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On Harnack's inequality for positive solutions of linear elliptic equations with discontinuous coefficients

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Abstract. *A new class of non-divergence structure pointwise non-uniformly degenerating elliptic equations of second order has been studied on the subject of Harnack's inequality for positive solutions.*

Keywords. non-divergence, strong solution, elliptic equation, Harnack's inequality

Mathematics Subject Classification (2010): 35B09, 35B45, 35B65, 35D35, 35B50

1 Introduction

The Harnack inequality plays fundamental role in the study of qualitative properties of solutions of elliptic and parabolic equations (see, e.g. in [15], [18], [20]). Notice, the Harnack inequality first time was proved by J.Moser [25], [26] in the study of regularity properties of solutions of second order divergent structure uniformly elliptic and parabolic equations. This result was extended to the non-divergence structure uniformly elliptic and parabolic equations with discontinues coefficients by N.V. Krylov and M.V. Safonov [19]. After, it was found different new proofs and extensions of the Harnack inequality. In this perspective it is appropriate to mention the Harnack inequality results for the class of uniformly degenerating divergent structure elliptic and parabolic equations, its quasi-linear analogues (see, e.g. in [11], [13], [14], [7]). A study of divergent structure equations essentially depends on existence of the appropriate embedding results (Cf. [8], [22], [24]) At the moment, this topics had an essential development in directions of non-uniformly degenerating elliptic and parabolic equations its quasi-linear and vector field analogues also with non-standard growth condition(see e.g. [2], [3], [12], [27], [4]). Even in [10], [23] the case of double-divergence structure elliptic equations has been considered for this subject.

In the resent stage of development of Harnack's inequality studies an increasing interest is being seen on the study of new classes of non-uniformly degenerating elliptic and parabolic equations. In this paper, we consider the Harnack inequality for positive strong solutions of a special class of non-divergence structure and point-wise non-uniformly degenerating elliptic equations. Before, the similar question for a class of power type degenerated

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equations was considered in [1] and /or in [5], [16], [21], where mainly the divergent elliptic equations is considered. Below, assuming some assumptions on the small coefficients $b_i(x)$ and $c(x)$ we show that the Harnack inequality result holds for solutions of non-divergent fully linear elliptic equation (1.1) of non-divergent structure under more general then [1] degeneration condition (Cf. [6] for this case when the equation is of divergent structure and small coefficients absent). The non-power functions ω_i are the main novelty of this study where a new approach also implemented. In proofs the technic of authors [19] (see, also in [18]) is developed to the case of non-uniformly degeneration. The results are announced in [9].

Let E_n be n -dimensional Euclidean space of points $x = (x_1, ..., x_n)$, $n \geq 3$ and D be a bounded domain lying in E_n and ∂D be its boundary, such that $\partial D \in C^2$ and $0 \in \overline{D}$. Consider in D second order elliptic equation

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)u_{ij} + \sum_{i=1}^{n} b_i(x)u_i + c(x)u(x) = 0, \tag{1.1}$$

assuming that $\|a_{ij}(x)\|$ be a real symmetric matrix with measurable elements in D satisfying

$$\gamma \sum_{i=1}^{n} \lambda_i(x)\xi_i^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \gamma^{-1} \sum_{i=1}^{n} \lambda_i(x)\xi_i^2 \tag{1.2}$$

for any $x \in D$ and $\xi \in E_n$, wherein $\gamma \in (0, 1]$ is a constant and $u_i = \frac{\partial u}{\partial x_i}$, $u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$, $\lambda_i(x) = g_i(\rho(x))$, $\rho(x) = \sum\limits_{i=1}^{n} \omega_i(|x_i|)$, $g_i(t) = \left(\frac{\omega_i^{-1}(t)}{t}\right)^2$; $j, i = 1, ..., n$. Where the functions $\omega_i(t)$ are strongly monotone, convex and continuous functions on $[0, \operatorname{diam} D]$ such that $\omega_i(0) = 0$; $\omega_i^{-1}(t)$ are inverse functions of $\omega_i(t)$. With respect to the $\omega_i(t)$; $i = 1, ..., n$ we also assume: there exist constants $\alpha, \beta, \eta \in (1, \infty)$, $q > n$, $A > 0$ such that

$$\alpha\omega_i(t) \leq \omega_i(\eta t) \leq \beta\omega_i(t), \quad t \in (0, \operatorname{diam} D),$$

$$\left(\frac{\omega_i^{-1}(t)}{t}\right)^{q-1} \int\limits_0^{\omega_i^{-1}(t)} \left(\frac{\omega_i(\tau)}{\tau}\right)^q d\tau \leq At, \;\; i = 1, ..., n. \tag{1.3}$$

Furthermore, suppose that

$$|b_i(x)| \leq b_0, \;\; -c_0 \leq c(x) \leq 0, \;\; i = 1, ..., n, \tag{1.4}$$

with b_0, c_0 are positive constants.

2 Notation and definitions

Denote $W^2_{2,\lambda}(D)$ the Banach space of functions $u(x)$ given in D and having finite norm

$$\|u\|^2_{W^2_{2,\lambda}(D)} = \int\limits_D \left(u^2(x) + \sum_{i=1}^{n} \lambda_i(x)u_i^2 + \sum_{i,j=1}^{n} \lambda_i(x)\lambda_j(x)u_{ij}^2 \right) dx. \tag{2.1}$$

Define $\dot{W}^2_{2,\lambda}(D)$ the closure of functions $u(x) \in C^\infty(\overline{D}), u\,|_{\partial D} = 0$ with respect to the norm of space $W^2_{2,\lambda}(D)$.

A function $u(x) \in \dot{W}^2_{2,\lambda}(D)$ is called strong solution of equation (1) in D, if it satisfies (1) a.e. in D. A function $u(x) \in W^2_{2,\lambda}(D)$ that is a solution of the inequality $Lu \geq 0$ is called L –sub elliptic function. A function $u(x)$ is called L-super elliptic in D if $-u(x)$ is an L -sub elliptic function in D. The notation $C(\cdots)$ means that the positive constant C depends on the content of parenthesize.

Let $R \in (0,1]$, $k > 0$ and $x^0 \in E_n$. Given a parallelepiped $\Pi_{R:K}(x^0) = \{x : |x_i - x_i^0| < K\omega_i^{-1}(R), i = 1, ..., n\}$ and let $B'_R(x^0) = \{x : |x - x^0| < R\}$ be an Euclidean ball. Denote $E_R^{x^0}(K)$ the ellipsoid $\left\{x : \sum\limits_{i=1}^{n} \frac{(x_i - x_i^0)^2}{(w_i^{-1}(R))^2} < K^2\right\}$ and set $B^1 = E_R^0(17)$, $B^2 = E_R^0(1)$, $B^3 = B^1 \setminus B^2 = E_R^0(1, 17)$, $B_R(x^0) = E_R^{x^0}(9)$, $B^4 = E_R^0\left(\frac{1}{4}\right)$, $B^5 = E_R^0(8.5, 9.5)$.

3 Main result

Theorem 3.1 *Let $u(x)$- be a positive solution of equation* (1.1) *in the domain D and the coefficients operator L satisfy the conditions* (1.2)-(1.4). *If $\overline{B}^1 \subset D$ and $R \leq R_0$ then it holds*

$$\sup_{B^4} u(x) \leq C_8(n, b_0, c_0) \inf_{B^4} u(x). \tag{3.1}$$

4 Auxiliary result

Following assertion is of known results that will be used in our proofs below (see, [9]).

Theorem 4.1 *Let D be a domain located in B^1 having limit points on ∂B^1 be such that its intersection with $B_R(0)$ is not empty. Let in D given a positive and continues $L-$sub elliptic function $u(x)$ vanishing on $\partial D \cap B^1$. Assume that numbers $\sigma > 0$, $R \leq 1$, $\eta > 0$ and the operator L satisfies conditions* (1.2)-(1.4)*. Then there exists positive constant $\eta = \eta(\gamma, n, b_0, c_0, \sigma)$ such that it holds estimate*

$$\sup_{D} u(x) \geq \big(1 + \eta\big) \sup_{D \cap B_R(0)} u(x) \tag{4.1}$$

provided that meas $H \geq \sigma$meas B^5 to be satisfied for Lebesgue measure of $H = B^5 \setminus D$.

The proof of this assertion given on Section 6.

5 Proofs

To prove the Harnack inequality we need on following Lemma 5.1 on growth of positive solutions of equation (1). Following main result has been obtained in this paper.

Lemma 5.1 *Let $x^0 \in B^1$ and in $B^1(x^0) := E_R^{x^0}(17)$ given a domain G having limit points on spheroid $\partial B^1(x^0)$ and intersecting $B_R(x^0)$. Let the coefficients of operator L satisfy the conditions (2)-(4) and in G is given a positive continuous and L sub-elliptic function $u(x)$ vanishing on $\partial G \cap B^1(x^0)$. Then for any $K > 1$ there exist $\delta_0(\gamma, n, b_0, c_0, K) > 0$ such that it holds an estimate*

$$\sup_{D} u(x) \geq K \sup_{G \cap B_R(x^0)} u(x) \tag{5.1}$$

provided that

$$meas\, G \leq \delta_0 meas\, B^1(x^0). \tag{5.2}$$

Proof. of Lemma 5.1. Let η_0 be the constant from Theorem 3.1, $\sigma = \frac{1}{2}$ and p be the list natural number such that $(1+\eta_0)^p \geq K$. Insert

$$\delta_0 = \frac{1}{2 \cdot 17^n} \left(\frac{36}{17p}\right)^n .$$

Split the layer $B^1(x^0) \setminus B_R(x^0)$ into p number smaller sub-layers by the ∂E_i with $E_i = E_R^{x^0}\left(9+\frac{8i}{p}\right)$, $i = 0, 1, ...p-1$. Evidently, ∂E_0 coincides with $\partial B_R(x^0)$. For $i = 0, 1, ...p-1$ denote $\sup\limits_{G \cap \partial E_i} u(x)$ as M_i and let $u(x)$ reaches M_i at the point $x^i \in \partial E_i$. Consider also ellipsoids $B_1^{(i)} := E_R^{x^i}\left(\frac{4}{p}\right)$, $B_2^{(i)} := E_R^{x^i}\left(\frac{36}{17p}\right)$, $i = 0, 1, ...p-1$. Denoting $E_p = B^1(x^0)$ we see that $B_1^{(i)} \subset E_{i+1}, i = 0, 1, ...p-1$. Also,

$$\text{meas}\left(B_2^{(i)} \setminus G\right) \geq \text{meas}\, B_2^{(i)} - \text{meas}\, G, \quad i = 0, 1, .., p-1. \tag{5.3}$$

On other hand,

$$\text{meas}\, B_2^{(i)} = \left(\frac{36}{17p}\right)^n \mathop{\Pi}_{i=1}^{n} \omega_i^{-1}(R) \left|S_1\right|, \tag{5.4}$$

where $|S_1|$ is volume of unit n-dimensional ball. In addition, according to (5.2) it holds

$$\text{meas}\, G \leq \delta_0 \text{meas}\, B^1(x^0) \leq \delta_0 \left|S_1\right| 17^n \mathop{\Pi}_{i=1}^{n} \omega_i^{-1}(R). \tag{5.5}$$

Using (5.4), (5.5) in (5.3) and taking into the account the value of δ_0 we infer

$$\begin{aligned} \text{meas}\left(B_2^{(i)} \setminus G\right) &\geq \left|S_1\right| \left(\frac{36}{17p}\right)^n \mathop{\Pi}_{i=1}^{n} \omega_i^{-1}(R) - \delta_0 \left|S_1\right| 17^n \mathop{\Pi}_{i=1}^{n} \omega_i^{-1}(R) \\ &= \left|S_1\right| \mathop{\Pi}_{i=1}^{n} \omega_i^{-1}(R) \left[\left(\frac{36}{17p}\right)^n - \frac{1}{2 \cdot 17^n}\left(\frac{36}{17p}\right)^n 17^n\right] \\ &= \frac{\left|S_1\right|}{2}\left(\frac{36}{17p}\right)^n \mathop{\Pi}_{i=1}^{n} \omega_i^{-1}(R) = \frac{1}{2}\text{meas}\, B_2^{(i)}, \quad i = 0, 1, ...p-1. \end{aligned}$$

Therefore, according to Theorem 3.1, it follows

$$M_{i+1} \geq (1+\eta) M_i, \quad i = 0, 1, ..., p-1,$$

where $M_p = \sup\limits_{G} u(x)$. Thus

$$M_p \geq (1+\eta)^p M_0$$

and the desired estimate (5.1) is ready.

Let G be a domain lying on $B^1(x^0)$, with $x^0 \in B^1$ and $R \leq R_0$. Define

$$A(G) = \{u(x) : x \in G, Lu \leq 0\} \text{ and } A^+(G) = \{u(x) : x \in G,\ u(x) \geq 0,\ Lu \leq 0\}.$$

Let further, for $\beta \in [0, 1]$ it is

$$A_\beta^R(x^0) = A^+\left(B^1(x^0)\right) \cap \left\{u(x) : \text{meas}\left(B^1(x^0) \cap [u(x) \geq 1]\right) \geq \beta \text{meas}\, B^1(x^0)\right\},$$

For $u \in A_\beta^R(x^0)$ set

$$\gamma_\beta^R(x^0) = \inf\left\{u(x) : x \in E_R^{x^0}\left(\frac{1}{2}\right)\right\}, \gamma_\beta^R = \inf_{x^0} \gamma_\beta^R(x^0)$$

and $\gamma(\beta) = \varliminf\limits_{R \to +0} \gamma_\beta^R$. It easily seen that $0 \leq \gamma(\beta) \leq 1$ and the function $\gamma(\beta)$ is nondecreasing on β. It is possible to show also that this function $\gamma(\beta)$ is continuous on $[0, 1]$.

Lemma 5.2 *Let $x^0 \in B^1$ the function $u(x) \in A^+\left(B^1(x^0)\right)$, and $R \le R_0$. If there exist $\beta \in [0,1]$ and $\varepsilon > 0$ such that*

$$\textit{meas}\left(B^1(x^0) \cap \{u(x) \ge \varepsilon\}\right) \ge \beta \textit{meas}\, B^1(x^0),$$

then

$$u(x) \ge \varepsilon\gamma(\beta), \quad \textit{as} \quad x \in E_R^{x^0}\left(\frac{1}{2}\right).$$

The assertion of Lemma 5.2 follows from definition of $\gamma(\beta)$.

Lemma 5.3 *Let the number $R \le R_0$, point $x^0 \in B^1$ and the function $u(x) \in A^+\left(B^1(x^0)\right)$. If there exist $\beta \in [0,1]$ and $\nu > 0$ such that $u(x^0) \ge \nu$ and*

$$\textit{meas}\left(B^1(x^0) \cap \left\{u(x) \le \frac{1}{2}\nu\right\}\right) \ge \beta \textit{meas}\, B^1(x^0),$$

then

$$\sup_{B^1(x^0)} u(x) \ge \frac{\nu}{2}\left(1 + \frac{1}{1-\gamma(\beta)}\right). \tag{5.6}$$

Proof. of Lemma 5.3. Suppose that inequality (5.6) fails. Then there exist $\varepsilon_1 > 0$ such that

$$\sup_{B^1(x^0)} W(x) \ge \frac{1}{1-\gamma(\beta)+\varepsilon_1} = a_1.$$

holds for the function $W(x) = \frac{2u(x)}{\nu} - 1$. Let further, $z(x) = 1 - \frac{W(x)}{a_1}$. For $z(x) \in A^+\left(B^1(x^0)\right)$ it holds $z(x) \ge 1$ since $u(x) \le \frac{\nu}{2}$. Applying Lemma 5.2 for $\varepsilon = 1$ we get $z(x^0) \ge \gamma(\beta)$. On other hand by assumptions $W(x^0) \ge 1$. Therefore,

$$1 - \frac{1}{a_1} \ge 1 - \frac{W(x^0)}{a_1} \ge \gamma(\beta), \quad \text{i.e.} \quad a_1 \ge \frac{1}{1-\gamma(\beta)}$$

and the last, contradicts to the above assumptions.
The contradiction proves Lemma 5.3.

Lemma 5.4 *It holds the limit equality*

$$\lim_{\beta \to 1-0} \gamma(\beta) = 1.$$

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Proof. of Lemma 5.4. First, rewrite the statement of Lemma 5.1. Let $u(x) \in A^+(G), \quad u|_{\partial G \cap B^1(x^0)} = 1$. Then for any $K > 0$ there exist $\delta_0(\gamma, n, b_0, c_0, K)$ such that $R \le R_1(\gamma, n, b_0, c_0, K)$ and the condition (5.1) is satisfied. Therefore,

$$\inf_{G \cap B_R(x^0)} u(x) \ge 1 - \frac{1}{K}. \tag{5.7}$$

Indeed, $G' = \{x : u(x) < 1\}$, $v(x) = 1 - u(x)$. We have $Lv = c(x) - Lu \ge -c_0$, therefore the function $v(x)$ is a $L-$ sub elliptic function on G'.

First, assume that $G' \cap B_R(x^0) \ne \varnothing$. It is possible two cases :

$$1) \sup_{G' \cap B_R(x^0)} v(x) > 0; \quad 2) \sup_{G' \cap B_R(x^0)} v(x) \le 0.$$

Let the case 1) takes place. Then according to Lemma 5.1 there exists δ_0 relevant to K such that

$$1-\inf_{G'}u(x)\geq K\left(1-\inf_{G'\cap B_R(x^0)}u(x)\right),$$

i.e.

$$\inf_{G'\cap B_R(x^0)}u(x)\geq\frac{K-1}{K}=1-\frac{1}{K}.$$

Let the case 2) takes place. Then $\inf\limits_{G'\cap B_R(x^0)}u(x)\geq 1-\frac{1}{K}$ and we come to the estimate (5.7). Since $u(x)\geq 1$ for $x\in G\backslash G'$. If $G'\cap B_R(x^0)=,$ then $u(x)\geq 1$ for $x\in G\cap B_R(x^0)$.

Therefore, the inequality (5.7) has been proved.

Now turn back to the proof of Lemma 5.4. Assume that its assertion does not hold. Fix arbitrary number $\varepsilon>0$. Then there exists an $a\in(0,1)$ such that for $\beta\in(1-\varepsilon,1)$ it holds $\gamma(\beta)<1-a$. Insert $K=\frac{4}{a}$ in (5.7) and choose proper δ_0 and R_1. Let $\varepsilon'=\min\{\varepsilon,\delta_0\}$. By definition of $\gamma(\beta)$ there exists a $R_2\leq R_1$ such that for $\beta\in(1-\varepsilon',1)$ it is $\gamma_\beta^{R_2}<1-\frac{a}{2}$. Fix arbitrary $\beta_0\in(1-\varepsilon',1)$. Then there exist $x^0\in B^1$ (for $R=R_2$)and a function $u(x)\in A_{\beta_0}^{R_2}(x^0)$, a point $x^1\in E_{R_2}^{x^0}\left(\frac{1}{2}\right)$ such that

$$u(x^1)<1-\frac{a}{4}. \tag{5.8}$$

Let $D'=\left\{x:x\in B^1(x^0),u(x)<1\right\}$. It is known that, $D'=B^1(x^0)\setminus\{u\geq 1\}$. By definition of the classes $A_{\beta_0}^{R_2}(x^0)$ we have

$$\text{meas}\,D'<(1-\beta_0)\text{meas}\,B^1(x^0)\leq\delta_0\text{meas}\,B^1(x^0).$$

Then according to (5.7) it follows

$$\inf_{D'\cap B_{R_2}(x^0)}u(x)\geq 1-\frac{a}{4}.$$

Taking into the account $u(x)\geq 1$ for $x\in B_{R_2}(x^0)\backslash D'$ it holds

$$\inf_{B_{R_2}(x^0)}u(x)\geq 1-\frac{a}{4},$$

and in particularly, $u(x^1)\geq 1-\frac{a}{4}$. The last inequality contradicts (5.8).

This proves Lemma 5.4.

Lemma 5.5 *Let* $R\leq R_0,\ \ H\in\left[\frac{1}{4},1\right],\ \ \sigma\in(0,1],\ 4H\omega_i^{-1}(R)\leq|x_i^2-x_i^1|\leq 8H\omega_i^{-1}(R),\ \ x_i^1+H\omega_i^{-1}(R)\leq x_i^0\leq x_i^2-H\omega_i^{-1}(R),\ i=1,...,n,\ x^0\in E_R^0(4),$ $N=\left\{x:x_i^1<x_i<x_i^2\right\}$. *Then there exists* $m(\gamma,n,b_0,c_0,H)$ *such that* $u(x)\geq\sigma^m$ *for* $x_i^1+H\omega_i^{-1}(R)\leq x_i\leq x_i^2-H\omega_i^{-1}(R),\ i=1,...,n$ *provided that* $u(x)\in A^+(N)$ *and* $u(x)\geq 1$ *for* $x\in E_R^{x^0}(\sigma)$.

Proof. of Lemma 5.5. Without loss of generality, we may assume that $x^0=0$ and $2\sigma^2<H^2$. Fix a point $x^*\in\overline{N}$ such that $x_i^1+H\omega_i^{-1}(R)\leq x_i^*\leq x_i^2-H\omega_i^{-1}(R)$, $i=1,...,n$. Denote $\zeta=\frac{H}{4},\ \ y=\frac{x^*}{2H}$. Consider the set

$$S=\left\{x:\sum_{i=1}^{n}\frac{(x_i-2Hy_i)^2}{\left(\omega_i^{-1}(R)\right)^2}<2\zeta H+\sigma^2\right\}.$$

It is not difficult to see that the set S is contained in N. Indeed,

$$\sum_{i=1}^{n}\frac{(x_i-2Hy_i)^2}{\left(\omega_i^{-1}(R)\right)^2}=\sum_{i=1}^{n}\frac{(x_i-x_i^*)^2}{\left(\omega_i^{-1}(R)\right)^2}<2\frac{H}{4}H+\frac{H^2}{2}=H^2.$$

Therefore, for the indicated points it is satisfied $|x_i-x_i^*|<H\omega_i^{-1}(R)$ i.e. $|x_i-x_i^*|<H\omega_i^{-1}(R)$, and $x_i<x_i^*+H\omega_i^{-1}(R)\le x_i^2$, $x_i-x_i^*>-H\omega_i^{-1}(R)$, $x_i>x_i^*-H\omega_i^{-1}(R)\ge x_i^1$, $i=1,...,n$. It follows from the convexity of N that the set S entirely located in $\overline{N}$. Note, the boundary of S is the set

$$\partial S=\left\{x:\sum_{i=1}^{n}\frac{(x_i-2Hy_i)^2}{\left(\omega_i^{-1}(R)\right)^2}=2\zeta H+\sigma^2\right\}.$$

For $x\in\overline{S}$ introduce a function

$$z_i(x)=\frac{x_i-2Hy_i}{\sqrt{2\zeta H+\sigma^2}},\quad i=1,...,n,$$

$$\varphi(x)=\frac{(1-r(x))^2}{(2\zeta H+\sigma^2)^d},\ r(x)=\frac{1}{2\zeta H+\sigma^2}\sum_{i=1}^{n}\frac{(x_i-2Hy_i)^2}{\left(\omega_i^{-1}(R)\right)^2},$$

where d-is a positive constant.

It is not difficult to see that $0\le r(x)\le 1$ for $x\in\overline{S}$, moreover $r|_{\partial S}=1$. We have

$$L\varphi=(2\zeta H+\sigma^2)^{-d-1}\left\{8\sum_{i,j=1}^{n}\frac{a_{ij}(x)z_iz_j}{\left(\omega_i^{-1}(R)\right)^2\left(\omega_j^{-1}(R)\right)^2}\right.$$

$$\left.+2(r-1)\left[2\sum_{i=1}^{n}\frac{a_{ii}(x)}{\left(\omega_i^{-1}(R)\right)^2}+2\sum_{i=1}^{n}b_i(x)\frac{x_i-2Hy_i}{\left(\omega_i^{-1}(R)\right)^2}+\frac{(r-1)C(x)(2\zeta H+\sigma^2)}{2}\right]\right\}. \tag{5.9}$$

From (1.2) it follows that

$$\sum_{i,j=1}^{n}\frac{a_{ij}(x)z_iz_j}{\left(\omega_i^{-1}(R)\right)^2\left(\omega_j^{-1}(R)\right)^2}\ge\gamma\sum_{i=1}^{n}\lambda_i(x)\frac{z_i^2}{\left(\omega_i^{-1}(R)\right)^4},$$

$$\sum_{i=1}^{n}\frac{a_{ii}(x)}{\left(\omega_i^{-1}(R)\right)^2}\le\frac{1}{\gamma}\sum_{i=1}^{n}\frac{\lambda_i(x)}{\left(\omega_i^{-1}(R)\right)^2} \tag{5.10}$$

On other hand for $x\in S$ and $i=1,...,n$

$$C_1(\gamma,n,H)\left(\frac{\omega_i^{-1}(R)}{R}\right)^2\le\lambda_i(x)\le C_2(\gamma,n,H)\left(\frac{\omega_i^{-1}(R)}{R}\right)^2. \tag{5.11}$$

Using (5.11) in (5.10), we infer

$$\begin{gathered}\sum_{i,j=1}^{n}\frac{a_{ij}(x)z_iz_j}{\left(\omega_i^{-1}(R)\right)^2\left(\omega_j^{-1}(R)\right)^2}\ge C_3(\gamma,n,H)r,\\ \sum_{i=1}^{n}\frac{a_{ii}(x)}{\left(\omega_i^{-1}(R)\right)^2}\le C_4(\gamma,n,H).\end{gathered} \tag{5.12}$$

Further, and using (1.4) it follows

$$\left|\sum_{i=1}^{n} b_i(x)\frac{x_i - 2Hy_i}{\left(\omega_i^{-1}(R)\right)^2}\right| \le b_0 \left(\sum_{i=1}^{n}\frac{(x_i - 2Hy_i)^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2}$$

$$\times\left(\sum_{i=1}^{n}\frac{1}{\left(\omega_i^{-1}(R)\right)^2}\right) \le C_5(\gamma, n, H), \tag{5.13}$$

and by analogy,

$$\left|\frac{r-1}{2}c(x)(2\zeta H + \sigma^2)\right| \le C_6(\gamma, n, H, c_0). \tag{5.14}$$

Applying (5.12)-(5.14) in (5.9), we get

$$L\varphi \ge (2\zeta H + \sigma^2)^{-d-1}\left\{8C_3 r - 2(1-r)(2C_4 + 2C_5 + C_6)\right\}.$$

Therefore, it follows that $\frac{2C_4+2C_5+C_6}{4C_3+2C_4+2C_5+C_6} \le r < 1$ and therefore, $\varphi(x)$ is a L- sub elliptic function in S.

Now, let $u(x) \in A^+(N)$ and for $x \in \overline{E}_R^{x^0}(\sigma)$ it is $u(x) \ge 1$. Insert an auxiliary function $\omega(x) = u(x) - \sigma^{2\alpha}\varphi(x)$. It is clear that, $\omega(x) \in A^+(S)$. Also

$$\omega(x)|_{\partial S} = u(x)|_{\partial S} - \sigma^{2d}\,\varphi(x)|_{\partial S} \ge 1 - \sigma^{2d}\left.\frac{(1-r)^2}{(2\zeta H + \sigma^2)^d}\right|_{\partial S}$$

$$= 1 - (1-r)^2|_{\partial S}\frac{\sigma^{2d}}{(2\zeta H + \sigma^2)^d} \ge 0.$$

By using of the maximum principle, for $x \in \overline{S}$ it is $\omega(x) \ge 0$. In particular, for a point x^* where $r = 0$ we get

$$u(x^*) \ge \sigma^{2d}\varphi(x^*) = \frac{\sigma^{2d}}{(2\zeta H + \sigma^2)^d} \ge \sigma^{2d}.$$

Now, it suffices to choose $m = 2d, d > 0$ and this finishes the proof of the Lemma 5.5.

Theorem 5.1 *Let $u(x)$ be a positive solution of equation* (1.1) *in the domain D. Let the coefficients of operator L satisfy the conditions* (1.2)-(1.4). *Then if $\overline{B^1} \subset D$ and $R \le R_0$ it holds*

$$u(0) \le C_7(\gamma, n, b_0, C_0)\inf_{B^4} u(x). \tag{5.15}$$

Proof. of Theorem 5.1. Let number m is that taken from preceding Lemma 5.5 corresponds to $H = 1$. Fix such m and find a $\beta \in (0,1)$ in accordance with Lemma 5.3

$$\frac{1}{2}\left(1 + \frac{1}{1 - \gamma(1-\beta)}\right) \ge 2^m. \tag{5.16}$$

For $r \in (0,1)$, $Q(r) = \left\{x : x \in \overline{E_R^0}(r)\right\}$, set $\nu(r) = u(0)(1-r)^{-m}$, $g(r) = \max_{Q(r)} u(x)$.

Let r_1- be the maximal root of equation $g(r) = v(r)$. It is easily seen that $g(0) = \nu(0)$, $\lim_{r\to 1-0}\nu(r) = \infty$ and the function $g(r)$ is bounded and continues, therefore there exist a solution of equation $g(r) = v(r)$ such that $r_1 < 1$.

Let $x^* \in Q(r)$, $\quad g(r_1) = v(r_1) = u(x^*)$, $\ F = \left\{x : x \in E_R^{x^*}\left(\frac{1-r_1}{2}\right)\right\}$. For $x \in \overline{F}$ we have

$$\left(\sum_{i=1}^{n} \frac{x_i^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2} \le \left(\sum_{i=1}^{n} \frac{x_i^*}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2} + \left(\sum_{i=1}^{n} \frac{(x_i - x_i^*)^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2}$$
$$\le r_1 + \frac{1-r_1}{2} = \frac{1+r_1}{2}.$$

On other hand,

$$\frac{(1-r_1)}{2}\omega_i^{-1}(R) < \frac{1+r_1}{2}\omega_i^{-1}(R), \ \ i = 1, ..., n.$$

Therefore, $\overline{F} \subset Q\left(\frac{1+r_1}{2}\right)$, i.e. using (5.15) and $\frac{1+r_1}{2} > r_1$ for $x \in \overline{F}$ it holds

$$u(x) \le g\left(\frac{1+r_1}{2}\right) < v\left(\frac{1+r_1}{2}\right) = u(0)\left(1 - \frac{1+r_1}{2}\right)^m = u(0)\left(\frac{1-r_1}{2}\right)^{-m}$$
$$= 2^m u(0)(1-r_1)^{-m} = 2^m v(r_1) < \frac{v(r_1)}{2}\left(1 + \frac{1}{1-\gamma(1-\beta)}\right). \tag{5.17}$$

Now, if we assume that

$$\text{meas}\left(F \cap \left[u \le \frac{v(r_1)}{2}\right]\right) \ge (1-\beta)\text{meas}\, F$$

then from equality $u(x^*) = u(r_1)$ and Lemma 5.3 it follows

$$\sup_F u(x) \ge \frac{v(r_1)}{2}\left(1 + \frac{1}{1-\gamma(1-\beta)}\right).$$

Last inequality contradicts to (5.17), where it has been used that $u(x)$ is a solution of (1.1).

Thus

$$\text{meas}\left(F \cap \left\{u \le \frac{v(r_1)}{2}\right\}\right) < (1-\beta)\text{meas}\, F,$$

i.e.

$$\text{meas}\left(F \cap \left\{u \ge \frac{v(r_1)}{2}\right\}\right) \ge \beta\text{meas}\, F. \tag{5.18}$$

Now, we will use Lemma 5.5. Two cases are possible: $r_1 > \frac{1}{3}$ and $r_1 \in \left(0, \frac{1}{3}\right]$. Let the first case takes place. Insert

$$x^0 = \frac{9r_1 - 1}{8r}x^*, \quad H = \frac{9r_1 - 1}{8}.$$

It is not difficult to see that $\frac{1}{4} \le H \le 1$. Now,

$$\sigma = \frac{1-r_1}{8}, \quad \text{and} \quad E_R^{x^0}(\sigma) \subset E_R^{x^*}\left(\frac{1-r_1}{4}\right).$$

Indeed, let $x \in E_R^{x^0}(\sigma)$, then

$$\left(\sum_{i=1}^{n} \frac{(x_i - x_i^0)^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^2 < \frac{(1-r_1)^2}{64}.$$

Therefore,

$$\left(\sum_{i=1}^{n}\frac{(x_i-x_i^*)^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2}\leq\left(\sum_{i=1}^{n}\frac{(x_i-x_i^0)^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2}$$

$$+\left(\sum_{i=1}^{n}\frac{(x_i^0-x_i^*)^2}{\left(\omega_i^{-1}(R)\right)^2}\right)^{1/2}\leq\frac{1-r_1}{8}+\frac{1-r_1}{8r_1}r_1=\frac{1-r_1}{4}.$$

Insert $x_i^1=-2H\omega_i^{-1}(R),\quad x_i^2=2H\omega_i^{-1}(R),\quad i=1,...,n$. Then from Lemmas 5.2 and 5.5 using (5.18) it follows that for $x\in B^4$ it holds

$$u(x)\geq\left(\frac{1-r_1}{8}\right)^m\frac{v(r_1)}{2}\gamma(\beta)=8^{-m}(1-r_1)^m\frac{1}{2}u(0)(1-r_1)^{-m}\gamma(\beta)$$

$$=2^{-3m-1}\gamma(\beta)u(0). \tag{5.19}$$

Let now, $r_1\in\left(0,\frac{1}{3}\right]$, σ and H are in the same meaning as above. Set $H=1$, then

$$x^0=\frac{7r_1+1}{8r_1}x^*,\ \ x_i^1=-\left(\frac{8r_1}{7r_1+1}+1\right)\omega_i^{-1}(R),$$

$$x_i^2=\left(\frac{8r_1}{7r_1+1}+1\right)\omega_i^{-1}(R),\ \ i=1,...,n.$$

Therefore, using Lemmas 5.2, 5.5 and inequality (5.18) again we come to the estimate (5.19). Therefore, the inequality (5.15) has been proved with $C_7=\frac{2^{3m+1}}{\gamma(\rho)}$.

Proof. of Theorem 3.1 easily follows from Theorem 5.1 (see, e.g. [20, Theorem 10.2]).

6 Proof of Theorem 4.1

Proof. Consider the ellipsoid $B^4=E_R^0\left(\frac{1}{4}\right)$. Make a change of variables $y_i=\frac{Rx_i}{\omega_i^{-1}(R)}, i=1,...,n$. Therefore, the image of B^4 under this transform will be the ball $B_{R/4}(0)$. Let $\widetilde{u}(y)$ is an image of $u(x)$ under this transform. Then the equation (1.1) in the variables y looks as following:

$$\widetilde{L}\,\widetilde{u}(y)=\sum_{i,j=1}^{n}\widetilde{a}_{ij}(y)\frac{\partial^2u}{\partial y_i\partial y_j}+\sum_{i=1}^{n}\widetilde{b}_i(y)\frac{\partial\,\widetilde{u}}{\partial y_i}+\widetilde{c}_i(y)\widetilde{u}=0,$$

where

$$\widetilde{a}_{ij}(y)=\frac{R^2}{\omega_i^{-1}(R)\omega_j^{-1}(R)}a_{ij}\left(\frac{y_1\omega_1^{-1}(R)}{R},...,\frac{y_n\omega_n^{-1}}{R}\right),\ \ i,j=1,...,n,$$

$$\widetilde{b}_i(y)=\frac{R}{\omega_i^{-1}(R)}b_i\left(\frac{y_1\omega_1^{-1}(R)}{R},...,\frac{y_n\omega_n^{-1}(R)}{R}\right),$$

$$\widetilde{c}(y)=c\left(\frac{y_1\omega_1^{-1}(R)}{R},...,\frac{y_n\omega_n^{-1}(R)}{R}\right),\quad i=1,...,n.$$

For $y \in B'_{R/4}(0)$ and arbitrary $\xi \in E_n$ according to (1.2) we have

$$\gamma \sum_{i=1}^{n} \widetilde{\lambda}_i \left(\frac{y_1 \omega_1^{-1}(R)}{R}, ..., \frac{y_n \omega_n^{-1}(R)}{R} \right) \frac{R^2}{(\omega_i^{-1}(R))^2} \xi_i^2$$

$$\leq \sum_{i,j=1}^{n} \widetilde{a}_i(y) \xi_i \xi_j \leq \gamma^{-1} \sum_{i,j=1}^{n} \widetilde{\lambda}_i(y) \frac{R^2}{(\omega_i^{-1}(R))^2} \xi_i^2. \tag{6.1}$$

On other hand, for $y \in B'_{R/4}(0)$ it holds the inequality

$$C_8(\gamma, n, b_0, c_0) \left(\frac{\omega_i^{-1}(R)}{R} \right)^2 \leq \widetilde{\lambda}_i(y) \leq C_9(\gamma, n, b_0, c_0) \left(\frac{\omega_i^{-1}(R)}{R} \right)^2, \;\; i = 1, ..., n. \tag{6.2}$$

Therefore and using (6.2) in (6.1), we have

$$\gamma C_8 \sum_{i,j=1}^{n} \xi_i^2 \leq \sum_{i,j=1}^{n} \widetilde{a}_{ij}(y) \xi_i \xi_j \leq \frac{C_9}{\gamma} \sum_{i,j=1}^{n} \xi_i^2.$$

In addition, $R \leq R_0 \leq 1$ and the conditions (1.3) and (1.4) is fulfilled. Therefore, $\left| \widetilde{b}_i(y) \right| \leq b_0, \;\; -c_0 \leq \widetilde{c}(y) \leq 0, \;\; i = 1, ..., n.$

The assumption meas $H \geq \sigma$meas B^5 on the complementary set near origin $H = B^5 \setminus D$. will be transferred to

$$\left(\prod_{i=1}^{n} \frac{\omega^{-1}(R)}{R} \right) \text{meas}\, \widetilde{H} \geq \sigma' \text{meas}\, B'_{R/4}(0) \left(\prod_{i=1}^{n} \frac{\omega^{-1}(R)}{R} \right),$$

where $\widetilde{H} = B'_R(0) \setminus \widetilde{D}$ and $\sigma' = c\sigma$. Therefore, meas $\widetilde{H} \geq \sigma'$meas $B'_R(0)$ which according to the growth property of positive solutions on domains with fat complementary of the uniformly elliptic equations of second order (see, e.g. [15], [18], [19]) it follows

$$\sup_{B'_R(0)} \widetilde{u}(y) \geq \big(1 + \eta_0\big) \widetilde{u}(0)$$

by some $\eta = \eta_0\big(\gamma, n, b_0, c_0\big)$. After inverse change of variables to x, we get

$$\sup_{B^4} u(x) \geq \; (1 + \eta)\, u(0),$$

which easiy implies Theorem 3.1.

Theorem 3.1 has been proved.

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