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Necessary conditions in delayed variational problems with free right end

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Abstract. *The results obtained in this paper can be obtained in a similar way for more complex problems, such as variation problems with a finite number of delays. It is also possible to prove the validity of similar results in a similar way for the problem of variation with both open ends.*

Keywords. Variational problem with a delayed argument · necessary conditions · weak local minimum · first and second variation of the functional.

Mathematics Subject Classification (2010): 35K20, 49J20

1 Introduction

The study of delayed argument processes are of great importance both from theoretical and practical point of view. A hundreds of papens and monographs in this direction were published (e.i. [2, 3, 6–8]).

The delayed variational problems were first studied by Kamensky. In his work [4] he has obtained the analogue of the Euler equation for weak local extremum in a delayed variational problem with closed ends.

In this paper we consider a delayed variational problem with closed left end and with free right end. Using the variational method, the first and second variations of the studied functional were determined. The analogues and transversality condition of the Euler equation, of the Legendre condition were obtained.

2 Problem statement

Let us consider the following variational problem:

$$J\left(x(\cdot)\right)=\int_{t_0}^{t_1} L\ \left(t,\, x(t),\, x(t-h),\, \dot{x}(t),\, \dot{x}(t-h)\right) dt+F(x(t_1))\rightarrow \min_{x(\cdot)}, \tag{2.1}$$

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$$x\,(t) = \varphi\,(t)\,,\ \ t \in [t_0 - h,\ t_0]. \tag{2.2}$$

Here t_0 and t_1 $R = (-\infty,\ +\infty)$ are the given points of the real axis, $x(\cdot) \in R^n$, R^n is an $n-$dimensional Euclidean space and $h = const > 0$, so that $t_1 - t_0 > h$, and the given function $\varphi(t),\ t \in [t_0 - h,\ t_0]$ is a second order continuously differentiable function.

In problem (2.1)-(2.2) $x(t) \in C^2\,([t_0 - h, t_1])$ is the desired function. The given function $L(t, x, y, \dot{x}, \dot{y})$ is said to be an integrant in problem (2.1), (2.2) and $L(\cdot)$ in a second order continuously differentiable function of its arguments.

Definition 2.1 *Every function $x(t)$ $t \in [t_0 - h, t_1]$ satistying the condition (2.2) and being an element of space $C^2\,([t_0 - h, t_1])$ is called an admissible function.*

3 Neceessary conditions

At first we introduce the notions of the first and second variations of the functional for problem (2.1), (2.2). Assume that $\bar{x}(\cdot)$ is an admissible function. We accept the following denotations:

$$\begin{array}{ll} y(t) = x(t-h), & z(t) = (x(t),\, y(t))^T, \\ \dot{z}(t) = (\dot{x}(t),\ \dot{y}(t))^T\,, & \overline{z}(t) = (\overline{x}(t),\ \overline{y}(t))^T. \end{array} \tag{3.1}$$

Here T indicates the transposition.

We use the following variation given first by Lagrange:

$$\tilde{x}\,(t) = \begin{cases} \bar{x}(t) + \varepsilon\delta x(t), & t \in [t_0,\ t_1], \\ \bar{x}(t), & t \in [t_0 - h,\ t_0], \end{cases} \tag{3.2}$$

here $\varepsilon \in R$ and $\delta x(t) \in C^2([t_0 - h, t_1])$, such that $\delta x(t) \equiv 0, t \in [t_0 - h,\ t_0]$.

Now using (3.1) and (3.2), we calculate the increment $J(\tilde{x}(\cdot)) - J(\bar{x}(\cdot)) =: \triangle_\varepsilon J(\bar{x}(\cdot))$ of the functional (2.1).

We can write the following:

$$\triangle_\varepsilon J(\bar{x}(\cdot)) = \int\limits_{t_0}^{t_1} \left[L\ (\,t,\ \bar{z}(t) + \varepsilon\delta z(t),\ \dot{\bar{z}}(t) + \varepsilon\,\delta\dot{z}(t)) - L(t, \bar{z}, \dot{\bar{z}}(t))\right] dt +$$

$$+ F(\bar{x}(t_1) + \varepsilon\delta x(t_1)) - F(\bar{x}(t_1)).$$

From the last one, by the Taylor formula we get:

$$\triangle_\varepsilon J(\bar{x}(\cdot)) = \varepsilon\delta J(\bar{x}(\cdot); \delta x(\cdot)) + \frac{1}{2}\varepsilon^2\delta^2 J(\bar{x}(\cdot); \delta x(\cdot)) + o(\varepsilon^2), \tag{3.3}$$

here $o(\varepsilon^2)/\varepsilon^2 \to 0$, if $\varepsilon \to 0$,

$$\delta J(\bar{x}(\cdot); \delta x(\cdot)) = \int\limits_{t_0}^{t_1} \left[\bar{L}_Z^T(t)\delta z(t) + \bar{L}_{\dot{Z}}^T(t)\delta\dot{z}(t)\right] dt + F_x^T(\bar{x}(t_1))\delta x(t_1), \tag{3.4}$$

$$\delta^2 J(\bar{x}(\cdot); \delta x(\cdot)) = \int\limits_{t_0}^{t_1} \left[\delta z^T(t)\bar{L}_{ZZ}(t)\delta z(t) + 2\delta z^T(t)\bar{L}_{Z\dot{Z}}(t)\delta\dot{z}(t)\ + \right.$$

$$\left. + \delta\dot{z}^T(t)\bar{L}_{\dot{Z}\dot{Z}}(t)\delta\dot{z}(t)\right] dt + \delta x^T(t_1)F_{xx}(\bar{x}(t_1))\delta x(t_1). \tag{3.5}$$

Note that the expressions $\bar{L}_Z(t),\ \bar{L}_{\dot{Z}}(t),\ \bar{L}_{ZZ}(t),\ \bar{L}_{Z\dot{Z}}(t)$ and $\bar{L}_{\dot{Z}\dot{Z}}(t)$ are calculated along $(t, \bar{z}(t), \dot{\bar{z}}(t))$.

Taking into account $z = (x, y)^T$, $\delta z(t) = (\delta x(t), \delta x(t-h))^T$ and $\delta \dot{z}(t) = (\delta \dot{x}(t), \delta \dot{x}(t-h))^T$ from (3.4), we can easily write the following:

$$\delta J(\bar{x}(\cdot); \delta x(\cdot)) = \int_{t_0}^{t_1} \left\{ \left[\bar{L}_x^T(t) + \bar{L}_y^T(t+h)\right] \delta x(t) + \right.$$
$$\left. + \left[\bar{L}_{\dot{x}}^T(t) + \bar{L}_{\dot{y}}^T(t+h)\right] \delta \dot{x}(t) \right\} dt + F_x^T(\bar{x}(t_1)) \delta x(t_1), \qquad (3.6)$$

here for $t > t_1$ $\bar{L}_y(t) = \bar{L}_{\dot{y}}(t) = 0$.

Definition 3.1 *The expression $\delta J(\bar{x}(\cdot); \delta x(\cdot))$ determined by (3.6), in the problem (2.1), (2.2) is called the first variation, the $\delta^2 J(\bar{x}(\cdot); \delta x(\cdot))$ determined by (3.5) is called the second variation.*

Using (3.3), we easily prove the following theorem.

Theorem 3.1 *Assume that the admisible function $\bar{x}(t)$ in problem (2.1), (2.2) is a weak local minimum. Then the followings are valid:*

$$\delta J(\bar{x}(\cdot); \delta x(\cdot)) = 0,\ \forall \delta x(\cdot) \in C^2([t_0 - h, t_1]), \qquad (3.7)$$

$$\delta^2 J(\bar{x}(\cdot); \delta x(\cdot)) \geq 0,\ \forall \delta x(\cdot) \in C^2([t_0 - h, t_1]). \qquad (3.8)$$

Note that the following theorem showing that the Euler equation, the analogues of the transverality conditions are obtained for problem (2.1), (2.2) by using the minimality conditions (3.7) and (3.8) and by means of certain yudgements (see [1], [5]) is valid.

Theorem 3.2 *Assume that the admissible function $\bar{x}(t),\ t \in [t_0 - h, t_1]$ is a weak local minimum in problem (2.1), (2.2). Then the following is valid:*

$$\begin{cases} \bar{L}_x(t) - \frac{d}{dt}\bar{L}_{\dot{x}}(t) = 0, & t \in [t_1 - h,\ t_1], \\ \bar{L}_x(t) + \bar{L}_y(t+h) - \frac{d}{dt}\left(\bar{L}_{\dot{x}}(t) + \bar{L}_{\dot{y}}(t+h)\right) = 0, & t \in [t_0,\ t_1 - h), \end{cases} \qquad (3.9)$$

$$\begin{cases} \xi^T \left[\bar{L}_{\dot{x}\dot{x}}(t) + \bar{L}_{\dot{y}\dot{y}}(t+h)\right] \xi \geq 0,\ \ \forall \xi \in R^n,\ \ \forall t \in [t_0,\ t_1], \\ \bar{L}_{\dot{y}\dot{y}}(t) = 0, \qquad t > t_1, \end{cases} \qquad (3.10)$$

$$\bar{L}_{\dot{x}}(t_1 - 0) + F_x(\bar{x}(t_1)) = 0. \qquad (3.11)$$

Note that the minimality condition (3.9) obtained here is called the analogous of the Euler equation, (3.10) of the Legendre condition, (3.11) of the transversality condition. The result similar to the condition (3.9) was obtained in [4, p. 391-398]) the result similar to (3.10) was obtained in [9].

Proof of Theorem 3.1. Since the admissible function $\bar{x}(t)$ is a weak local minimum in problem (2.1), (2.2), there exists wich a number $\varepsilon_0 \geq 0$ that the inequality $\triangle_\varepsilon J(\bar{x}(\cdot)) \geq 0$,

$\forall\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ is valid. Taking into account this inequality and $\forall\varepsilon \in (-\varepsilon_0, \varepsilon_0), \varepsilon \to 0$, we can write the following:

$$\delta J(\bar{x}(\cdot); \delta x(\cdot)) \geq 0 \;\; and \;\; \delta J(\bar{x}(\cdot); \delta x(\cdot)) \leq 0,$$
$$\forall \delta x(\cdot) \in C^2([t_0 - h, t_1]).$$

Thus, $\delta J(\bar{x}(\cdot); \delta x(\cdot)) = 0$, i.e. we prove the validity of equality (3.7).

We now show the validity of the inequality (3.8).

Taking into account (3.7) in (3.3), along the admisible function $\bar{x}(\cdot)$, i.e. for a weak local minimum we can write the following inequality:

$$\frac{1}{\varepsilon^2} \triangle_\varepsilon J(\bar{x}(\cdot)) = \frac{1}{2} \delta^2 J(\bar{x}(\cdot); \delta x(\cdot)) + \frac{o(\varepsilon^2)}{\varepsilon^2} \geq 0, \forall\varepsilon \in (-\varepsilon_0, \varepsilon_0).$$

Here, as $\varepsilon \to 0$ we obtain the proof of the inequality (3.8). So, Theorem 3.1 is proved.

Proof of Theorem 3.2. At first we prove (3.9) and (3.11). Taking into account the statement (3.7) of Theorem 3.1 and expression (3.6), we can write the following:

$$\delta J(\bar{x}(\cdot); \delta x(\cdot)) = \int_{t_0}^{t_1} \left\{ \left[\bar{L}_x^T(t) + \bar{L}_y^T(t+h) \right] \delta x(t) + \left[\bar{L}_{\dot{x}}^T(t) + \bar{L}_{\dot{y}}^T(t+h) \right] \delta \dot{x}(t) \right\} dt +$$
$$+ F_x^T \left(\bar{x}(t_1) \right) \delta x(t_1) = 0, \;\; \forall \delta x(t) \in C^2([t_0 - h, t_1]), \tag{3.12}$$

here

$$\bar{L}_y(t) = \bar{L}_{\dot{y}}(t) \equiv 0, \;\; \forall t \in (t_1, +\infty). \tag{3.13}$$

Applying the method of integration by parts, we can write equation (3.12) as follows:

$$\delta J(\bar{x}(\cdot); \delta x(\cdot)) = \int_{t_0}^{t_1} \left\{ \bar{L}_x^T(t) + \bar{L}_y^T(t+h) - \frac{d}{dt} \left(\bar{L}_{\dot{x}}^T(t) + \bar{L}_{\dot{y}}^T(t+h) \right) \right\} \delta x(t) dt +$$
$$+ \left(\bar{L}_{\dot{x}}^T(t) + \bar{L}_{\dot{y}}^T(t+h) \right) \delta x(t) \,\Big|_{t_0}^{t_1} + F_x^T(\bar{x}(t_1)) \delta x(t_1) = 0. \tag{3.14}$$

Here as variation $\delta x(\cdot)$ we accept the condition $\delta x(t_0) = \delta x(t_1) = 0$. Then we prove that by the Lagrange lemma (see, e.i. [1, p. 61]), (3.9) is valid at the points $t \in [t_0, \; t_1]$.

Furthermore, taking into account (3.9), and also for $\delta x(t_0) = 0$ and $t > t_1$ the equations $\bar{L}_y(t) = \bar{L}_{\dot{y}}(t) = 0$, from (3.14) we get:

$$\left[\bar{L}_{\dot{x}}^T(t_1 - 0) + F_x^T(\bar{x}(t_1)) \right] \delta x(t_1) = 0, \forall \delta x(t_1) \in R^n.$$

Here we get the proof of the equation (3.11).

We now prove the validity of statement (3.10) of Theorem 3.2. Assume that $\bar{x}(t)$ is a weak local minimum. Then, by the statement (3.8) of Theorem 3.1, taking into account $\dot{\bar{z}} = (\dot{\bar{x}}, \dot{\bar{y}})$ and (3.5), we consider new expression of $\delta^2 J(\bar{x}(\cdot); \delta x(\cdot))$:

$$\delta^2 J(\bar{x}(\cdot); \delta x(\cdot)) = \int_{t_0}^{t_1} \left[\delta z^T(t) \bar{L}_{ZZ}(t) \delta z(t) + 2 \delta z^T(t) \bar{L}_{Z\dot{Z}}(t) \delta \dot{z}(t) + \delta \dot{x}^T(t) \bar{L}_{\dot{x}\dot{x}}(t) \delta \dot{x}(t) + \right.$$
$$\left. + 2 \delta \dot{x}^T(t) \bar{L}_{\dot{x}\dot{y}}(t) \delta \dot{y}(t) + \delta \dot{y}^T(t) \bar{L}_{\dot{y}\dot{y}}(t) \delta \dot{y}(t) \right] dt + \delta x^T(t_1) F_{xx}(x(t_1)) \delta x(t_1) \geq 0, \tag{3.15}$$

$$\forall \delta z(t) = (\delta x(t), \delta y(t)) \in C^2([t_0 - h, t_1]).$$

Let us consider the following variation:

$$\delta x(t) = \begin{cases} \xi \sin^2 \pi \frac{t-\theta+\varepsilon}{2\varepsilon}, & t \in [\theta - \varepsilon,\ \theta + \varepsilon], \\ 0, & t \in [t_0,\ t_1] \backslash [\theta - \varepsilon,\ \theta + \varepsilon], \end{cases} \tag{3.16}$$

here $\theta \in (t_0,\ t_1)$, $\varepsilon > 0$, $\varepsilon < \frac{h}{2}$, $\xi \in R^n$, $[\theta - \varepsilon,\ \theta + \varepsilon] \subset [t_0, t_1]$.

It is clear that $\delta y(t) = 0, t \in [\theta - \varepsilon,\ \theta + \varepsilon]$, and also $\delta \dot{y}(t) = 0, t \in [\theta - \varepsilon,\ \theta + \varepsilon]$ and $\delta x(t_1) = 0$. Take these and (3.16) into account in (3.15). Then $\delta^2 J(\bar{x}(\cdot); \delta x(\cdot))$ is as follows:

$$\begin{aligned} \delta^2 J(\bar{x}(\cdot); \delta x(\cdot)) = \int\limits_{\theta}^{\theta+\varepsilon} \{ & \delta x^T(t) \left[\bar{L}_{xx}(t) + \bar{L}_{yy}(t+h) \right] \delta x(t) + \\ & +2 \left[\delta x^T(t) (\bar{L}_{x\dot{x}}(t) + \bar{L}_{y\dot{y}}(t+h)) \delta \dot{x}(t) \right] + \\ & + \delta \dot{x}^T(t) \left[\bar{L}_{\dot{x}\dot{x}}(t) + \bar{L}_{\dot{y}\dot{y}}(t+h) \right] \delta \dot{x}(t) \} \, dt \geq 0, \end{aligned} \tag{3.17}$$

here $\bar{L}_{yy}(t) = \bar{L}_{y\dot{y}}(t) = \bar{L}_{\dot{y}\dot{y}}(t) = 0$, $t > t_1$.

In the last inequality we take into account (3.16) and also

$$\delta \dot{x}(t) = \begin{cases} \xi \frac{\pi}{2\varepsilon} \sin 2\pi \frac{t-\theta+\varepsilon}{2\varepsilon}, & t \in [\theta - \varepsilon,\ \theta + \varepsilon], \\ 0 \in R^n, & t \in [t_0,\ t_1] \backslash [\theta - \varepsilon,\ \theta + \varepsilon], \end{cases} \tag{3.18}$$

and prove the validity of (3.10). Assume the contrary, i.e. exists such a point $\theta \in (t_0,\ t_1)$ that at the certain point $\xi \in R^n$

$$Q(t) \mid_{t=\theta} = \xi \left[\bar{L}_{\dot{x}\dot{x}}(\theta) + \bar{L}_{\dot{y}\dot{y}}(\theta + h) \right] \xi < -q, \ \ q > 0. \tag{3.19}$$

According to the feature of a continuous function, for a rather small $\varepsilon > 0$ the inequality (3.19) is valid for $\forall t \in (\theta - \varepsilon,\ \theta + \varepsilon)$ as well.

Taking into account the continuity of the integrand expression (3.6), (3.18), and also (3.19), we can write the following evaluation of the second variation of $\delta^2 J(\bar{x}(\cdot); \delta x(\cdot))$ in (3.17) as follows:

$$\delta^2 J(\bar{x}(\cdot); \delta x(\cdot)) < m_1 + m_2 \varepsilon - \frac{q\pi^2}{\varepsilon}, \tag{3.20}$$

here $m_1 > 0,\ m_2 > 0$ are certain numbers.

From (3.20) for a rather small $\varepsilon > 0$ we obtain $\delta^2 J(\bar{x}(\cdot); \delta x(\cdot)) < 0$. This contradicts (3.17). So, we prove that statement (3.10) of Theorem 3.2 is valid.

So, Theorem 3.2 was proved.

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