Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **42** (4), 102-111 (2022).

# On absolute and uniform convergence of a biorthogonal series in root functions of an odd order differential operator

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Received: 06.05.2022 / Revised: 21.09.2022 / Accepted: 31.10.2022

**Abstract.** *In the paper we study absolute and uniform convergence of biorthogonal expansion of the function from the class $W_2^1(G)$, $G=(0,1)$, in root functions of an odd order ordinary differential operator. Sufficient conditions for absolute and uniform convergence were established and estimation for the rate of uniform convergence of these expansions on the segment $\overline{G}=[0,1]$ was found.*

**Keywords.** absolute convergence, uniform convergence, root functions.

**Mathematics Subject Classification (2010):** 34L10, 42A20

## 1 Introduction and problem statement

On the interval $G=(0,1)$ we consider an odd order formal differential operator

$$L\,u=u^{(2m+1)}+P_2(x)\,u^{(2m-1)}+...+P_{2m+1}(x)\,u\,, \qquad m=1,2,...$$

with complex-valued coefficients $P_l(x)\in W_1^{2m+1-l}\,(G)\,,\quad l=\overline{2,\,2m+1}$.

We denote by$D\,(G)$ a class of functions absolutely continuous with their derivatives up to the $2m$ -th order inclusively, on $\overline{G}=[0,1]$.

Let us consider an arbitrary $\{u_k\,(x)\}\,_{k=1}^{\infty}$ consisting of eigen and associated (root) functions of the operator $L$, corresponding to the system of eigen-values $\{\lambda_k\}\,_{k=1}^{\infty}$ and require that together with each root function of order $r\geq 1$ this system includes the corresponding root functions of order less than $r$ and the rank of the eigen functions is uniformly bounded. This means that $u_k(x)\in D\,(G)$ and satisfies almost everywhere in $G$ the equation

$$L\,u_k+\lambda_k\,u_k=\theta_k\,u_{k-1}\ ,$$

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where $\theta_k$ either equals 0 (in this case $u_k(x)$ is an eigen-function), or 1 (in this case we require $\lambda_k = \lambda_{k-1}$ and call $u_k(x)$ an associated function), $\theta_1 = 0$ (see [3]).

We will require that the system of the root functions $\{u_k(x)\}_{k=1}^{\infty}$ and corresponding system of eigen-values $\{\lambda_k\}_{k=1}^{\infty}$ satisfy the conditions $A$:
1) the system $\{u_k(x)\}_{k=1}^{\infty}$ is complete and minimal in $L_2(G)$;
2) the Carleman and the "sum of units" conditions are fulfilled

$$|Im\,\mu_k| \leq const, \qquad k = 1, 2, ..., \tag{1.1}$$

$$\sum_{\tau \leq \rho_k \leq \tau+1} 1 \leq const, \quad \forall \tau \geq 0, \quad \rho_k = Re\,\mu_k\,, \tag{1.2}$$

where

$$\mu_k = \begin{cases} (-i\,\lambda_k)^{1/(2m+1)} & for \quad Im\,\lambda_k \geq 0\,, \\ (i\,\lambda_k)^{1/(2m+1)} & for \quad Im\,\lambda_k < 0\,, \end{cases}$$

$$\left(\rho\, e^{i\varphi}\right)^{1/(2m+1)} = \rho^{1/(2m+1)}\, e^{i\varphi/(2m+1)}\,, \qquad -\pi < \varphi \leq \pi.$$

3) the system $\{\upsilon_k(x)\}_{k=1}^{\infty}$, biorthogonally conjugated to $\{u_k(x)\}_{k=1}^{\infty}$, is the system of root functions of the formally adjoint operator

$$L^*\upsilon = (-1)^{2m+1}\,\upsilon^{(2m+1)} + (-1)^{2m-1}\left(\overline{P_2}\,\upsilon\right)^{(2m-1)} + ... + \overline{P_{2m+1}}\,\upsilon,$$

i.e.

$$L^*\upsilon_k + \overline{\lambda_k}\,\upsilon_k = \theta_{k+1}\,\upsilon_{k+1};$$

4) for the systems $\{u_k(x)\}_{k=1}^{\infty}$ and $\{\upsilon_k(x)\}_{k=1}^{\infty}$ the following "a priori" estimations are fulfilled:

$$\theta_k\,\|u_{k-1}\|_2 \leq const\,(1 + |\mu_k|)^{2m}\,\|u_k\|_2\,, \tag{1.3}$$

$$\theta_{k+1}\,\|\upsilon_{k+1}\|_2 \leq const\,(1 + |\mu_k|)^{2m}\,\|\upsilon_k\|_2\,, \tag{1.4}$$

where $\|\cdot\|_p = \|\cdot\|_{L_p(G)}$.
5) for any $\tau \geq 0$ the following estimations are fulfilled

$$\sum_{0 \leq \rho_k \leq \tau} \|u_k\|_\infty^2\,\|u_k\|_2^{-2} \leq const\,(1+\tau)\,, \tag{1.5}$$

$$\sum_{0 \leq \rho_k \leq \tau} \|\upsilon_k\|_\infty^2\,\|\upsilon_k\|_2^{-2} \leq const\,(1+\tau)\,; \tag{1.6}$$

6) for any number $k$ the following estimation is fulfilled

$$\|u_k\|_2\,\|\upsilon_k\|_2 \leq const. \tag{1.7}$$

Let $f(x)$ be an arbitrary function from the class $W_2^1(G)$. Let us compose a partial sum of its biorthogonal expansion in the system of root functions of the operator $L$:

$$\sigma_\nu\,(x, f) = \sum_{\rho_k \leq \nu} f_k\,u_k(x)\,, \qquad \nu > 0\,,$$

where biorthogonal coefficients $f_k$ are determined by the formula:

$$f_k = (f, \upsilon_k) = \int_G f(x)\,\overline{\upsilon_k(x)}\,dx\,.$$

In this paper we prove the following theorem on absolute and uniform convergence of biorthogonal expansion of the function $f(x)$ in the system of root functions $\{u_k(x)\}_{k=1}^{\infty}$.

**Theorem 1.1.** *Let the systems $\{u_k(x)\}_{k=1}^{\infty}$ and $\{\mu_k\}_{k=1}^{\infty}$ satisfy the conditions **A**, for the function $f(\cdot) \in W_2^1(G)$ and the biorthogonal system the following conditions be fulfilled:*

$$\left| f(x)\ \overline{\upsilon_k^{(2m)}(x)} \Big|_0^1 \right| \leq\ C(f)\ |\mu_k|^{\delta}\ \|\upsilon_k\|_{\infty}\ , \tag{1.8}$$

*where $0 \leq \delta < 2m\,,\quad \rho_k \geq 1$.*

*Then biorthogonal expansion of the function $f(x)$ converges absolutely and uniformly on $\overline{G} = [0,1]$ and the following estimation is valid:*

$$\begin{aligned} \|\,\sigma_{\nu}\,(\cdot\,,\,f) - f\,\|_{C\,[0,1]}\ \leq\ const\ \Big\{ C(f)\ \Big(1+\tfrac{1}{2m-\delta}\Big)^2 \nu^{\delta-2m} + \nu^{-\frac{1}{2}}\ \|\,f'\,\|_2 \\ + \textstyle\sum_{l=2}^{2m} \nu^{\frac{1}{2}-l}\ \|\,Q_l\, f\|_2 + \nu^{-2m}\ \|\,Q_{2m+1}\|_1 \Big\}\ , \qquad \nu \geq 2\,, \end{aligned} \tag{1.9}$$

*where*

$$Q_l(x) = \sum_{s=0}^{l-2} (-1)^{2m-l+s}\ C_{2m+1-l+s}^{s}\, P_{l-s}^{(s)}(x)\,,$$

$l = \overline{2,\ 2m+1}\,,$ *const is independent of the function $f(x)$.*

**Corollary.** *If in the theorem 1.1 the function $f(x)$ satisfies the condition $f\,(0) = f\,(1) = 0$, then its biorthogonal expansion converges absolutely and uniformly on $\overline{G} = [0,1]$ and the following estimations are valid:*

$$\|\,\sigma_{\nu}\ (\cdot\,,\,f) - f\,\|_{C[0,1]}\ \leq\ const\,\nu^{-\frac{1}{2}}\ \|\,f\,\|_{W_2^1(G)}\ , \qquad \nu \geq 2\,; \tag{1.10}$$

$$\|\,\sigma_{\nu}\ (\cdot\,,\,f) - f\,\|_{C[0,1]}\ =\ o\left(\nu^{-\frac{1}{2}}\right)\,, \qquad \nu \to +\infty\,, \tag{1.11}$$

*where the symbol “o” depends on $f(x)$, while const is independent of $f(x)$.*

Note that such results for even order operators were established in [4], [11], [5], [2], [6]. For *m=1,* i.e. for a third order operator, these results were proved in [1], [7].

## 2 Some auxiliary statements.

It follows from the results of the papers [8-10] that subject to the condition $A$, each of the systems

$$\left\{ u_k(x)\ \|\,u_k\,\|_2^{-1} \right\}_{k=1}^{\infty} \quad \text{and} \quad \left\{ \upsilon_k(x)\ \|\,\upsilon_k\,\|_2^{-1} \right\}_{k=1}^{\infty}$$

form a Riesz basis in the space $L_2(G)$. Consequently, for them the Bessel inequality in the space $L_2(G)$ is valid.

**Statement 2.1.** *The Fourier coefficients $f_k$ of biorthogonal expansion of the function $f(\cdot) \in W_2^1(G)$ have the following representation*

$$f_k = (f,\,\upsilon_k) = \frac{1}{\lambda_k} \sum_{i=0}^{n_k} \frac{f(x)\ \overline{\upsilon_{k+i}^{(2m)}(x)}\Big|_0^1}{\lambda_k^i} - \frac{1}{\lambda_k} \sum_{i=0}^{n_k} \frac{\left(f',\,\upsilon_{k+i}^{(2m)}\right)}{\lambda_k^i}$$

$$+\frac{1}{\lambda_k}\sum_{i=0}^{n_k}\frac{1}{\lambda_k^i}\sum_{l=2}^{2m+1}\left(Q_l\,f,\,\upsilon_{k+i}^{(2m+1-l)}\right),\qquad \lambda_k\neq 0, \tag{2.1}$$

*where $n_k$ is the order of the root function $\upsilon_k(x)$.*

**Proof.** Let the function $\upsilon_k(x)$ be an associated function of the operator $L^*$, $\overline{\lambda_k}$ be a corresponding eigen value,$\lambda_k\neq 0$. By the definition of the function $\upsilon_k(x)$ the following equality holds

$$\upsilon_k=-\frac{L^*\,\upsilon_k}{\overline{\lambda_k}}+\theta_{k+1}\frac{\upsilon_{k+1}}{\overline{\lambda_k}}=$$

$$=-\frac{1}{\overline{\lambda_k}}\left((-1)^{(2m+1)}\,\upsilon_k^{(2m+1)}+(-1)^{(2m-1)}\,\left(\overline{P_2}\,\upsilon_k\right)^{(2m-1)}+...+\overline{P_{2m+1}}\,\upsilon_k\right)+\theta_{k+1}\frac{\upsilon_{k+1}}{\overline{\lambda_k}}\ ,$$

allowing for which we obtain

$$(\upsilon_k\,,\,f)=-\frac{1}{\overline{\lambda_k}}\,(L^*\,\upsilon_k\,,\,f)+\frac{\theta_{k+1}}{\overline{\lambda_k}}\,(\upsilon_{k+1}\,,\,f)$$

$$=\frac{1}{\overline{\lambda_k}}\left(\upsilon_k^{(2m+1)},\,f\right)+(-1)^{2m-1}\frac{1}{\overline{\lambda_k}}\left(\left(\overline{P_2}\,\upsilon_k\right)^{(2m-1)},\,f\right)\ +...$$

$$+(-1)\frac{1}{\overline{\lambda_k}}\left(\overline{P_{2m+1}}\,\upsilon_k\,,\,f\right)+\frac{\theta_{k+1}}{\overline{\lambda_k}}\,(\upsilon_{k+1}\,,\,f)=\frac{1}{\overline{\lambda_k}}\left(\upsilon_k^{(2m+1)},\,f\right)+\frac{1}{\overline{\lambda_k}}\left(\upsilon_k^{(2m-1)},\,Q_2\,f\right)$$

$$+\frac{1}{\overline{\lambda_k}}\left(\upsilon_k^{(2m-2)},\,Q_3\,f\right)+...+\frac{1}{\overline{\lambda_k}}\,(\upsilon_k\,,\,Q_{2m+1}\,f)+\frac{\theta_{k+1}}{\overline{\lambda_k}}\,(\upsilon_{k+1}\,,\,f)\ .$$

Consequently, we have the recurrent relation

$$(\upsilon_k,\,f)=\frac{1}{\overline{\lambda_k}}\left[\left(\upsilon_k^{(2m+1)},\,f\right)+\sum_{l=2}^{2m+1}\left(\upsilon_k^{(2m+1-l)},\ Q_l\,f\right)\right]+\frac{\theta_{k+1}}{\overline{\lambda_k}}\,(\upsilon_{k+1}\,,\,f)\ .$$

Taking into account, $\theta_{k+1}=\theta_{k+2}=...=\theta_{k+n_k}=1\,,\ \ \theta_{k+n_k+1}=0$, from this recurrent relation we have:

$$(\upsilon_k\,,\,f)=\frac{1}{\overline{\lambda_k}}\sum_{i=0}^{n_k}\frac{(\upsilon_{k+i}\,,\,f)}{\overline{\lambda_k^i}}+\frac{1}{\overline{\lambda_k}}\sum_{i=0}^{n_k}\sum_{l=2}^{2m+1}\frac{1}{\overline{\lambda_k^i}}\left(\upsilon_{k+i}^{(2m+1-l)}\,,\ Q_l\,f\right).$$

At first, making integration by parts in the expressions $\left(\upsilon_{k+i}^{(2m+1)}\,,\,f\right)$ , $i=\overline{0,n_k}$, and then taking complex conjugation, we obtain the formula (2.1).

Statement 2.1 is proved.

**Statement 2.2.** *Subject to the conditions* (1.1), (1.2), (1.4), (1.6) *and 3) the systems*

$$\left\{\upsilon_k^{(s)}\,(x)\ \,\|\,\upsilon_k\,\|_2^{-1}\ \mu_k^{-s}\right\}\ ,\ \ \mu_k\neq 0\ ,\ \ \ s=\overline{1\,,\,2m\,,}$$

*are Bessel systems in the space $L_2(G)$, i.e. for an arbitrary function $f\in L_2(G)$ the following Bessel inequality is valid:*

$$\left(\sum_{\mu_k\neq 0}\left|\left(f\,,\,\upsilon_k^{(s)}\ \|\,\upsilon_k\,\|_2^{-1}\ \mu_k^{-s}\right)\right|^2\right)^{1/2}\ \leq\ M\ \|\,f\,\|_2\ , \tag{2.2}$$

*where $M>0$ is a constant independent of $f$.*

This statement was proved in [8-9].

**Statement 2.3.** *Subject to the conditions* (1.5)–(1.6) *for any* $\mu \geq 2$ *and* $\delta > 0$*the following estimations are valid*

$$\sum_{\rho_k \geq \mu} |\mu_k|^{-(1+\delta)} \, \| u_k \|_\infty^2 \, \| u_k \|_2^{-2} \leq C_1(\delta) \, \mu^{-\delta}, \tag{2.3}$$

$$\sum_{\rho_k \geq \mu} |\mu_k|^{-(1+\delta)} \, \| \upsilon_k \|_\infty^2 \, \| \upsilon_k \|_2^{-2} \leq C_2(\delta) \, \mu^{-\delta}, \tag{2.4}$$

*where* $C_1(\delta)$ , $C_2(\delta)$ *are positive constants independent of* $\mu$*.*

**Proof.** Prove the estimations (2.3). We fix any natural number $p$. Then

$$\sum_{\mu \leq \rho_k \leq [\mu]+p} |\mu_k|^{-(1+\delta)} \| u_k \|_\infty^2 \, \| u_k \|_2^{-2} \leq \sum_{[\mu] \leq \rho_k \leq [\mu]+p} |\rho_k|^{-(1+\delta)} \| u_k \|_\infty^2 \, \| u_k \|_2^{-2}$$

$$\leq \sum_{n=[\mu]}^{[\mu]+p} n^{-(1+\delta)} \sum_{n \leq \rho_k < n+1} \| u_k \|_\infty^2 \, \| u_k \|_2^{-2} . \tag{2.5}$$

Denoting

$$b_n = n^{-(1+\delta)}, \quad a_n = \sum_{n \leq \rho_k < n+1} \| u_k \|_\infty^2 \, \| u_k \|_2^{-2} , S_n = a_1 + a_2 + ... + a_n$$

and using the Abel transformations, from (2.5) we get

$$\sum_{\mu \leq \rho_k \leq [\mu]+\rho} |\mu_k|^{-(1+\delta)} \, \| u_k \|_\infty^2 \, \| u_k \|_2^{-2}$$

$$\leq \sum_{n=[\mu]}^{[\mu]+p} a_n \, b_n = \sum_{n=[\mu]}^{[\mu]+p-1} S_n \, (b_n - b_{n+1}) + S_{[\mu]+p} \, b_{[\mu]+p} - S_{[\mu]-1} \, b_{[\mu]}$$

$$\leq \sum_{n=[\mu]}^{[\mu]+p-1} S_n \left( n^{-(1+\delta)} - (n+1)^{-(1+\delta)} \right) + ([\mu]+p)^{-(1+\delta)} \, S_{[\mu]+p} + [\mu]^{-(1+\delta)} \, S_{[\mu]-1}.$$

Considering here the estimation $S_n \leq const \; (n+1)$ that follows from (1.5), we obtain the inequality:

$$\sum_{\mu \leq \rho_k \leq [\mu]+p} |\mu_k|^{-(1+\delta)} \, \| u_k \|_\infty^2 \, \| u_k \|_2^{-2}$$

$$\leq \; const \sum_{n=[\mu]}^{[\mu]+p-1} (n+1) \, \frac{(1+\delta) \, (n+1)^{\delta}}{(n \, (n+1))^{1+\delta}} + \; const \; ([\mu]+p) \; ([\mu]+p)^{-(1+\delta)}$$

$$+ \; const \; [\mu] \; [\mu]^{-(1+\delta)} \leq \; const \left( (1+\delta) \sum_{n=[\mu]}^{\infty} n^{-(1+\delta)} + [\mu]^{-\delta} \right) \leq C_1(\delta) \, \mu^{-\delta},$$

where $C_1(\delta) = \; const \; \left(1 + \delta^{-1}\right)$.

Consequently, by the arbitrariness of the natural number $p$ the estimation (2.3) is valid. The estimation (2.4) is proved in the same way. Statement 2.3 is proved.

Denote

$$I_l\,(\mu\,,\,x) \equiv \;I_l\,(\mu) \;=\; \sum_{\rho_k \geq \mu} \left| \lambda_k^{-1} \sum_{j=0}^{n_k} \lambda_k^{-j} \,\left(Q_l\,f,\;\upsilon_{k+j}^{(2m+1-l)}\right) \right| \;\;|\,u_k\,(x)\,|\;\;,$$

where

$$\mu \geq 2\,; \quad l = \overline{2,\;2m+1}\,; \quad x \in \overline{G\,.}$$

**Statement 2.4.** *Subject to the conditions* **A** *the following estimations are valid*:

$$I_l\,(\mu) \;\leq\; const\;\mu^{\frac{1}{2}-l}\;\|Q_l\,f\|_2\;, \qquad l = \;\overline{2\,,\,2m}\;; \tag{2.6}$$

$$I_{2m+1}\,(\mu) \;\leq\; const\;\mu^{-2m}\;\|Q_{2m+1}\,f\|_1\;. \tag{2.7}$$

**Proof.** Considering $|\,\lambda_k\,| \;=\; |\,\mu_k\,|^{2m+1}$ and applying the estimations (1.4), (1.6), we obtain:

$$I_l\,(\mu) \leq \sum_{\rho_k\geq\mu} |\lambda_k|^{-1} \sum_{j=0}^{n_k} |\lambda_k|^{-j} \left| \left(Q_l f, \upsilon_{k+j}^{(2m+1-l)}\right) \right| \;\; |u_k(x)|$$

$$\leq \sum_{\rho_k\geq\mu} |\lambda_k|^{-1}\, \|u_k\|_\infty \sum_{j=0}^{n_k} |\mu_k|^{-(2m+1)j}$$

$$\times \left| \left(Q_l f,\; \|\upsilon_{k+j}\|_2^{-1}\, \mu_k^{l-2m-1} \upsilon_{k+j}^{(2m+1-l)}\right) \right| \;\; |\mu_k|^{2m+1-l}\, \|\upsilon_{k+j}\|_2$$

$$\leq const \sum_{\rho_k\geq\mu} \|u_k\|_\infty\, \|\upsilon_k\|_2\, |\mu|_k^{-l} \sum_{j=0}^{n_k} \left| \left(Q_l f, \|\upsilon_{k+j}\|_2^{-1}\, \mu_k^{l-2m-1} \upsilon_{k+j}^{(2m+1-l)}\right) \right|$$

$$\leq const \sum_{\rho_k\geq\mu} \|u_k\|_\infty\, \|u_k\|_2^{-1}\, |\mu_k|^{-l} \sum_{j=0}^{n_k} \left| \left(Q_l f, \mu_k^{l-2m-1}\, \|\upsilon_{k+j}\|_2^{-1}\, \upsilon_{k+j}^{(2m+1-l)}\right) \right|, \quad l = \overline{2,\,2m}\,.$$

We apply the Cauchy-Bunyakovskii inequality for the sum. As a result we obtain the following inequality:

$$I_l\,(\mu) \leq const \left( \sum_{\rho_k\geq\mu} |\mu_k|^{-2l}\, \|u_k\|_\infty^2\, \|u_k\|_2^{-2} \right)^{1/2} \times$$

$$\times \left( \sum_{\rho_k\geq\mu} \left( \sum_{j=0}^{n_k} \left| \left(Q_l f, \mu_k^{l-2m-1}\, \|\upsilon_{k+j}\|_2^{-1}\, \upsilon_{k+j}^{(2m+1-l)}\right) \right| \right)^2 \right)^{1/2}.$$

Hence, by $\sup\limits_k n_k < \infty$ (this follows from the condition (1.2)) and statements 2.2 and 2.3 (see (2.2) and (2.3)) we obtain the inequality

$$I_l\,(\mu) \leq const\mu^{\frac{1}{2}-l} \left( \sup_k n_k \right) \;\|Q_l f\|_2 \leq const\,\mu^{\frac{1}{2}-l}\, \|Q_l f\|_2\,, \quad l = \overline{2,\;2m}\,.$$

The estimation (2.6) is established.

To prove the estimation (2.7), at first we apply the Holder inequality for $p = 1\,,\ q = \infty$, then take into account $\sup\limits_k n_k < \infty$ and apply anti a priori estimation (1.4). As a result, we obtain

$$I_{2m+1}(\mu) \le \sum_{\rho_k \ge \mu} |\mu_k|^{-(2m+1)} \left( \sum_{j=0}^{n_k} |\mu_k|^{-(2m+1)j} \ \|\upsilon_{k+j}\|_\infty \right) \ \|Q_{2m+1} f\|_1 \,.$$

Hence, by the Cauchy-Bunyakovsky inequality, condition (1.7) and statement 3, it follows that

$$I_{2m+1}(\mu) \le const \left( \sum_{\rho_k \ge \mu} |\mu_k|^{-(2m+1)} \|u_k\|_\infty^2 \|u_k\|_2^{-2} \right)^{1/2}$$

$$\times \left( \sum_{\rho_k \ge \mu} |\mu_k|^{-(2m+1)} \|\upsilon_k\|_\infty^2 \|\upsilon_k\|_2^{-2} \right)^{1/2} \quad \|Q_{2m+1} f\|_1$$

$$\le const\, \mu^{-2m} \ \|Q_{2m+1} f\|_1 \,.$$

Statement 2.4 is proved.

**Statement 2.5.** *Subject to the conditions A and (1.8), the following estimation is valid:*

$$A(x,\mu) \equiv \sum_{\rho_k \ge \mu} |\lambda_k|^{-1} |u_k(x)| \left| \sum_{i=0}^{n_k} |\lambda_k|^{-i} \left| f(t) \ \overline{\upsilon_{k+i}^{(2m)}(t)} \right|_0^1 \right|$$

$$\le const\ C(f) \ \left( 1 + \frac{1}{2m-\delta} \right)^2 \nu^{\delta - 2m}\,, \ x \in \overline{G}, \quad \mu \ge 2\,. \tag{2.8}$$

**Proof.** It follows from the condition (1.8) that

$$A(x,\mu) \le \ C(f) \sum_{\rho_k \ge \mu} |\mu_k|^{\delta - 2m - 1} \ \|u_k\|_\infty \left( \sum_{i=0}^{n_k} |\mu_k|^{-(2m+1)\,i} \ \|\upsilon_{k+i}\|_\infty \right) \ .$$

Applying conditions (1.4), (1.7) and considering $\sup\limits_k n_k < \infty$, we obtain

$$A(x,\mu) \le \ const\, C(f) \sum_{\rho_k \ge \mu} |\mu_k|^{\delta - 2m - 1} \ \|u_k\|_\infty \ \|\upsilon_k\|_\infty$$

$$\le \ const \ C(f) \sum_{i=0}^{n_k} \left( |\mu_k|^{\frac{(\delta-2m-1)}{2}} \ \|u_k\|_\infty \ \|u_k\|_2^{-1} \right) \ \left( |\mu_k|^{\frac{(\delta-2m-1)}{2}} \ \|\upsilon_k\|_\infty \ \|\upsilon_k\|_2^{-1} \right) \ .$$

By the Cauchy-Bunyakovsky inequality and statement 2.3, from the last relation we have

$$A(x,\mu) \le const\, C(f) \ C_1\,(2m-\delta) \ C_2\,(2m-\delta) \ \mu^{-(2m-\delta)}$$

$$\le const\, C(f) \ \left( 1 + \frac{1}{2m-\delta} \right)^2 \mu^{-(2m-\delta)}\,,$$

where $C_1\,(2m-\delta)\ ,\ C_2(2m-\delta)$ are constants from statement 2.3 that do not exceed the value

$const\left(1+\frac{1}{2m-\delta}\right)$. Uniform convergence of the series $A\,(x\,,\mu)$ and estimation (2.8) are proved. Statement 2.5 is proved.

**Statement 2.6.** *Subject to the conditions* **A**, *the series*

$$B\,(x,\mu)\equiv\sum_{\rho_k\geq\mu}|\lambda_k|^{-1}\,|u_k(x)\mid\sum_{i=0}^{n_k}|\lambda_k|^{-i}\left|\,\left(f'\ ,\ \upsilon_{k+i}^{(2m)}\right)\right|$$

uniformly converges on $\overline{G}=[0,1]$ and the following estimation is valid:

$$B\,(x,\mu)\leq\ const\ \mu^{-\frac{1}{2}}\left\|f'\right\|_2\,,\quad \mu\geq 2\,. \tag{2.9}$$

**Proof.** Allowing for the equality $|\mu_k|^{2m+1}=|\lambda_k|$ we represent the series $B\,(x,\mu)$ in the form

$$B\,(x,\mu)=\sum_{\rho_k\geq\mu}|\mu_k|^{-1}\left(\sum_{i=0}^{n_k}\ \left|\left(f',\,\mu_k^{-2m}\ \ \|\upsilon_{k+i}\|_2^{-1}\ \upsilon_{k+i}^{(2m)}\right)\right|\ \ |\mu_k|^{-(2m+1)\,i}\ \ \|\upsilon_{k+i}\|_2\right)|u_k(x)|$$

Here we apply anti a priori estimation (1.4) and the condition 1.7.

$$B\,(x,\mu)\leq const\sum_{\rho_k\geq\mu}|\mu_k|^{-1}\ \|u_k\|_\infty\ \left(\sum_{i=0}^{n_k}\left|\left(f',\,\mu_k^{-2m}\ \ \|\upsilon_{k+i}\|_2^{-1}\,\upsilon_{k+i}^{(2m)}\right)\right|\ \|\upsilon_k\|_2\right)$$

$$\leq\ const\sum_{\rho_k\geq\mu}\|u_k\|_\infty\ \ \|u_k\|_2^{-1}\ \ |\mu_k|^{-1}\ \left(\sum_{i=0}^{n_k}\left|\left(f'\,,\,\mu_k^{-2m}\ \ \|\upsilon_{k+i}\|_2^{-1}\ \ \upsilon_{k+i}^{(2m)}\right)\right|\right).$$

Hence, by the Cauchy-Bunyakovsky inequality, statements 2.2 and 2.3 we obtain

$$B\,(x\,,\,\mu)\ \leq\ const\ \left(\sum_{\rho_k\geq\mu}\mu_k^{-2}\ \|u_k\|_\infty^2\ \|u_k\|_2^{-2}\right)^{1/2}$$

$$\times\left(\sum_{\rho_k\geq\mu}\left(\sum_{i=0}^{n_k}\ \left|\left(f'\,,\,\mu_k^{-2m}\ \ \|\upsilon_{k+i}\|_2^{-1}\ \ \upsilon_{k+i}^{(2m)}\right)\right|\right)^2\right)^{1/2}$$

$$\leq\ const\ \mu^{-1/2}\left(\sum_{\rho_k\geq\mu}n_k\sum_{i=0}^{n_k}\left|\ \left(f',\,\mu_k^{-2m}\ \ \|\upsilon_{k+i}\|_2^{-1}\ \upsilon_{k+i}^{(2m)}\right)\ \right|^2\right)^{1/2}$$

$$\leq\ const\ \left(\sup_k n_k\right)^2\ \mu^{-\frac{1}{2}}\left\|f'\right\|_2\ \leq\ const\ \mu^{-\frac{1}{2}}\ \left\|f'\right\|_2\ .$$

Statement 2.6 is proved.

## 3 Proof of the theorem 1.1.

Let us prove uniform convergence of the series

$$\sum_{k=1}^{\infty} |f_k| \ \| u_k(x) \| \tag{3.1}$$

on $\overline{G} = [0,1]$. For that we consider its remainder

$$R(\mu, x) = \sum_{\rho_k \geq \mu} |f_k| \ | u_k(x) |$$

and prove that uniformly with respect to $x \in \overline{G}$

$$\lim_{\mu \to +\infty} R(\mu, x) = 0. \tag{3.2}$$

By statement 2.1, the following inequality is fulfilled:

$$R(\mu, x) \leq A(x, \mu) + B(x, \mu) + \sum_{l=2}^{2m+1} I_l(\mu, x).$$

Here applying statements 2.4, 2.5 and 2.6 (estimations (2.6) (2.9)) for $R(\mu, x)$ we obtain the following inequality uniform with respect to $x \in \overline{G}$

$$0 \leq R(\mu, x) \leq const \left\{ C(f) \left(1 + \frac{1}{2m-\delta}\right)^2 \mu^{\delta - 2m} + \mu^{-\frac{1}{2}} \|f'\|_2 + \sum_{l=2}^{2m} \mu^{\frac{1}{2}-l} \|Q_l f\|_2 + \mu^{-2m} \|Q_{2m+1} f\|_1 \right\}.$$

Consequently the relation (3.2) is valid and the series (3.1) uniformly converges on $\overline{G}$. The uniform convergence of the series (3.1) implies uniform convergence of biorthogonal expansion of the function $f(x)$ in the system $\{ u_k(x) \}_{k=1}^{\infty}$. Since the system $\{ u_k(x) \}_{k=1}^{\infty}$ is complete in $L_2(G)$, the function $f(x)$ is absolutely continuous $\overline{G} = [0,1]$, then its biorthogonal series converges uniformly just to $f(x)$, i.e. the following equality is valid

$$f(x) = \sum_{k=1}^{\infty} f_k \, u_k(x), \quad x \in \overline{G}.$$

We now estimate the difference $f(x) - \sigma_\nu(x, f)$ in the metrics $C\,[0,1]$.

$$\| f - \sigma_\nu(\cdot, f) \|_{C[0,1]} = \left\| \sum_{\rho_k > \nu} f_k \, u_k \right\|_{C[0,1]} \leq \max_{x \in \overline{G}} \sum_{\rho_k \geq \nu} |f_k| \ |u_k(x)| = \max_{x \in \overline{G}} R(\nu, x)$$

$$\leq const \left\{ C(f) \left(1 + \frac{1}{2m-\delta}\right)^2 \nu^{\delta-2m} + \nu^{-\frac{1}{2}} \|f'\|_2 + \sum_{l=2}^{2m} \nu^{\frac{1}{2}-l} \|Q_l f\|_2 + \nu^{-2m} \|Q_{2m+1} f\|_1 \right\}.$$

The theorem 1.1 is proved.

If the function $f(\cdot) \in W_2^1(G)$ satisfies the condition $f(0) = f(1) = 0$, then the estimation (1.10) follows from the estimation (1.9). There with it suffices to take into account $C(f) = 0$, $\|Q_l f\|_2 \leq \|Q_l\|_2 \|f\|_\infty$, $l = \overline{2, 2m}$; $\|Q_{2m+1} f\|_1 \leq \|Q_{2m+1}\|_1 \|f\|_\infty$ and

$\|f\|_\infty \leq \|f'\|_2$. And for the justification of the estimation (1.11) the attention should be paid to the estimation of the series $B\ (x\ ,\ \nu)$ and to take into account

$$\sum_{\rho_k \geq \nu} n_k \sum_{i=0}^{n_k} \left| \left( f',\ \mu_k^{-2m}\ \|\upsilon_{k+i}\|_2^{-1}\ \upsilon_{k+i}^{(2m)} \right) \right|^2 = o\ (1)$$

as $\nu \to +\infty$, because $\sup\limits_k n_k < \infty$ and the system $\left\{ \mu_k^{-2m}\ \|\upsilon_{k+i}\|_2^{-1}\ \upsilon_{k+i}^{(2m)}\ (x) \right\}_{\rho_k > 0,\, 0 \leq i \leq n_k}$ is Bessel in $L_2(G)$.

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