and extend their results by removing the hypothesis that $\omega$ is concave and reducing the condition $w \in Dini(1/2)$ to a weaker condition $w \in Dini(1)$.

**Theorem 2.1** *[18] Let* $w \in Dini(1)$ *and* $T$ *be an* $m$*-linear operator with an* $m$*-linear Calderón-Zygmund kernel of type* $\omega$*. Suppose that for some* $1 \leq q_1, \ldots, q_m \leq \infty$*, and some* $0 < q < \infty$ *with* $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$*,* $T$ *maps* $L_{q_1}(\mathbb{R}^n) \times \cdots \times L_{q_m}(\mathbb{R}^n)$ *into* $L_{q,\infty}(\mathbb{R}^n)$*. Then* $T$ *can be extended to a bounded operator from* $L_1(\mathbb{R}^n) \times \cdots \times L_1(\mathbb{R}^n)$ *to* $L_{\frac{1}{m},\infty}(\mathbb{R}^n)$*.*

**Remark 2.1** Perez et al. [23] proved the same results in Theorems 1.3 and 1.4 in case $\omega(t) = t^\varepsilon$ for some $\varepsilon > 0$. We also note that similar results for $T_{\Sigma \mathbf{b}}$ were proved in [18] for commutators of the linear Calderón-Zygmund operator of type $\omega$, see [17].

The following result holds.

**Theorem 2.2** *[18] Let $T$ be an $m$-linear $w-CZO$ and $w \in \text{Dini}(1)$. Let also $p_1, \dots, p_m \in [1,\infty)$, $p \in (0,\infty)$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$.*

(1) *If $1 < p_j < \infty$ for all $j = 1, \dots, m$, then there exists a constant $C > 0$ such that*

$$\|T(\mathbf{f})\|_{L_p} \le C \prod_{j=1}^{m} \|f_j\|_{L_{p_j}}.$$

(2) *If $1 \le p_j < \infty$ for all $j = 1, \dots, m$, and at least one of the $p_j = 1$, then there exists a constant $C > 0$ such that*

$$\|T(\mathbf{f})\|_{WL_p} \le C \prod_{j=1}^{m} \|f_j\|_{L_{p_j}}.$$

In [8] and [10] the first author defined the generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$, the generalized local Morrey spaces $M_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ and their weak versions as follows.

**Definition 2.3** *Let $1 \le p < \infty$ and $\varphi$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$. We denote by $M_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ and $M_{p,\varphi}(\mathbb{R}^n)$ the generalized local Morrey space, the generalized Morrey space respectively, the spaces of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite norms*

$$\|f\|_{M_{p,\varphi}^{\{x_0\}}} = \sup_{r>0} \varphi(x_0,r)^{-1} \, |B(x_0,r)|^{-\frac{1}{p}} \, \|f\|_{L_p(B(x_0,r))},$$

$$\|f\|_{M_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, |B(x,r)|^{-\frac{1}{p}} \, \|f\|_{L_p(B(x,r))},$$

*respectively.*

*Furthermore, by $WM_{p,\varphi}(\mathbb{R}^n)^{\{x_0\}}$ and $WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space and weak generalized Morrey space respectively of all functions $f \in WL_p^{\rm loc}(\mathbb{R}^n)$ for which,*

$$\|f\|_{WM_{p,\varphi}^{\{x_0\}}} = \sup_{r>0} \varphi(x_0,r)^{-1} \, |B(x_0,r)|^{-\frac{1}{p}} \, \|f\|_{WL_p(B(x_0,r))} < \infty,$$

$$\|f\|_{WM_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, |B(x,r)|^{-\frac{1}{p}} \, \|f\|_{WL_p(B(x,r))} < \infty,$$

*respectively.*

**Remark 2.2** (1) If $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$ with $0 < \lambda < n$, then $M_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n) = L_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n)$ is the local Morrey space, $M_{p,\varphi}(\mathbb{R}^n) = L_{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey space and $WM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n) = WL_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n)$ is the weak local Morrey space, $WM_{p,\varphi}(\mathbb{R}^n) = WL_{p,\lambda}(\mathbb{R}^n)$ is the weak Morrey space.

(2) If $\varphi(x,r) \equiv |B(x,r)|^{-\frac{1}{p}}$, then $M_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n) = M_{p,\varphi}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ is the Lebesgue space and $WM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n) = WM_{p,\varphi}(\mathbb{R}^n) = WL_p(\mathbb{R}^n)$ is the weak Lebesgue space.

Let us recall the definition and some properties of $BMO$. A locally integrable function $b$ belongs to $BMO$ if

$$\|b\|_* := \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|} \int_B |b(x) - b_B| dx < \infty, \tag{2.1}$$

where $b_B = |B|^{-1} \int_B b(y) dy$.

**Lemma 2.1** *[9] Let $b \in BMO$, $1 \le p < \infty$, and $r_1, r_2 > 0$. Then,*

$$\Big(\frac{1}{|B(x_0,r_1)|}\int_{B(x_0,r_1)}|b(y)-b_{B(x_0,r_2)}|^p dy\Big)^{1/p} \le C\|b\|_*\Big(1+\Big|\ln\frac{r_1}{r_2}\Big|\Big), \tag{2.2}$$

*where $C > 0$ is independent of $f$, $x_0$, $r_1$, and $r_2$.*

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w g(t) := \int_t^\infty g(s)\, w(s)\, ds,\ H_w^* g(t) := \int_t^\infty \Big(1+\ln\frac{s}{t}\Big)^m g(s)\, w(s)\, ds,\ t>0,$$

where $w$ is a weight. The following theorem was proved in [9].

**Theorem 2.3** *[9] Let $v_1$, $v_2$ and $w$ be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t)\, H_w g(t) \le C \sup_{t>0} v_1(t)\, g(t)$$

*holds for some $C > 0$ for all non-negative and non-decreasing $g$ on $(0,\infty)$ if and only if*

$$B := \sup_{t>0} v_2(t)\int_t^\infty \frac{w(s)\, ds}{\sup_{s<\tau<\infty} v_1(\tau)} < \infty.$$

**Theorem 2.4** *[9] Let $v_1$, $v_2$ and $w$ be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t)\, H_w^* g(t) \le C \sup_{t>0} v_1(t)\, g(t)$$

*holds for some $C > 0$, for all non-negative and non-decreasing $g$ on $(0,\infty)$ if and only if*

$$B := \sup_{t>0} v_2(t)\int_t^\infty \Big(1+\ln\frac{s}{t}\Big)^m \frac{w(s)\, ds}{\sup_{s<\tau<\infty} v_1(\tau)} < \infty.$$

## 3 Proof of Main Results

We first prove a theorem below, such that we prove Theorem 1.1 with the help of this theorem.

**Theorem 3.1** *Let $m \ge 2$, $x_0 \in \mathbb{R}^n$, $T_m$ be an $m$-linear $w-CZO$, and $w \in Dini(1)$. If $p_1,\dots,p_m \in (1,\infty)$ and $p \in (0,\infty)$ with $1/p = \sum_{i=1}^m 1/p_i$, then the inequality*

$$\|T_m(\overrightarrow{f})\|_{L_p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{2r}^\infty t^{-\frac{n}{p}} \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t} \tag{3.1}$$

*holds for any balls $B(x_0,r)$, and for all $\overrightarrow{f} \in L_{p_1}^{loc}(\mathbb{R}^n)\times\cdots\times L_{p_m}^{loc}(\mathbb{R}^n)$.*

*If $p_1,\dots,p_m \in [1,\infty)$, $\min\{p_1,\dots,p_m\} = 1$, and $p \in (0,\infty)$ with $1/p = \sum_{i=1}^m 1/p_i$, then the inequality*

$$\|T(\mathbf{f})\|_{WL_p,B(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{2r}^\infty t^{-\frac{n}{p}} \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t} \tag{3.2}$$

*holds for any balls $B(x_0,r)$ and for all $\overrightarrow{f} \in L_{p_1}^{loc}(\mathbb{R}^n)\times\cdots\times L_{p_m}^{loc}(\mathbb{R}^n)$.*

**Proof.** Let $(p_1,\ldots,p_m) \in (1,\infty)^m$ and $p \in (0,\infty)$ with $1/p = \sum_{i=1}^m 1/p_i$. For the moment, we denote the multilinear singular integral operator on $L_{p_1}(\mathbb{R}^n)\times\cdots\times L_{p_m}(\mathbb{R}^n)$ by $T_0$ to avoid confusion.

For arbitrary $x \in \mathbb{R}^n$, set $B = B(x_0,r)$ for the ball centered at $x$ with a radius $r$, $2B = B(x_0,2r)$. We represent $\overrightarrow{f} = (f_1,\ldots,f_m)$ as

$$f_j = f_j^0 + f_j^\infty, \quad f_j^0 = f_j\chi_{2B}, \quad f_j^\infty = f_j\chi_{\complement_{(2B)}}, \quad j = 1,\ldots,m. \tag{3.3}$$

Then we write

$$\begin{aligned}\prod_{i=1}^m f_i(y_i) = \prod_{i=1}^m \big(f_i^0(y_i) + f_i^\infty(y_i)\big) &= \sum_{\beta_1,\ldots,\beta_m\in\{0,\infty\}} f_1^{\beta_1}(y_1)\ldots f_m^{\beta_m}(y_m)\\ &= \prod_{i=1}^m f_i^0(y_i) + \sum_{\beta_1,\ldots,\beta_m}{}' f_1^{\beta_1}(y_1)\ldots f_m^{\beta_m}(y_m),\end{aligned}$$

where each term in $\sum'$ contains at least one $\beta_i \neq 0$.

For all $\overrightarrow{f} \in L^{loc}_{p_1}(\mathbb{R}^n)\times\cdots\times L^{loc}_{p_m}(\mathbb{R}^n)$, we define

$$T(\overrightarrow{f})(x) := T_0(f_1^0,\ldots,f_m^0)(x) + \sum_{\beta_1,\ldots,\beta_m}{}' T(f_1^{\beta_1},\ldots,f_m^{\beta_m})(x), \tag{3.4}$$

where $\beta_1,\ldots,\beta_m \in \{0,\infty\}$ and each term in $\sum'$ contains at least one $\beta_i \neq 0$.

First we show that $T(\overrightarrow{f})(x)$ is well-defined $a.e.\ x$ and independent of the choice $B$ containing $x$. As $T_0$ is bounded on $L_{p_1}(\mathbb{R}^n)\times\cdots\times L_{p_m}(\mathbb{R}^n)$ provided by Theorem 2.1 and $(f_1^0,\ldots,f_m^0) \in L_{p_1}(\mathbb{R}^n)\times\cdots\times L_{p_m}(\mathbb{R}^n)$, $T_0(f_1^0,\ldots,f_m^0)$ is well-defined. Next, we show that the second-term of the right-hand side defining $T(\overrightarrow{f})(x)$ converges absolutely for any $\overrightarrow{f} \in L^{loc}_{p_1}(\mathbb{R}^n)\times\cdots\times L^{loc}_{p_m}(\mathbb{R}^n)$ and almost every $x \in \mathbb{R}^n$.

For the other terms, let us first deal with the case when $\beta_1 = \cdots = \beta_m = \infty$.

When $x \in B$, $y_i \in \complement(2B)$, we have $\frac{1}{2}|x_0-y_i| \le |x-y_i| \le \frac{3}{2}|x_0-y_i|$, and therefore,

$$\begin{aligned}\sum_{\beta_1,\ldots,\beta_m}{}' \Big|T(f_1^{\beta_1},\ldots,f_m^{\beta_m})(x)\Big| &\lesssim \int_{(\complement_{(2B)})^m} \frac{|f_1(y_1)\cdots f_m(y_m)|}{|(x_0-y_1,\ldots,x_0-y_m)|^{mn}} d\overrightarrow{y}\\ &\lesssim \int_{(\complement_{(2B)})^m} \prod_{i=1}^m \frac{|f_i(y_i)|}{|x_0-y_i|^n} dy_i.\end{aligned}$$

By Fubini's theorem we have

$$\begin{aligned}&\int_{(\complement_{(2B)})^m} \prod_{i=1}^m \frac{|f_i(y_i)|}{|x_0-y_i|^{mn}} dy_i \approx \int_{(\complement_{(2B)})^m} \prod_{i=1}^m |f_i(y_i)| \int_{|x_0-y_i|}^\infty \frac{dt}{t^{n+1}} dy_i\\ &\approx \int_{2r}^\infty \prod_{i=1}^m \int_{2r\le|x_0-y_i|<t} |f_i(y_i)| dy_i \frac{dt}{t^{n+1}} \lesssim \int_{2r}^\infty \prod_{i=1}^m \int_{B(x_0,t)} |f_i(y_i)| dy_i \frac{dt}{t^{n+1}}.\end{aligned}$$

Applying Hölder's inequality, we get

$$\int_{\left(^{\complement}(2B)\right)^m} \prod_{i=1}^{m} \frac{|f_i(y_i)|}{|x_0 - y_i|^n} dy_i \lesssim \int_{2r}^{\infty} \prod_{i=1}^{m} \|f_i\|_{L_{p_i}(B(x_0,t))} \|1\|_{L_{p'_i}(B(x_0,t))} \frac{dt}{t^{n+1}}$$
$$\approx \int_{2r}^{\infty} t^{-\frac{n}{p}} \prod_{i=1}^{m} \|f_i\|_{L_{p_i}(B(x_0,t))} |\frac{dt}{t}. \tag{3.5}$$

We now consider the cases when exactly $l$ of the $\beta_l$'s are $\infty$ for some $1 \le l < m$. We only give the arguments for one of these cases. The rest are similar and can easily be obtained from the arguments below by permuting the indices. To this end we may assume that $\beta_1 = \ldots = \beta_l = \infty$ and $\beta_{l+1} = \ldots = \beta_m = 0$. Recall the fact that $|x_0 - y_i| \approx |x - y_i|$ for $x \in B$, $y_i \in {}^{\complement}(2B)$ and $1 \le i \le l$. We have

$$\begin{aligned}
&\left|T(f_1^\infty, \ldots, f_l^\infty, f_{l+1}^0, \ldots, f_m^0)(x)\right| \\
&\lesssim \int_{\left(^{\complement}(2B)\right)^l} \int_{\left(2B\right)^{m-l}} \frac{|f_1(y_1)\cdots f_m(y_m)|\, d\overrightarrow{y}}{(|x_0 - y_1| + \cdots + |x_0 - y_l|)^{mn}} \\
&\lesssim r^{-(m-l)n} \int_{\left(^{\complement}(2B)\right)^l} \frac{|f_1(y_1)\cdots f_l(y_l)|\, dy_1 \ldots dy_l}{(|x_0 - y_1| + \cdots + |x_0 - y_l|)^{ln}} \\
&\times \int_{\left(2B\right)^{m-l}} |f_1(y_{l+1})\cdots f_m(y_m)|\, dy_{l+1} \ldots dy_m \\
&\lesssim \int_{\left(^{\complement}(2B)\right)^l} \prod_{i=1}^{l} \frac{|f_i(y_i)|}{|x_0 - y_i|^n} dy_i \; r^{-n(m-l)} \int_{\left(2B\right)^{m-l}} \prod_{i=l+1}^{m} |f_i(y_i)| dy_i.
\end{aligned}$$

Applying Hölder's inequality, we get

$$\int_{\left(^{\complement}(2B)\right)^l} \prod_{i=1}^{l} \frac{|f_i(y_i)|}{|x_0 - y_i|^n} dy_i \lesssim \int_{2r}^{\infty} \prod_{i=1}^{l} \|f_i\|_{L_{p_i}(B(x_0,t))} \|1\|_{L_{p'_i}(B(x_0,t))} \frac{dt}{t^{n+1}}$$
$$\le \int_{2r}^{\infty} t^{-\sum\limits_{i=1}^{l} \frac{n}{p_i}} \prod_{i=1}^{l} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t} \tag{3.6}$$

and

$$r^{-n(m-l)} \int_{\left(2B\right)^{m-l}} \prod_{i=l+1}^{m} |f_i(y_i)| dy_i \lesssim \prod_{i=l+1}^{m} \|f_i\|_{L_{p_i}(2B)} \|1\|_{L^{p'_i}(2B)} \int_{2r}^{\infty} \frac{dt}{t^{n+1}}$$
$$\le \int_{2r}^{\infty} \prod_{i=l+1}^{m} \|f\|_{L_{p_i}(B(x_0,t))} \|1\|_{L_{p'_i}(B(x_0,t))} \frac{dt}{t^{n+1}}$$
$$\le \int_{2r}^{\infty} t^{-\sum\limits_{i=l+1}^{n} \frac{n}{p_i}} \prod_{i=l+1}^{m} \|f\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t}. \tag{3.7}$$

Therefore, from (3.5), (3.6) and (3.7) we get second-term of the right-hand side $\sum\limits_{\beta_1,\ldots,\beta_m}' T(f_1^{\beta_1}, \ldots, f_m^{\beta_m})(x)$ converges absolutely for any $\overrightarrow{f} \in L_{p_1}^{\mathrm{loc}}(\mathbb{R}^n) \times \cdots \times L_{p_m}^{\mathrm{loc}}(\mathbb{R}^n)$, and almost every $x \in \mathbb{R}^n$, and therefore we get the right-hand side of (3.4) is finite.

We will show that the definition is independent of the choice of $B$. Let $\mathcal{B} = \{B(x,r) : x \in \mathbb{R}^n,\ r > 0\}$. For simplicity, let's assume that $m = 2$. That is, if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then

$$\begin{aligned} & T_0(f_1\chi_{2B_1}, f_2\chi_{2B_1})(x) + T(f_1\chi_{2B_1}, f_2\chi_{\complement_{(2B_1)}})(x) \\ & + T(f_1\chi_{\complement_{(2B_1)}}, f_2\chi_{2B_1})(x) + T(f_1\chi_{\complement_{(2B_1)}}, f_2\chi_{\complement_{(2B_1)}})(x) \\ & = T_0(f_1\chi_{2B_2}, f_2\chi_{2B_2})(x) + T(f_1\chi_{2B_2}, f_2\chi_{\complement_{(2B_2)}})(x) \\ & + T(f_1\chi_{\complement_{(2B_2)}}, f_2\chi_{2B_2})(x) + T(f_1\chi_{\complement_{(2B_2)}}, f_2\chi_{\complement_{(2B_2)}})(x). \end{aligned} \tag{3.8}$$

Actually, let $B_3 \in \mathcal{B}$ be selected so that $2B_1 \cup 2B_2 \subset B_3$.

Since $f_1\chi_{2B_1}, f_1\chi_{B_3\backslash 2B_1} \in L_{p_1,w_1}(\mathbb{R}^n)$, $f_2\chi_{2B_1}, f_2\chi_{B_3\backslash 2B_1} \in L_{p_2,w_2}(\mathbb{R}^n)$, the linearity of $T_0$ on $L_{p_1,w_1}(\mathbb{R}^n) \times L_{p_2,w_2}(\mathbb{R}^n)$ yields

$$\begin{aligned} & T_0(f_1\chi_{2B_1}, f_2\chi_{2B_1})(x) + T(f_1\chi_{2B_1}, f_2\chi_{\complement_{(2B_1)}})(x) + T(f_1\chi_{\complement_{(2B_1)}}, f_2\chi_{2B_1})(x) \\ & + T(f_1\chi_{\complement_{(2B_1)}}, f_2\chi_{\complement_{(2B_1)}})(x) = T_0(f_1\chi_{2B_1}, f_2\chi_{2B_1})(x) + T(f_1\chi_{2B_1}, f_2\chi_{B_3\backslash 2B_1})(x) \\ & + T(f_1\chi_{2B_1}, f_2\chi_{\complement_{B_3}})(x) + T(f_1\chi_{B_3\backslash 2B_1}, f_2\chi_{2B_1})(x) + T(f_1\chi_{\complement_{B_3}}, f_2\chi_{2B_1})(x) \\ & + T(f_1\chi_{B_3\backslash 2B_1}, f_2\chi_{B_3\backslash 2B_1})(x) + T(f_1\chi_{\complement_{B_3}}, f_2\chi_{B_3\backslash 2B_1})(x) \\ & + T(f_1\chi_{B_3\backslash 2B_1}, f_2\chi_{\complement_{B_3}})(x) + T(f_1\chi_{\complement_{B_3}}, f_2\chi_{\complement_{B_3}})(x) \\ & = T_0(f_1\chi_{2B_1}, f_2\chi_{2B_1})(x) + T_0(f_1\chi_{2B_1}, f_2\chi_{B_3\backslash 2B_1})(x) \\ & + T(f_1\chi_{2B_1}, f_2\chi_{\complement_{B_3}})(x) + T_0(f_1\chi_{B_3\backslash 2B_1}, f_2\chi_{2B_1})(x) + T(f_1\chi_{\complement_{B_3}}, f_2\chi_{2B_1})(x) \\ & + T_0(f_1\chi_{B_3\backslash 2B_1}, f_2\chi_{B_3\backslash 2B_1})(x) + T(f_1\chi_{\complement_{B_3}}, f_2\chi_{B_3\backslash 2B_1})(x) \\ & + T(f_1\chi_{B_3\backslash 2B_1}, f_2\chi_{\complement_{B_3}})(x) + T(f_1\chi_{\complement_{B_3}}, f_2\chi_{\complement_{B_3}})(x) \\ & = T_0(f_1\chi_{B_3}, f_2\chi_{B_3})(x) + T(f_1\chi_{B_3}, f_2\chi_{\complement_{B_3}})(x) \\ & + T(f_1\chi_{\complement_{B_3}}, f_2\chi_{B_3})(x) + T(f_1\chi_{\complement_{B_3}}, f_2\chi_{\complement_{B_3}})(x). \end{aligned} \tag{3.9}$$

Similarly, we also get

$$\begin{aligned} & T_0(f_1\chi_{2B_2}, f_2\chi_{2B_2})(x) + T(f_1\chi_{2B_2}, f_2\chi_{\complement_{(2B_2)}})(x) + T(f_1\chi_{\complement_{(2B_2)}}, f_2\chi_{2B_2})(x) \\ & + T(f_1\chi_{\complement_{(2B_2)}}, f_2\chi_{\complement_{(2B_2)}})(x) = T_0(f_1\chi_{2B_2}, f_2\chi_{2B_2})(x) + T(f_1\chi_{2B_2}, f_2\chi_{B_3\backslash 2B_2})(x) \\ & + T(f_1\chi_{2B_2}, f_2\chi_{\complement_{B_3}})(x) + T(f_1\chi_{B_3\backslash 2B_2}, f_2\chi_{2B_2})(x) + T(f_1\chi_{\complement_{B_3}}, f_2\chi_{2B_2})(x) \\ & + T(f_1\chi_{B_3\backslash 2B_2}, f_2\chi_{B_3\backslash 2B_2})(x) + T(f_1\chi_{\complement_{B_3}}, f_2\chi_{B_3\backslash 2B_2})(x) \\ & + T(f_1\chi_{B_3\backslash 2B_2}, f_2\chi_{\complement_{B_3}})(x) + T(f_1\chi_{\complement_{B_3}}, f_2\chi_{\complement_{B_3}})(x) \\ & = T_0(f_1\chi_{2B_2}, f_2\chi_{2B_2})(x) + T_0(f_1\chi_{2B_2}, f_2\chi_{B_3\backslash 2B_2})(x) \\ & + T(f_1\chi_{2B_2}, f_2\chi_{\complement_{B_3}})(x) + T_0(f_1\chi_{B_3\backslash 2B_2}, f_2\chi_{2B_2})(x) + T(f_1\chi_{\complement_{B_3}}, f_2\chi_{2B_2})(x) \\ & + T_0(f_1\chi_{B_3\backslash 2B_2}, f_2\chi_{B_3\backslash 2B_2})(x) + T(f_1\chi_{\complement_{B_3}}, f_2\chi_{B_3\backslash 2B_2})(x) \\ & + T(f_1\chi_{B_3\backslash 2B_2}, f_2\chi_{\complement_{B_3}})(x) + T(f_1\chi_{\complement_{B_3}}, f_2\chi_{\complement_{B_3}})(x) \\ & = T_0(f_1\chi_{B_3}, f_2\chi_{B_3})(x) + T(f_1\chi_{B_3}, f_2\chi_{\complement_{B_3}})(x) \\ & + T(f_1\chi_{\complement_{B_3}}, f_2\chi_{B_3})(x) + T(f_1\chi_{\complement_{B_3}}, f_2\chi_{\complement_{B_3}})(x). \end{aligned} \tag{3.10}$$

Since $T_m$ is an $m$-linear operator, we split $T_m(\overrightarrow{f})$ as follows:

$$\left|T_m(\overrightarrow{f})(y)\right| \le \left|T_m(f_1^0,\dots,f_m^0)(y)\right| + \Big|\sum_{\beta_1,\dots,\beta_m}{}' T_m(f_1^{\beta_1},\dots,f_m^{\beta_m})(y)\Big|,$$

where $\beta_1,\dots,\beta_m \in \{0,\infty\}$ and each term in $\sum'$ contains at least one $\beta_i \neq 0$. Then,

$$\begin{aligned}\|T_m(\overrightarrow{f})\|_{L_p(B(x,r))} &\le \|T_m(f_1^0,\dots,f_m^0)\|_{L_p(B(x,r))}\\ &\quad + \|\sum_{\beta_1,\dots,\beta_m}{}' T_m(f_1^{\beta_1},\dots,f_m^{\beta_m})\|_{L_p(B(x,r))},\end{aligned}$$

and

$$\begin{aligned}\|T_m(\overrightarrow{f})\|_{WL_p(B(x,r))} &\le \|T_m(f_1^0,\dots,f_m^0)\|_{WL_p(B(x,r))}\\ &\quad + \|\sum_{\beta_1,\dots,\beta_m}{}' T_m(f_1^{\beta_1},\dots,f_m^{\beta_m})\|_{WL_p(B(x,r))}.\end{aligned}$$

Applying Theorem 2.1 we get for $p_1,\dots,p_m \in (1,\infty)$, and $p \in (0,\infty)$ with $1/p = \sum_{i=1}^m 1/p_i$

$$\|T_m(\overrightarrow{f^0})\|_{L_p(B(x,r))} \le \|T_m(\overrightarrow{f^0})\|_{L_p(\mathbb{R}^n)} \lesssim \prod_{i=1}^m \|f_i^0\|_{L_{p_i}(\mathbb{R}^n)} \lesssim \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x,2r))}$$

and for $p_1,\dots,p_m \in [1,\infty)$, $\min\{p_1,\dots,p_m\} = 1$, and $p \in (0,\infty)$ with $1/p = \sum_{i=1}^m 1/p_i$

$$\|T_m(\overrightarrow{f^0})\|_{WL_p(B(x,r))} \le \|T_m(\overrightarrow{f^0})\|_{WL_p(\mathbb{R}^n)} \lesssim \prod_{i=1}^m \|f_i^0\|_{L_{p_i}(\mathbb{R}^n)} \lesssim \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x,2r))}.$$

On the other hand, we have

$$\begin{aligned}\prod_{i=1}^m \|f_i\|_{L_{p_i}(2B)} &\approx r^{\frac{n}{p}} \prod_{i=1}^m \|f_i\|_{L_{p_i}(2B)} \int_{2r}^\infty \frac{dt}{t^{\frac{n}{p_i}+1}}\\ &\le r^{\frac{n}{p}} \int_{2r}^\infty t^{-\frac{n}{p}} \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x,t))} \frac{dt}{t}. \qquad (3.11)\end{aligned}$$

Thus

$$\|T_m(\overrightarrow{f^0})\|_{L_p(B(x,r))} \lesssim r^{\frac{n}{p}} \int_{2r}^\infty t^{-\frac{n}{p}} \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x,t))} \frac{dt}{t}. \qquad (3.12)$$

For the other terms, let us first deal with the case when $\beta_1 = \cdots = \beta_m = \infty$.

When $|x-y_i| \le r$, $|z-y_i| \ge 2r$, we have $\frac{1}{2}|z-y_i| \le |x-y_i| \le \frac{3}{2}|z-y_i|$, and therefore,

$$|T_m(f_1^\infty,\dots,f_m^\infty)(z)| \lesssim \int_{\left(\complement_{B(x,2r)}\right)^m} \frac{|f_1(y_1)\cdots f_m(y_m)|}{|(x-y_1,\dots,x-y_m)|^{mn}} dy_1\dots dy_m$$
$$\lesssim \int_{\left(\complement_{B(x,2r)}\right)^m} \prod_{i=1}^m \frac{|f_i(y_i)|}{|x-y_i|^n} dy_i$$

and

$$\|T_m(f_1^\infty,\dots,f_m^\infty)\|_{L_p(B(x,r))} \le \int_{\left(\complement_{B(x,2r)}\right)^m} \prod_{i=1}^m \frac{|f_i(y_i)|}{|x-y_i|^n} dy_i \, \|\chi_{B(x,r)}\|_{L_p}$$
$$\lesssim r^{\frac{n}{p}} \int_{\left(\complement_{B(x,2r)}\right)^m} \prod_{i=1}^m \frac{|f_i(y_i)|}{|x-y_i|^n} dy_i.$$

By Fubini's theorem we have

$$\int_{\left(\complement_{B(x,2r)}\right)^m} \prod_{i=1}^m \frac{|f_i(y_i)|}{|x-y_i|^n} dy_i \approx \int_{\left(\complement_{B(x,2r)}\right)^m} \prod_{i=1}^m |f_i(y_i)| \int_{|x-y_i|}^\infty \frac{dt}{t^{n+1}} dy_i$$
$$\approx \int_{2r}^\infty \prod_{i=1}^m \int_{2r\le|x-y_i|<t} |f_i(y_i)| dy_i \frac{dt}{t^{n+1}} \lesssim \int_{2r}^\infty \prod_{i=1}^m \int_{B(x,t)} |f_i(y_i)| dy_i \frac{dt}{t^{n+1}}.$$

Applying Hölder's inequality, we get

$$\int_{\left(\complement_{B(x,2r)}\right)^m} \prod_{i=1}^m \frac{|f_i(y_i)|}{|x-y_i|^n} dy_i \lesssim \int_{2r}^\infty \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x,t))} \|1\|_{L_{p_i'}(B(x,t))} \frac{dt}{t^{n+1}}$$
$$\le \int_{2r}^\infty t^{-\frac{n}{p}} \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x,t))} \frac{dt}{t}. \tag{3.13}$$

Moreover, for all $p_i \in [1,\infty)$, $i=1,\dots,m$, the inequality

$$\|T_m(f_1^\infty,\dots,f_m^\infty)\|_{L_p(B(x,r))} \lesssim r^{\frac{n}{p}} \int_{2r}^\infty t^{-\frac{n}{p}} \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x,t))} \frac{dt}{t} \tag{3.14}$$

is valid.

We now consider the cases when exactly $l$ of the $\beta_l$'s are $\infty$ for some $1 \le l < m$. We only give the arguments for one of these cases. The rest are similar and can easily be obtained from the arguments below by permuting the indices. To this end we may assume that $\beta_1 = \ldots = \beta_l = \infty$ and $\beta_{l+1} = \ldots = \beta_m = 0$. Recall the fact that $|x-y_i| \approx |z-y_i|$

for $z \in B(x,r)$, $y_i \in {}^{\complement}B(x,2r)$ and $1 \le i \le l$. We have

$$\begin{aligned}
&\left|T_m(f_1^\infty,\dots,f_l^\infty,f_{l+1}^0,\dots,f_m^0)(z)\right| \\
&\lesssim \int_{\left({}^{\complement}B(x,2r)\right)^l}\int_{\left(B(x,2r)\right)^{m-l}} \frac{|f_1(y_1)\cdots f_m(y_m)|\,dy_1\dots dy_m}{(|x-y_1|+\dots+|x-y_l|)^{mn}} \\
&\lesssim r^{-(m-l)n}\int_{\left({}^{\complement}B(x,2r)\right)^l} \frac{|f_1(y_1)\cdots f_l(y_m)|\,dy_1\dots dy_l}{(|x-y_1|+\dots+|x-y_l|)^{ln}} \\
&\times \int_{\left(B(x,2r)\right)^{m-l}} |f_1(y_{l+1})\cdots f_m(y_m)|\,dy_{l+1}\dots dy_m \\
&\lesssim \int_{\left({}^{\complement}B(x,2r)\right)^l}\prod_{i=1}^{l}\frac{|f_i(y_i)|}{|x-y_i|^n}dy_i\; r^{-n(m-l)}\int_{\left(B(x,2r)\right)^{m-l}}\prod_{i=l+1}^{m}|f_i(y_i)|dy_i.
\end{aligned}$$

Applying Hölder's inequality, we get

$$\begin{aligned}
&\int_{\left({}^{\complement}B(x,2r)\right)^l}\prod_{i=1}^{l}\frac{|f_i(y_i)|}{|x-y_i|^n}dy_i \lesssim \int_{2r}^{\infty}\prod_{i=1}^{l}\|f_i\|_{L_{p_i}(B(x,t))}\,\|1\|_{L_{p_i'}(B(x,t))}\,\frac{dt}{t^{n+1}} \\
&\le \int_{2r}^{\infty} t^{-\sum_{i=1}^{l}\frac{n}{p_i}}\prod_{i=1}^{l}\|f_i\|_{L_{p_i}(B(x,t))}\,\frac{dt}{t}
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
&r^{-n(m-l)}\int_{\left(B(x,2r)\right)^{m-l}}\prod_{i=l+1}^{m}|f_i(y_i)|dy_i \\
&\lesssim \prod_{i=l+1}^{m}\|f_i\|_{L_{p_i}(B(x,2r))}\,\|1\|_{L_{p_i'}(B(x,2r))}\int_{2r}^{\infty}\frac{dt}{t^{nl+1}} \\
&\le \int_{2r}^{\infty}\prod_{i=l+1}^{m}\|f\|_{L_{p_i}(B(x,t))}\,\|1\|_{L_{p_i'}(B(x,t))}\,\frac{dt}{t^{nl+1}} \\
&\le \int_{2r}^{\infty}\prod_{i=l+1}^{m}\|f\|_{L_{p_i}(B(x,t))}\,|B(x,t)|^{-\frac{1}{p_i}}\,\frac{dt}{t}.
\end{aligned} \tag{3.16}$$

From (3.15) and (3.16) we get

$$\begin{aligned}
&\|T_m(f_1^\infty,\dots,f_l^\infty,f_{l+1}^0,\dots,f_m^0)\|_{L_p(B(x,r))} \\
&\lesssim r^{\frac{n}{p}}\int_{\left({}^{\complement}B(x,2r)\right)^l}\prod_{i=1}^{l}\frac{|f_i(y_i)|}{|x-y_i|^n}dy_i\; r^{-n(m-l)}\int_{\left(B(x,2r)\right)^{m-l}}|f_i(y_i)|dy_i \\
&\lesssim r^{\frac{n}{p}}\int_{2r}^{\infty}t^{-\frac{n}{p}}\prod_{i=1}^{m}\|f\|_{L_{p_i}(B(x,t))}\,\frac{dt}{t}.
\end{aligned}$$

Thus we get the following inequality:

$$\Big\|\sum_{\beta_1,\dots,\beta_m}{}' T_m(f_1^{\beta_1},\dots,f_m^{\beta_m})\Big\|_{L_p(B(x,r))} \lesssim r^{\frac{n}{p}}\int_{2r}^{\infty}t^{-\frac{n}{p}}\prod_{i=1}^{m}\|f\|_{L_{p_i}(B(x,t))}\,\frac{dt}{t}.$$

Consequently, the inequality (3.1) is valid.

**Proof of Theorem 1.1**. For $(p_1,\dots,p_m)\in(1,\infty)^m$ from Theorem 2.3 and Theorem 3.1 (see, inequality (3.1)) we get

$$\|T(\mathbf{f})\|_{M^{\{x_0\}}_{p,\varphi_2}}\lesssim\sup_{r>0}\varphi_2(x_0,r)^{-1}\int_r^\infty t^{-\frac{n}{p}}\prod_{i=1}^m\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t}$$
$$\lesssim\sup_{r>0}t^{-\frac{n}{p}}\prod_{i=1}^m\varphi_{1i}(x,r)^{-1}\|f_i\|_{L_{p_i}(B(x_0,r))}\lesssim\prod_{i=1}^m\|f_i\|_{M^{\{x_0\}}_{p_i,\varphi_{1i}}}.$$

For $(p_1,\dots,p_m)\in[1,\infty)^m$, $\min\{p_1,\dots,p_m\}=1$ from Theorem 2.3 and Theorem 3.1 (see, inequality (3.2)) we get

$$\|T(\mathbf{f})\|_{WM^{\{x_0\}}_{p,\varphi_2}}\lesssim\sup_{r>0}\varphi_2(x_0,r)^{-1}\int_r^\infty t^{-\frac{n}{p}}\prod_{i=1}^m\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t}$$
$$\lesssim\sup_{r>0}t^{-\frac{nm}{p}}\prod_{i=1}^m\varphi_{1i}(x_0,r)^{-1}\|f_i\|_{L_{p_i}(B(x_0,r))}\lesssim\prod_{i=1}^m\|f_i\|_{M^{\{x_0\}}_{p_i,\varphi_{1i}}}.$$

This completes the proof of Theorem 1.1.

Now we prove a theorem below such that we prove Theorem 1.2 with the help of this theorem.

**Theorem 3.2** *Let $m\ge 2$, $x_0\in\mathbb{R}^n$, $T$ be an $m$-linear $\omega-CZO$, and $\omega$ satisfy* (1.5). *If $p_1,\dots,p_m\in(1,\infty)$ and $p\in(0,\infty)$ with $1/p=\sum\limits_{i=1}^m 1/p_i$, then the inequality*

$$\|T_{\Pi\mathbf{b}}(\mathbf{f})\|_{L_p(B(x_0,r))}\lesssim\|\mathbf{b}\|_*\,r^{\frac{n}{p}}\times\int_{2r}^\infty\Big(1+\ln\frac{t}{r}\Big)^m t^{-\frac{n}{p}}\prod_{i=1}^m\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t}\tag{3.17}$$

*holds for any balls $B(x_0,r)$ and for all $\overrightarrow{f}\in L^{loc}_{p_1}(\mathbb{R}^n)\times\cdots\times L^{loc}_{p_m}(\mathbb{R}^n)$.*

**Proof.** For simplicity of the expansion, we only present the case $m=2$.

We represent $f_i$ as $f_i=f_i^0+f_i^\infty$, where $f_i^0=f_i\chi_{B(x_0,2r)}$, $i=1,2$, and $\chi_{B(x_0,2r)}$ denotes the characteristic function of $B(x_0,2s)$. Then,

$$\|T_{\Pi\mathbf{b}}(\mathbf{f})\|_{L^p(B(x_0,r))}\le\Big(\int_{B(x_0,r)}|T_{\Pi\mathbf{b}}(f_1^0,f_2^0)(x)|^p dx\Big)^{1/p}$$
$$+\Big(\int_{B(x_0,r)}|T_{\Pi\mathbf{b}}(f_1^0,f_2^\infty)(x)|^p dx\Big)^{1/p}+\Big(\int_{B(x_0,r)}|T_{\Pi\mathbf{b}}(f_1^\infty,f_2^0)(x)|^p(x) dx\Big)^{1/p}$$
$$+\Big(\int_{B(x_0,r)}|T_{\Pi\mathbf{b}}(f_1^\infty,f_2^\infty)(x)|^p dx\Big)^{1/p}=I+II+III+IV.\tag{3.18}$$

Since $T_{\Pi\mathbf{b}}$ bounded from $L_{p_1}\times L_{p_2}$ to $L_p$, we get

$$\Big(\int_{B(x_0,r)}|T_{\Pi\mathbf{b}}(f_1^0,f_2^0)(x)|^p dx\Big)^{1/p}\lesssim\|\mathbf{b}\|_*\prod_{i=1}^m\|f_i\|_{L_{p_i}(B(x_0,2r))}.\tag{3.19}$$

Then, by (3.11), we get

$$I \lesssim \|\mathbf{b}\|_* r^{\frac{n}{p}} \int_{2r}^{\infty} t^{-\frac{n}{p}} \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t}. \tag{3.20}$$

Owing to the symmetry of II and III, we only estimate II. Taking $\lambda_i = (b_i)_{B(x_0,s)}$, then

$$\begin{aligned} &T_{\Pi\mathbf{b}}(f_1^0, f_2^\infty)(x) = (b_1(x)-\lambda_1)(b_2(x)-\lambda_2)T(f_1^0, f_2^\infty)(x) \\ &- (b_1(x)-\lambda_1)T(f_1^0,(b_2-\lambda_2)f_2^\infty)(x) - (b_2(x)-\lambda_2)T((b_1-\lambda_1)f_1^0, f_2^\infty)(x) \\ &+ T((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^\infty)(x) = II_1 + II_2 + II_3 + II_4. \end{aligned} \tag{3.21}$$

Similar to the estimate of (3.14), for any $x \in B(x_0, s)$, we can deduce

$$\sup_{x\in B(x_0,r)} |T(f_1^0, f_2^\infty)(x)| \lesssim \int_{2r}^{\infty} t^{-\frac{n}{p}} \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t}. \tag{3.22}$$

Applying Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned} &\Big(\int_{B(x_0,r)} |(b_1(x)-\lambda_1)(b_2(x)-\lambda_2)|^p dx\Big)^{1/p} \\ &\lesssim \prod_{i=1}^{2} \Big(\int_{B(x_0,r)} |b_i(x)-\lambda_i|^{2p} dx\Big)^{1/2p} \lesssim \|\mathbf{b}\|_*. \end{aligned} \tag{3.23}$$

Then, by (3.22) and (3.23) we have

$$\begin{aligned} &\Big(\int_{B(x_0,r)} |II_1|^p dx\Big)^{1/p} \le \Big(\int_{B(x_0,r)} |(b_1(x)-\lambda_1)(b_2(x)-\lambda_2)|^p dx\Big)^{1/p} \\ &\times \sup_{x\in B(x_0,r)} |T(f_1^0, f_2^\infty)(x)| \lesssim \|\mathbf{b}\|_* r^{\frac{n}{p}} \int_{2r}^{\infty} t^{-\frac{n}{p}} \prod_{i=1}^{2} \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t}. \end{aligned} \tag{3.24}$$

Applying the size condition, we deduce that, for any $x \in B(x_0, r)$,

$$\begin{aligned} &|T(f_1^0,(b_2-\lambda_2)f_2^\infty)(x)| \\ &\lesssim \int_{B(x_0,2r)} \int_{\mathbb{R}^n\setminus B(x_0,2r)} \frac{|f_1(y_1)(b_2(y_2)-\lambda_2)f_2(y_2)|}{(|x-y_1|+|x-y_2|)^{2n}} dy_1 dy_2 \\ &\lesssim \sum_{j=1}^{\infty} (2^{j+1}r)^{-2n} \int_{B(x_0,2^{j+1}r)} |f_1(y_1)| dy_1 \int_{B(x_0,2^{j+1}r)} |(b_2(y_2)-\lambda_2)f_2(y_2)| dy_2. \end{aligned} \tag{3.25}$$

Note that

$$\begin{aligned} &\int_{B(x_0,2^{j+1}r)} |f_1(y_1)| dy_1 \le C\|f_1\|_{L_{p_1}(B(x_0,2^{j+1}r))} \|1\|_{L_{p_1'}(B(x_0,2^{j+1}r))}, \\ &\int_{B(x_0,2^{j+1}r)} |(b_2(y_2)-\lambda_2)f_2(y_2)| dy_2 \\ &\le C\|f_2\|_{L_{p_2}(B(x_0,2^{j+1}r))} \|b_2(\cdot)-\lambda_2\|_{L_{p_2'}(B(x_0,2^{j+1}r))}. \end{aligned} \tag{3.26}$$

Then,

$$\sup_{x\in B(x_0,r)} |T(f_1^0,(b_2-\lambda_2)f_2^\infty)(x)| \le C\int_{2r}^\infty \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B(x_0,t))}$$
$$\times \|1\|_{L_{p'_1}(B(x_0,t))} \|b_2(\cdot)-\lambda_2\|_{L_{p'_2}(B(x_0,t))} \frac{dt}{t^{n+1}}. \tag{3.27}$$

From Lemma 2.1, we get

$$\|b_2(\cdot)-\lambda_2\|_{L_{p'_2}(B(x_0,r))} \le C\Big(\int_{B(x_0,r)} |b_2(z)-\lambda_2|^{p'_2}dz\Big)^{1/p'_2}$$
$$\le C\Big(1+\Big|\ln\frac{t}{r}\Big|\Big)\|b_2\|_* |B(x_0,r)|^{1/p'_2}. \tag{3.28}$$

Note that

$$\prod_{i=1}^2 \|1\|_{L_{p'_i}(B(x_0,r))} = |B(x_0,r)|^2 \prod_{i=1}^2 |B(x_0,r)|^{-\frac{1}{p_i}}. \tag{3.29}$$

From (3.27), (3.28), and (3.29), we can deduce

$$\sup_{x\in B(x_0,r)} |T(f_1^0,(b_2-\lambda_2)f_2^\infty)(x)|$$
$$\lesssim \|b_2\|_* \int_{2r}^\infty \Big(1+\ln\frac{t}{r}\Big) r^{-\frac{n}{p}} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t}. \tag{3.30}$$

Applying Lemma 2.1 and using Hölder's inequality, we have

$$\Big(\int\limits_{B(x_0,r)} |b_1(x)-\lambda_1|^p dx\Big) \lesssim \|b_1\|_* \prod_{i=1}^2 |B(x_0,r)|^{\frac{1}{p_i}}. \tag{3.31}$$

Then, by (3.30),

$$\Big(\int_{B(x_0,r)} |II_2|^p dx\Big)^{1/p} \le \Big(\int_{B(x_0,r)} |(b_1(x)-\lambda_1)|^p dx\Big)^{1/p}$$
$$\times \sup_{x\in B(x_0,r)} |T(f_1^0,(b_2-\lambda_2)f_2^\infty)(x)| \lesssim \prod_{i=1}^2 \|b_i\|_* |B(x_0,r)|^{\frac{1}{p_i}}$$
$$\times \int_{2r}^\infty \Big(1+\ln\frac{t}{r}\Big)\Big(\prod_{i=1}^2 \|f_i\|_{L_{p_i}(B(x_0,t))} |B(x_0,r)|^{-\frac{1}{p_i}}\Big)\frac{dt}{t} \tag{3.32}$$
$$\approx \|\mathbf{b}\|_* r^{\frac{n}{p}} \int_{2r}^\infty t^{-\frac{n}{p}} \prod_{i=1}^2 \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t}.$$

Similarly, we also have that

$$\Big(\int_{B(x_0,r)}|II_3|^p dx\Big)^{1/p} \le C\prod_{i=1}^{2}\Big(\|b_i\|_*\,|B(x_0,r)|^{\frac{1}{p_i}}\Big)$$
$$\times\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\Big(\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\,|B(x_0,t)|^{-\frac{1}{p_i}}\Big)\frac{dt}{t} \tag{3.33}$$
$$\approx \|\mathbf{b}\|_*\,r^{\frac{n}{p}}\int_{2r}^{\infty}t^{-\frac{n}{p}}\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t}.$$

Applying the size condition again, for any $x\in B(x_0,r)$, with the same method of estimate for (3.30), we have

$$|T((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^\infty)(x)| \lesssim \sum_{j=1}^{\infty}(2^{j+1}r)\prod_{i=1}^{2}\int_{B(x_0,2^{j+1}r)}|(b_i(y)-\lambda_i)f_i(y_i)|dy_i$$
$$\lesssim \int_{2r}^{\infty}\Big(\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\,\|b_i(\cdot)-\lambda_i\|_{L_{p_2'}(B(x_0,t))}\Big)\frac{dt}{t^{n+1}} \tag{3.34}$$
$$\lesssim \|\mathbf{b}\|_*\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)^2 t^{-\frac{n}{p}}\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t}.$$

Then,

$$\Big(\int_{B(x_0,r)}|II_4|^p dx\Big)^{1/p} \lesssim \sup_{x\in B(x_0,r)}|\big((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^\infty\big)(x)|$$
$$\lesssim \|\mathbf{b}\|_*\,r^{\frac{n}{p}}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)^2 t^{-\frac{n}{p}}\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t}. \tag{3.35}$$

Then, combining (3.24), (3.32), (3.33), and (3.35), we get

$$\Big(\int_{B(x_0,r)}|II|^p dx\Big)^{1/p} \lesssim \|\mathbf{b}\|_*\,r^{\frac{n}{p}} \tag{3.36}$$
$$\times\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)^2 t^{-\frac{n}{p}}\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t}.$$

Finally, we still decompose $T_{\Pi\mathbf{b}}(f_1^\infty,f_2^\infty)(x)$ as follows:

$$T_{\Pi\mathbf{b}}(f_1^\infty,f_2^\infty)(x) = (b_1(x)-\lambda_1)(b_2(x)-\lambda_2)T(f_1^\infty,f_2^\infty)(x)$$
$$-(b_1(x)-\lambda_1)T(f_1^\infty,(b_2-\lambda_2)f_2^\infty)(x)-(b_2(x)-\lambda_2)T((b_1-\lambda_1)f_1^\infty,f_2^\infty)(x)$$
$$+T((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^\infty)(x) = IV_1+IV_2+IV_3+IV_4. \tag{3.37}$$

Because each term $IV_j$ is completely analogous to $II_j$, $j=1,2,3,4$ with a bit difference, we get the following estimate without details:

$$\Big(\int_{B(x_0,r)}|IV|^p dx\Big)^{1/p} \lesssim \|\mathbf{b}\|_*\,r^{\frac{n}{p}} \tag{3.38}$$
$$\times\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)^2 t^{-\frac{n}{p}}\prod_{i=1}^{2}\|f_i\|_{L_{p_i}(B(x_0,t))}\frac{dt}{t}.$$

Summing up the above estimates, (3.17) is proved.

**Proof of Theorem 1.2**. For $(p_1, \ldots, p_m) \in (1, \infty)^m$ from Theorem 2.4 and Theorem 3.2 (see, inequality (3.17)) we get

$$\|T_{\Pi \mathbf{b}}(\mathbf{f})\|_{M^{\{x_0\}}_{p,\varphi_2}} \lesssim \sup_{r>0} \varphi_2(x_0, r)^{-1} \int_r^\infty \Big(1 + \ln \frac{t}{r}\Big)^m t^{-\frac{n}{p}} \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x_0,t))} \frac{dt}{t}$$
$$\lesssim \sup_{r>0} t^{-\frac{n}{p}} \prod_{i=1}^m \varphi_{1i}(x, r)^{-1} \|f_i\|_{L_{p_i}(B(x_0,r))} \lesssim \prod_{i=1}^m \|f_i\|_{M^{\{x_0\}}_{p_i,\varphi_{1i}}}.$$

This completes the proof of Theorem 1.2.

The proof of Theorem 1.3 can be done using the method in Theorem 1.2, with similar argument.

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