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Mixed problem for systems of one-dimensional wave equations with a nonlinear boundary condition and a nonstandard internal source

Gulshan Kh. Shafieva⋆

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Abstract. *In this paper, we study a mixed problem for systems of one-dimensional semilinear wave equations with a nonstandard internal source and nonlinear boundary conditions. Existence theorems for local as well as global solutions are proved.*

Keywords. mixed problem, system of wave equations, nonlinear boundary condition, local solution, global solution.

Mathematics Subject Classification (2010): 35L20,35L70, 35B44

1 Introduction

Nonlinear wave equations with a nonlinear source having a constant growth have been studied in the works of various authors [1-10].

Recently, much attention has been paid to the study of nonlinear models of partial differential equations with variable exponents nonlinearity [11–17] . Mathematical models of physical processes such as the flow of electrorheological fluids or fluids with temperature-dependent viscosity, filtration in porous media, nonlinear viscoelasticity, etc., are reduced to nonlinear hyperbolic equations with nonstandard nonlinearity. More detailed information on these issues can be found in [11–15]. To date, there are several works devoted to the study of hyperbolic problems with a non-standard internal source [16,20,21]. In [18,19], initial-boundary value problems with the Dirichlet boundary condition for a nonlinear wave equation with a variable growth order of nonlinearity were studied. It is proved that for some relations between the growth orders of a nonlinear source and nonlinear dissipation, the solution of the corresponding initial-boundary value problem blow-up in a finite time.

Various problems of mechanics and optical physics are reduced to the study of mixed problems for wave equations with nonstationary boundary conditions [20], see also [22–28].

In this paper, we study the initial-boundary value problem for systems of semi-linear hyperbolic equations with nonlinear boundary conditions and a nonlinear source with variable growth order. The existence of local as well as global solutions is proved.

⋆ Corresponding author

G.Kh. Shafieva
Institute of Mathematics and Mechanics of NAS of Azerbaijan
Baku State University
E-mail: gulshan.shafiyeva@mail.ru

The paper is structured as follows: Section 2 introduces some notation and formulates the main results; Section 3 proves the existence and uniqueness of local solutions of the problem under consideration; Section 4 investigates the global resolvability solutions.

2 Problem statement and main results

In this paper, we are consider for the following initial-boundary value problem:

$$u_{i_{tt}}-(a_i(x)u_{i_x})_x=\theta f_i(x,u_1,u_2), 0\le x\le l,\quad t>0, \tag{2.1}$$

$$u_i(x,0)=u_{i0}(x),\quad u_{i_t}(x,0)=u_{i1}(x),\quad 0\le x\le l, \tag{2.2}$$

$$u_i(0,t)=0,\quad t>0, \tag{2.3}$$

$$a_i(l)u_{i_x}(l,t)+|u_{i_t}(l,t)|^{r_i-1}u_{i_t}(l,t)=\mu g_i(u_1(l,t),u_2(l,t)),\quad t>0, \tag{2.4}$$

where $i=1,2,\ r_1\ge 1,\ r_2\ge 1,\ \theta\in\mathbb{R},\ \mu\in\mathbb{R},\ u_1=u_1(x,t),\ u_2=u_2(x,t), u_{10}(x)$, $u_{20}(x),\ u_{11}(x),\ u_{21}(x),\ a_1(x),\ a_2(x)$ -real functions,

$$a_i(x)\in C^1[0,l],\quad a_i(x)\ge a_{i0}>0,\quad 0\le x\le l,\quad i=1,2, \tag{2.5}$$

$$\begin{aligned} f_i(x,\ u_1,u_2)&=|u_1+u_2|^{2p(x)}(u_1+u_2)\\ &+|u_1|^{p(x)+(-1)^i}|u_2|^{p(x)-(-1)^i}u_i,\quad i=1,2, \end{aligned} \tag{2.6}$$

$$\begin{aligned} g_i(\alpha_1,\alpha_2)&=|\alpha_1+\alpha_2|^{2q}(\alpha_1+\alpha_2)\\ &+|\alpha_1|^{q+(-1)^i}|\alpha_2|^{q-(-1)^i}\alpha_i,\quad i=1,2,\quad \alpha_1,\alpha_2\in\mathbb{R}. \end{aligned} \tag{2.7}$$

Suppose that the $p(\cdot)$ is a measurable function satisfies the Hölder logarithmic continuity condition, i.e. for any $x,y\in[0,l]$ such that

$$|x-y|<\delta,\quad 0<\delta<1$$

the inequality,

$$|p(x)-p(y)|\le\frac{C}{|\log|x-y||}, \tag{2.8}$$

is satisfied, where $C>0$ is independent of x,y. Let us also assume that

$$2\le p_1\le p(x)\le p_2<+\infty,\quad 0\le x\le l, \tag{2.9}$$

where $p_1=\operatorname*{ess\,inf}_{x\in[0,l]}p(x),\quad p_2=\operatorname*{ess\,sup}_{x\in[0,l]}p(x)$.

Let's assume that

$$q\ge 2, \tag{2.10}$$

We introduce the energy functional is determined by the equality

$$E(t)=E_0(t)-R(u_1(\cdot,t),u_2(\cdot,t)), \tag{2.11}$$

where

$$E_0(t)=\frac{1}{2}\sum_{i=1}^{2}\left[\|u_{i_t}(\cdot,t)\|_2^2+\left\|\sqrt{a_i(x)}u_{i_x}(\cdot,t)\right\|_2^2\right],$$

$$R(u_1,u_2)=F(u_1,u_2)+G(u_1,u_2),$$

$$F(u_1,u_2)=\int_0^l\frac{\theta}{2(p(x)+1)}|u_1(x,t)+u_2(x,t)|^{2(p(x)+1)}dx$$

$$+\int_0^l \frac{\theta}{p(x)+1} \left|u_1(x,t)\cdot u_2(x,t)\right|^{p(x)+1} dx,$$

$$G(u_1,u_2)=\frac{\mu}{2(q+1)}\left|u_1(l,t)+u_2(l,t)\right|^{2(q+1)}+\frac{\mu}{q+1}\left|u_1(l,t)\cdot u_2(l,t)\right|^{q+1},$$

and $\|\cdot\|_2$ - is norm in space $L_2(0,1)$.

We present some notation and definitions. Let $H^1=\{u;\, u,u_x\in L_2(0,l)\}$ denote the Sobolev space with the usual scalar products and norm. Moreover, ${}_0H^1$ denotes the closure in H^1 of $C^\infty[0,l)$.

Definition 2.1 *A strong solution to problem* (2.1) - (2.4) *in the domain* $(0,l)\times\ (0,T)$ *is a pair* $(u_1(x,t),u_2(x,t))$ *functions such that* $u_i(\cdot)\in L_\infty(0,T;\ H^2\bigcap{}_0H^1)$, $u_{it}(\cdot)\in L_\infty(0,T;{}_0H^1)$, $u_{itt}(\cdot)\in L_\infty(0,T;L_2(0,l))$, $i=1,2$ *and for almost all* $(x,t)\in(0,l)\times\ (0,T)$ *satisfying system* (2.1)*, as well as initial condition* (2.2) *and boundary conditions* (2.3)*,* (2.4)*.*

Definition 2.2 *By a weak solution to problem* (2.1) - (2.4) *we mean a pair* $(u_1(x,t),u_2(x,t))$ *functions such that*

a) $u_i(\cdot)\in C_w\left([0,T];\ {}_0H^1\right)$, $\quad u_{i_t}(\cdot)\in C_w\left([0,T];L_2(0,l)\right)$, $\quad i=1,2;$

b) The trace of u_{it} *in* $\{l\}\times(0,T)$ *belongs to* $L^{r_i}(0,T)$*, i.e.* $u_{it}(l,t)\in L^{r_i}(0,T)$, $\quad i=1,2;$

c) For all $\eta_1(\cdot),\eta_2(\cdot)\in C_w\left([0,T];{}_0H^1\right)$*, where* $\eta_{1t}(\cdot),\eta_{2t}(\cdot)\in C_w\left([0,T];L_2(0,l)\right)$, $\eta_{i_t}(l,\cdot)\in L^{r_i}(0,T)$, $\eta_i(x,T)=0$, $0\le x\le l$, $i=1,2$ *the following equalities hold:*

$$\int_0^T\int_0^l\left[-u_{i_t}(x,t)\eta_{i_t}(x,t)+u_{i_x}(x,t)\eta_{i_x}(x,t)\right]dxdt$$

$$+\int_0^T\left|u_{i_t}(l,t)\right|^{r_i-1}u_{i_t}(l,t)\eta_i(l,t)dt+\int_0^l u_{i1}(x)\eta_i(x,0)dx$$

$$=\theta\int_0^T\int_0^l f_i(x,u_1,u_2)\eta_i(x,t)dxdt+\mu\int_0^T g_i(u_1(l,t),u_2(l,t))\eta_i(l,t)dt;$$

and $\lim\limits_{t\to0}\left\langle u_i(\cdot,t)-u_{i0}(\cdot),\eta_i(\cdot,t)\right\rangle_{{}_0H^1}=0.$

Here $C_w\left([0,T];\ Y\right)$ denotes the space of weakly continuous functions with values in a Banach space Y.

We define the following classes of functions:

$$C^1_{T,i}=\{v:v\in C([0,T];{}_0H^1),v_t(\cdot)\in C([0,T];L_2(0,l)),v_t(l,\cdot)\in L^{r_i}(0,T)\},i=1,2,$$

$$C^2_T=\Big\{v:v\in C([0,T];H^2\bigcap{}_0H^1\ ),$$

$$v_t(\cdot)\in C([0,T];{}_0H^1),v_{tt}(\cdot)\in C([0,T];L_2(0,l))\Big\}.$$

The following theorems on the local solvability of the problem (2.1) - (2.4) are valid.

Theorem 2.1 *Let the conditions* (2.5) - (2.10) *be satisfied. Then for any initial data* $u_{i0}\in{}_0H^1$, $u_{i1}\in L_2(0,l),i=1,2$ *there exists such* $T'\in(0,T]$*, that problem* (2.1) - (2.4) *has a weak solution* $(u_1(x,t),u_2(x,t))$ *in the domain of* $(0,l)\times(0,T')$*, and* $u_i(\cdot)\in C^1_{T,i}$, $\quad i=1,2$*. Moreover, for the weak local solutions the following energy equality hold:*

$$E(t)+\sum_{i=1}^2\int_0^t\left|u_{i_\tau}(l,\tau)\right|^{r_i+1}d\tau=E(0),\quad 0\le t\le T'. \tag{2.12}$$

The following global solvability theorems are also proved.

Theorem 2.2 *Let the conditions* (2.5) - (2.10) *and (2.13) be satisfied. Assume that*

$$\theta \leq 0, \quad \mu \leq 0. \tag{2.13}$$

Then for any initial data $u_{i0} \in H^2 \bigcap {}_0H^1, \quad u_{i1} \in {}_0H^1, \quad i = 1,2$ *and* $T > 0$ *the problem* (2.1)-(2.4) *has a global strong solution* $(u_1(x,t), u_2(x,t))$ *in the domain of* $(0,l) \times (0,T')$, *such that* $u_i(\cdot) \in C_T^2$ *and the energy equality* (2.12) *is satisfied.*

Theorem 2.3 *Let the conditions* (2.5) - (2.10) *satisfied. Assume that*

$$\theta \leq 0, \ \mu > 0, \ 2q + 1 \leq \min\{r_1, r_2\}. \tag{2.14}$$

Then for any $u_{i0} \in H^2 \bigcap {}_0H^1, \quad u_{i1} \in {}_0H^1, \quad i = 1,2$ *and* $T > 0$ *the problem (2.1)-*(2.4) *has a global strong solution* $(u_1(x,t), u_2(x,t))$ *such that* $u_i(\cdot) \in C^2_{T,}$ *and the energy equality* (2.12) *is satisfied.*

3 Local Solvability of Problem (2.1) - (2.4)

Let us consider the following initial - boundary value problem:

$$u_{i_{tt}} - (a_i(x) u_{i_x})_x = \theta f_i(x, u_1, u_2), 0 < x < l, \quad t > 0, \tag{3.1}$$

$$u_i(x,0) = u_{i0}(x), \quad u_{i_t}(x,0) = u_{i1}(x), \quad 0 < x < l, \tag{3.2}$$

$$u_i(0,t) = 0, \quad t > 0, \tag{3.3}$$

$$a_i(l) u_{i_x}(l,t) + \chi_i(u_{i_t}(l,t)) = \mu g_i(u_1(l,t), u_2(l,t)), \quad t > 0, \tag{3.4}$$

where $i = 1,2, \ r_1 \geq 1, \ r_2 \geq 1, \ \theta \in \mathbb{R}, \ \mu \in \mathbb{R}, u_1 = u_1(x,t), \ u_2 = u_2(x,t), \ u_{10}(x),$ $u_{20}(x), \ u_{11}(x), \ u_{21}(x), \ a_1(x), \ a_2(x)$ - are real-valued functions, moreover $a_i(x), \quad i = 1,2$ satisfy condition (2.5), and the $f_i, g_i, \quad i = 1,2$ satisfy conditions (2.6) - (2.10), $\chi_i, \quad i = 1,2$ is a real function satisfying the following conditions

$$\chi_i(\cdot) \in C(R), \quad \chi_i(0) = 0, \quad i = 1,2 \tag{3.5}$$

and there is a constant $\chi_0 > 0$ such that for any $\xi_i, \eta_i \in \mathbb{R}, i = 1,2$ the following inequality holds:

$$\sum_{i=1}^{2} [\chi_i(\xi_i) - \chi_i(\eta_i)] (\xi_i - \eta_i) \geq \chi_0 \sum_{i=1}^{2} |\xi_i - \eta_i|^2. \tag{3.6}$$

Let us define the class of functions

$$C^1_{T',\chi_i} = \{v : v \in C([0,T']; H^2 \bigcap {}_0H^1), \quad v_t(\cdot) \in C([0,T']; {}_0H^1),$$
$$v_{tt}(\cdot) \in C([0,T']; L_2(0,l)), \quad \chi_i(v_{i_t}(l,\cdot)) v_{i_t}(l,\cdot) \in L_1(0,T')\}, \quad i = 1,2.$$

Theorem 3.1 *Let the conditions* (2.5) - (2.10), (3.5) *and* (3.6) *be satisfied. Then for any initial data* $u_{i0} \in H^2 \bigcap {}_0H^1, \quad u_{i1} \in {}_0H^1$ *,* $i = 1,2$ *for which* $a_i(l) u_{i0x}(l) + \chi_i(u_{i0_t}(l)) = g_i(u_{10}(l), u_{20}(l)), \quad i = 1,2$ *there exists such* $T' \in (0,T]$, *that problem (3.1)* - (3.4) *has a unique strong solution* $(u_1(x,t), u_2(x,t))$ *in the domain* $(0,l) \times (0,T')$.

For any initial data $u_{i0} \in {}_0H^1, \quad u_{i1} \in L_2(0,l), \quad i = 1,2$ *there exist such* $T' \in (0,T]$, *that problem* (3.1) - (3.4) *has a unique weak solution* $(u_1(x,t), u_2(x,t))$ *in the domain* $(0,l) \times (0,T')$, *and* $u_i(\cdot) \in C^1_{T',\chi_i}, \quad i = 1,2$. *Moreover, weak local solutions for* $0 \leq t \leq T'$ *the following energy equality hold:*

$$E(t) + \sum_{i=1}^{2} \int_0^t \chi_i(u_{i_\tau}(l,\tau)) u_{i_\tau}(l,\tau) d\tau = E(0). \tag{3.7}$$

Proof. Let $K > 0, \quad K_1 = \frac{1}{\sqrt{l}}K$. Define the following cut functions:

$$g_{iK}(\xi_1,\xi_2) = \begin{cases} g_i(\xi_1,\xi_2), & |\xi_1| \le K, \quad |\xi_2| \le K, \\ g_i(K\frac{\xi_1}{|\xi_1|},\xi_2), & |\xi_1| > K, \quad |\xi_2| \le K, \\ g_i(\xi_1, K\frac{\xi_2}{|\xi_2|}), & |\xi_1| \le K, \quad |\xi_2| > K, \\ g_i(K\frac{\xi_1}{|\xi_1|}, K\frac{\xi_2}{|\xi_2|}), & |\xi_1| > K, \quad |\xi_2| > K, \end{cases}$$

$$f_{iK}(x,u_1,u_2) = \begin{cases} f_i(x,u_1,u_2), & \|u_1\|_{{}_0H^1} \le K_1, \quad \|u_2\|_{{}_0H^1} \le K_1, \\ f_i(x,K_1\frac{u_1}{\|u_1\|_{{}_0H^1}},u_2), & \|u_1\|_{{}_0H^1} > K_1, \quad \|u_2\|_{{}_0H^1} \le K_1, \\ f_i(x,u_1,K_1\frac{u_2}{\|u_2\|_{{}_0H^1}}), & \|u_1\|_{{}_0H^1} \le K_1, \quad \|u_2\|_{{}_0H^1} > K_1, \\ f_i(x,K_1\frac{u_1}{\|u_1\|_{{}_0H^1}},K_1\frac{u_2}{\|u_2\|_{{}_0H^1}}), & \|u_1\|_{{}_0H^1} > K_1, \quad \|u_2\|_{{}_0H^1} > K_1 \end{cases}.$$

Consider for the following initial - boundary value problem:

$$u_{i_{tt}} - (a_i(x)u_{i_x})_x = \theta f_{i,K}(x,u_1,u_2), 0 < x < l, \quad t > 0, \tag{3.8}$$

$$u_i(x,0) = u_{i0}(x), \quad u_{i_t}(x,0) = u_{i1}(x), \quad 0 < x < l, \tag{3.9}$$

$$u_i(0,t) = 0, \quad t > 0, \tag{3.10}$$

$$a_i(l)u_{i_x}(l,t) + \chi_i(u_{i_t}(l,t)) = \mu g_{iK}(u_1(l,t),u_2(l,t)), \quad t > 0, \tag{3.11}$$

(see [3–8]).

In the space $\mathcal{H}=\left[{}_0H^1 \times L_2(0,l)\right]^2$ we introduce the following scalar product:

$$\langle w^1, w^2\rangle = \sum_{i=1}^{2}\left[\int_0^l a_i(x)u^1_{i_x}(x)u^2_{i_x}(x)dx + \int_0^l v^1_i(x)v^2_i(x)dx\right].$$

In the space $L_2(0,l)$ we define the linear operator ${}^i_0\Delta$:

$$D({}^i_0\Delta) = \left\{y : y \in H^2(0,l),\, y(0) = 0,\, y'(l) = 0\right\},$$

$${}^i_0\Delta y(x) = -(a_i(x)y_x(x))_x,\ x \in (0,l),\, i = 1,2,$$

as well as the linear operator $N_i : \mathbb{R} \to H^2 : \alpha \in \mathbb{R},\, \alpha \to h_i = N_i\alpha$, where $i = 1,2$, $(a_i(x)h_{i_x}(x))_x = 0, \quad 0 < x < l,\ h_i(0) = 0, \quad h_{i_x}(l) = \frac{\alpha}{a_i(l)}$, i.e. $h_i(x) = \alpha\int_0^x \frac{ds}{a_i(s)}$. Hence we get that $(N_i\alpha)_x = \alpha\frac{1}{a_i(x)}, \quad 0 \le x \le l, \quad i = 1,2$.

Using the definition of the adjoint operator, we have $\langle N_i\alpha, u\rangle_{L_2(0,l)} = (\alpha, N_i^*u)_{\mathbb{R}} = \alpha N_i^*u$, where $N_i^*u = \int_0^l u(x)\int_0^x \frac{ds}{a_i(s)}dx, \quad i = 1,2$.

Let $z \in D({}^i_0\Delta),\ \alpha \in \mathbb{R}$, then

$$N_i^*\, {}^i_0\Delta z\cdot\alpha = ({}^i_0\Delta z, N_i\alpha)_{L_2(0,l)} = \left({}^i_0\Delta z,\ \alpha\int_0^x \frac{ds}{a_i(s)}\right)_{L_2(0,l)} = z(l)\cdot\alpha.$$

In the space $\mathcal{H}$ we define the operator $A_K(\cdot)$ in the following way:

$$A_K(w) = \left\{-v_1, {}^1_0\Delta\left(u_1 + N_1\left[\chi_1(\Gamma_l v_1) - \mu g_{1K}(\Gamma_l u_1, \Gamma_l u_2)\right]\right),\right.$$

$$\left. -v_2, {}^2_0\Delta\left(u_2 + N_2\left[\chi_2(\Gamma_l v_2) - \mu g_{2K}(\Gamma_l u_1, \Gamma_l u_2)\right]\right)\right\},$$

$$D(A_K) = \{w : w = (u_1,v_1,u_2,v_2) \in \mathcal{H}, u_i, v_i \in {}_0H^1,$$

$$\left. u_i + N_i\left[\chi_i(\Gamma_l v_i) - \mu g_{iK}(\Gamma_l u_i)\right] \in D({}^i_0\Delta), \quad i = 1,2\right\},$$

also a nonlinear operator $F_K(\cdot)$:

$$F_K(w) = (0,\ -\theta f_{1K_1}(x,u_1,u_2),\ 0,\ -\theta f_{2K_1}(x,u_1,u_2)),$$

where $w = (u_1,v_1,u_2,v_2) \in \mathcal{H}$, Γ_l- trace operator from ${}_0H^1$ at the point $x = l$.

Lemma 3.1 *The nonlinear mappings* $(\xi_1, \xi_2) \to g_{iK}(\xi_1, \xi_2), \quad i = 1, 2$ *satisfy the Lipschitz condition, i.e. for any* $(\xi_1, \xi_2), (\eta_1, \eta_2) \in \mathbb{R}^2$ *the inequalities*

$$|g_{iK}(\xi_1, \xi_2) - g_{iK}(\eta_1, \eta_2)| \le c_g(K) \sum_{i=1}^{2} |\xi_i - \eta_i| ,$$

where $c_g(K) \ge 0$.

Lemma 3.2 *The nonlinear operator* $w \to F_K(w) : \mathcal{H} \to \mathcal{H}$ *satisfies the Lipschitz condition, i.e. for any* $w_1, w_2 \in \mathcal{H}$ *the inequality*

$$\|F_K(w_2) - F_K(w_1)\|_{\mathcal{H}} \le c_F(K)\|w_2 - w_1\|_{\mathcal{H}},$$

where $c_F(K) \ge 0$.

The proof of Lemmas 3.1 and 3.2 is carried out as a proof of similar assertions from [3].

Lemma 3.3 $A_K(\cdot)$ *– maximally accretive operator.*

Proof. Let $w^i = \left(u_1^i, v_1^i, u_2^i, v_2^i\right) \in D(A_K), \quad i = 1, 2$. Then

$$\left\langle A_K(w^2) - A_K(w^1), w^2 - w^1 \right\rangle_{\mathcal{H}}$$

$$= \sum_{i=1}^{2} \int_0^l \left\{ \left[\chi_i(v_i^2(l)) - \mu g_{iK}(u_1^2(l), u_2^2(l))\right] - \left[\chi_i(v_i^1(l)) - \mu g_{iK}(u_1^1(l), u_2^1(l))\right] \right\}$$

$$\times (v_i^2(x) - v_i^1(x))_x dx$$

$$= \sum_{i=1}^{2} \left[\chi_i(v_i^2(l)) - \chi_i(v_i^1(l))\right] (v_i^2(l) - v_i^1(l))$$

$$-\mu \sum_{i=1}^{2} \left[g_{iK}(u_1^2(l), u_2^2(l)) - g_{iK}(u_1^1(l), u_2^1(l))\right] (v_i^2(l) - v_i^1(l))$$

$$\ge \chi_0 \sum_{i=1}^{2} \left|v_i^2(l) - v_i^1(l)\right|^2 - |\mu| \sum_{i=1}^{2} c_g(K) \left|u_i^2(l) - u_i^1(l)\right| \left|v_i^2(l) - v_i^1(l)\right| .$$

Applying Young's inequality with the parameter $0 < \varepsilon < \chi_0$,we get that

$$\left\langle A_K(w^2) - A_K(w^1), w^2 - w^1 \right\rangle_{\mathcal{H}}$$

$$\ge (\chi_0 - \varepsilon) \sum_{i=1}^{2} \left|v_i^2(l) - v_i^1(l)\right|^2 - \sum_{i=1}^{2} \frac{|\mu|\, c_g(K)}{4\varepsilon} \left|u_i^2(l) - u_i^1(l)\right|^2$$

$$\ge -\frac{|\mu|\, c_g(K)}{4\varepsilon} \|w^2 - w^1\|_{\mathcal{H}}^2.$$

Hence we get that for any $\varepsilon \in (0, \chi_0)$ operator $A_K(\cdot) + \frac{|\mu| c_g(K)}{4\varepsilon} I$, where $I-$ the identity operator is an accretive operator.

Now let's prove that $A_K(\cdot) + \omega I$ maximally accretive operator, where $\omega = \frac{|\mu| c_g(K)}{4\varepsilon}$.
Consider the equation

$$A_K(w) + \lambda w = h, \tag{3.12}$$

where $\lambda > 0, \quad h = (h_{11}, h_{12}, h_{21}, h_{22})^{tr} \in \mathcal{H}$ given element, $w = (u_1, v_1, u_2, v_2)^{tr} \in D(A_K(\cdot))$. Obviously, (3.12) is equivalent to the boundary value problem

$$\begin{cases} -v_i + \lambda u_i = h_{i1} \\ {}_0^i\Delta(u_i + N_i[\chi_i(\Gamma_l v_i) + \mu g_{iK}(\Gamma_l u_1, \Gamma_l u_2)]) + \lambda v_i = h_{i2} \\ u_i(0) = 0, \quad v_i(0) = 0 \\ u_i'(l) + \chi_i(v_i(l)) - \mu g_{iK}(u_1(l), u_2(l)) = 0 \end{cases}, \quad i = 1, 2. \tag{3.13}$$

Hence we have

$$u_i = \frac{1}{\lambda} v_i + \frac{1}{\lambda} h_{i1},$$

$${}_0^i\Delta(u_i + N_i[\chi_i(\Gamma_l v_i) + \mu g_{iK}(\Gamma_l u_1, \Gamma_l u_2)]) + \lambda^2 u_i = \lambda h_{i1} + h_{i2}, \quad i = 1, 2.$$

It is easy to see that

$$z_i = u_i + N_i[\chi_i(\Gamma_l v_i) + \mu g_{iK}(\Gamma_l u_1, \Gamma_l u_2)]), \quad i = 1, 2 \tag{3.14}$$

is solutions to the boundary value problem

$$z_i'' - \lambda^2 z_i = \eta_i(x), \tag{3.15}$$

$$z_i(0) = 0, \quad z_i'(l) = 0, \tag{3.16}$$

where

$$\eta_i(x) = -\lambda h_{i1}(x) - h_{i2}(x) - \lambda^2 N_i \alpha_i,$$

$$\alpha_i = \chi_i(\Gamma_l v_i) + \mu g_{iK}(\Gamma_l u_1, \Gamma_l u_2), \quad i = 1, 2. \tag{3.17}$$

The next function is a solution to problem (3.15), (3.16)

$$z_i(x) = \frac{1}{\lambda} \int_0^x sh\lambda(x - \tau)\eta_i(\tau)d\tau - \frac{1}{\lambda} \frac{sh\lambda x}{ch\lambda l} \int_0^l ch\lambda(x - \tau)\eta_i(\tau)d\tau, \quad i = 1, 2.$$

Hence we get that

$$z_i(l) = \alpha_i \left\{ \int_0^l \frac{dx}{a_i(x)} - \frac{1}{ch\lambda l} \int_0^l \frac{ch\lambda\tau}{a_i(\tau)} d\tau \right\} + B_i(l, \lambda), \tag{3.18}$$

where $B_i(l, \lambda) = -\frac{1}{\lambda} \int_0^l sh\lambda(l - \tau)\eta_{1i}(\tau)d\tau + \frac{1}{\lambda} th\lambda l \int_0^l ch\lambda(l - \tau)\eta_{1i}(\tau)d\tau, \quad i = 1, 2.$

On the other hand, it follows from (3.14) that

$$z_i(l) = u_i(l) + \alpha_i \int_0^l \frac{dx}{a_i(x)} = \frac{1}{\lambda} v_i(l) + \frac{1}{\lambda} h_{i1}(l) + \alpha_i \int_0^l \frac{dx}{a_i(x)}, \quad i = 1, 2. \tag{3.19}$$

Comparing (3.18) and (3.19) we find

$$\chi_i(v_i(l)) + \Re_i(\lambda) v_i(l) + \mu g_{iK}(\frac{1}{\lambda} v_1(l) + \frac{1}{\lambda} h_{11}(l), \frac{1}{\lambda} v_2(l) + \frac{1}{\lambda} h_{21}(l)) = C_i(l, \lambda),$$

where $\Re_i(\lambda) = \frac{ch\lambda l}{\lambda \int_0^l \frac{ch\lambda\tau}{a_i(\tau)} d\tau} > 0, C_i(l, \lambda) = \Re_i(\lambda) \left\{ B_i(l, \lambda) + \frac{1}{\lambda} h_{i1}(l) \right\}, \quad i = 1, 2.$

To display $\varPhi(\xi) = (\chi_1(\xi_1) + \Re_1(\lambda)\xi_1, \chi_2(\xi_2) + \Re_2(\lambda)\xi_2)$ have

$$\left\langle \Phi(\xi^2) - \Phi(\xi^1), \xi^2 - \xi^1 \right\rangle_{\mathbb{R}^2}$$

$$= \sum_{i=1}^2 \left[\chi_i(\xi_i^2) - \chi_i(\xi_i^1) \right] \left(\xi_i^2 - \xi_i^1 \right) + \sum_{i=1}^2 \left[\Re_i(\lambda)\xi_i^2 - \Re_i(\lambda)\xi_i^1 \right] \left(\xi_i^2 - \xi_i^1 \right)$$

$$\geq \chi_0 \sum_{i=1}^{2} \left|\xi_i^2 - \xi_i^1\right|^2 + \min\left\{\Re_1(\lambda), \Re_2(\lambda)\right\} \sum_{i=1}^{2} \left|\xi_i^2 - \xi_i^1\right|^2$$

$$\geq \chi_0 \sum_{i=1}^{2} \left|\xi_i^2 - \xi_i^1\right|^2 = \chi_0 \left\|\xi^2 - \xi^1\right\|_{\mathbb{R}^2}^2 . \tag{3.20}$$

Denoting $\tilde{G}(\xi,\lambda) = (\tilde{g}_{1K}(\xi_1,\xi_2), \tilde{g}_{2K}(\xi_1,\xi_2))$, where $\tilde{g}_{iK}(\xi_i) = \mu g_{iK}(\frac{1}{\lambda}\xi_1 + \frac{1}{\lambda}h_{11}(l), \frac{1}{\lambda}\xi_2 + \frac{1}{\lambda}h_{21}(l))$, $\quad i = 1,2$ we get that

$$\left\|\tilde{G}(\xi^2,\lambda) - \tilde{G}(\xi^1,\lambda)\right\|_{\mathbb{R}^2}^2$$

$$= |\mu| \sum_{i=1}^{2} \left| g_{iK}(\frac{1}{\lambda}\xi_1^2 - h_{11}(l), \frac{1}{\lambda}\xi_2^2 + \frac{1}{\lambda}h_{21}(l)) \right.$$

$$\left. - g_{iK}(\frac{1}{\lambda}\xi_1^1 - h_{11}(l), \frac{1}{\lambda}\xi_2^1 + \frac{1}{\lambda}h_{21}(l)) \right|^2$$

$$\leq |\mu| \sum_{i=1}^{2} c_g(K) \left|\frac{1}{\lambda}\xi_i^2 - \frac{1}{\lambda}\xi_i^1\right|^2 \leq \frac{|\mu|\, c_g(K)}{\lambda} \sum_{i=1}^{2} \left|\xi_i^2 - \xi_i^1\right|^2$$

$$= |\mu| \left(\frac{c_g(K)}{\lambda}\right)^2 \left\|\xi^2 - \xi^1\right\|_{\mathbb{R}^2}^2 . \tag{3.21}$$

Hence

$$\left\langle \left[\Phi(\xi^2) + \tilde{G}(\xi^2,\lambda)\right] - \left[\Phi(\xi^1) + \tilde{G}(\xi^1,\lambda)\right], \xi^2 - \xi^1 \right\rangle_{\mathbb{R}^2} \geq \frac{\chi_0}{2} \left\|\xi^2 - \xi^1\right\|_{\mathbb{R}^2}^2 ,$$

where $\xi^i = (\xi_1^i . \xi_2^i), \quad i = 1,2, \quad \lambda > \frac{2c_g(K)}{\sqrt{\chi_0}}\sqrt{|\mu|}$.

Thus, under conditions (3.5), (3.6) $\Phi(\cdot) + \tilde{G}(\cdot,\lambda)$ maximally accretive operator [25,26]. Therefore, at

$$\lambda > \frac{2c_g(K)}{\sqrt{\chi_0}}\sqrt{|\mu|},$$

$$\Phi(\xi) + \tilde{G}(\xi,\lambda) = M(l,\lambda), \tag{3.22}$$

where $M(l,\lambda) = (C_1(l,\lambda), C_2(l,\lambda))$ has a solution. Let the $\xi^0 = (\xi_1^0, \xi_2^0) \in \mathbb{R}^2$ be a solution to the equation (3.22), then setting $v_i(l) = \xi_i^0, \quad u_i(l) = \xi_i^0 + h_{i1}(l), \quad i = 1,2$ we have $\alpha_i = \chi_i(\xi_i^0) + \mu g_{iK}(\xi_i^0 + h_{i1}(l)), \quad i = 1,2$. Considering this, in (3.17) we find the function $z_i(x)$, and from (3.14) we get that $u_i(x) = z_i(x) - N_i\alpha_i, \quad i = 1,2$.

In this way, $A_K(\cdot) + \omega I$ is the maximally accretive operator. The lemma is proven. Problem (3.1) - (3.4) can be written as the following Cauchy problem in the space $\mathcal{H}$:

$$\begin{cases} w' + A_K(w) + F_K(w) = 0, \\ w(0) = w_0, \end{cases} \tag{3.23}$$

where $w_0 = (u_{10}(x), u_{11}(x), u_{20}(x), u_{21}(x)), \quad 0 \leq x \leq l$.

By virtue of Theorem 4.1 from [24], for any $w_0 \in D(A_K)$ and $K > 0$ problem (3.23) has a unique strong solution $w(\cdot) \in C^1([0,T_K];\mathcal{H}) \bigcap C([0,T_K]; D(A_K))$, and by virtue of Theorem 4.1A from [27], problem (3.23) has a weak solution $w(\cdot) \in C^1([0,T_K];\mathcal{H})$, if $w_0 \in \mathcal{H}$.

Identity (3.7) for strong solutions is proved by direct differentiation and for weak solutions the same identity is obtained by approximating them by strong solutions and passing to the limit.

Since $A_K(\cdot)+\omega I$ is a maximally accretive operator, then from (3.23) we obtain $\|w(t)\|^2_{\mathcal{H}} \leq \|w_0\|^2_{\mathcal{H}} e^{2[\omega+C_F(K)]t}$. Hence it follows that $\|w(t)\|^2_{\mathcal{H}}< K^2, \quad 0 \leq t \leq T^*$, if $\|w_0\|_{\mathcal{H}}< K$, where $T^* = T(\|w_0\|_{\mathcal{H}}) = \frac{1}{\omega+c_F(K)} Ln \frac{K}{\|w_0\|_{\mathcal{H}}}$. Hence, $g_{iK}(u_1(l,t),\ u_2(l,t)) = g_i(u_1(l,t), u_2(l,t))$ and $f_{iK}(x, u_1(x,t), u_2(x,t)) = f_i(x, u_1(x,t), u_2(x,t))$ at $0 \leq t \leq T^*$.

Therefore, the function $u(x,t)$ is the solution of the problem (2.1) - (2.4) in the domain $(0,l) \times (0,T')$. Theorem 3.1 is proved.

Note 1 As can be seen from the proof of Theorem 3.1 that , if $w_0 \in \mathcal{H}$, the corresponding weak solution is the limit of strong solutions. In addition, for large enough $K, \quad D(A_K)$ does not depend on K and is a dense set in $\mathcal{H}$ (see [4,7]).

Note 2 Let $T = +\infty$. As follows from proof of Theorem 3.1 that , if T_0 the length of the maximum interval of existence of a weak local solution, then the following alternative is valid:

1 $T_0 = +\infty$, or
2 $\lim\limits_{t\to T_0-0} \|w(t)\|_{\mathcal{H}}$=$+\infty$.

Proof Theorem 2.1. Let $w_0 = (u_{10}, u_{11}, u_{20}, u_{21}) \in \mathcal{H}$. Consider a mixed problem:

$$u_{in_{tt}} - (a_i(x)u_{in_x})_x = \theta f_{i,K_1}(x, u_{1n}, u_{2n}), \quad 0 < x < l,\ \ 0 \leq t \leq T, \tag{3.24}$$

$$u_{in}(x,0) = u_{i0n}(x), \quad u_{in_t}(x,0) = u_{i1n}(x), \quad 0 < x < l, \tag{3.25}$$

$$u_{in}(0,t) = 0, \quad 0 \leq t \leq T, \tag{3.26}$$

$$\begin{aligned} &a_i(l)u_{in_x}(l,t) + [|u_{in_t}(l,t)|^{r_i-1} + \frac{1}{n}]u_{in_t}(l,t) \\ &= \mu g_i(u_{1n}(l,t), u_{2n}(l,t)), \quad 0 \leq t \leq T, \end{aligned} \tag{3.27}$$

where $\ i = 1,2,\ r_1 \geq 1,\ r_2 \geq 1, \lambda \in \mathbb{R},\ \mu \in \mathbb{R}, a_1(\cdot), a_2(\cdot)$ given functions satisfying condition (2.5), $\ f_1, f_2, g_1, g_2$ satisfy conditions (2.6) - (2.10).

If in (3.3) instead of $\chi_i, \quad i = 1,2$ take $\chi_i(\xi) = \left[|\xi|^{r_i-1} + \frac{1}{n}\right]\xi$, then by virtue of (2.5) - (2.10) all the conditions of Theorem 2.1 are satisfied, where

$$u_{in0} \in H^2 \bigcap {}_0H^1,\ u_{in1} \in {}_0H^1,\ i = 1,2, \tag{3.28}$$

$$\begin{aligned} &a_i(l)u_{in0_x}(l) + [|u_{in1_t}(l)|^{r_i-2} + \frac{1}{n}]u_{in1_t} \\ &= \mu g_i(u_{1n0}(l,t), u_{2n0}(l,t)), \quad i = 1,2, \end{aligned} \tag{3.29}$$

$$\|u_{in0} - u_{i0}\|_{{}_0H^1} \to 0 \ \text{ and } \ \|u_{in1} - u_{i1}\|_{L_2(0,l)} \to 0 \text{ at } n \to \infty,\ \ i = 1,2. \tag{3.30}$$

Considering Note 3.1, we get that there is a sequence (u_{in0}, u_{in1}) satisfying conditions (3.28) - (3.30).

By virtue of Theorem 2.1, there is such $T_n \in (0,T]$, that the problem (3.24) - (3.27) has a unique strong solution $(u_{1n}(x,t), u_{1n}(x,t))$ in the domain $(0,l) \times (0,T_n)$, and $u_{in}(\cdot) \in C^1_{T_n}, \quad i = 1,2$ and the following identity is true

$$E_n(t) + \sum_{i=1}^{2} \int_0^t \left\{ |u_{in_\tau}(l,\tau)|^{r_i+1} + \frac{1}{n}\,|u_{in_\tau}(l,\tau)|^2 \right\} d\tau = E_n(0),\ 0 \leq t \leq T_n, \tag{3.31}$$

where $T_n = \frac{1}{\omega+c_F(K)} Ln \frac{K}{\|w_{n0}\|_{\mathcal{H}}}$, $E_n(t) = E_{n0}(t) - R(u_{1n}(\cdot,t), u_{2n}(\cdot,t))$, $E_{n0}(t) = \frac{1}{2}\sum_{i=1}^{2}\left[\|u_{in_t}(\cdot,t)\|_2^2 + \left\|\sqrt{a_i(x)}u_{in_x}(\cdot,t)\right\|_2^2\right]$.

From (3.31) we have

$$E'_{n0}(t) + \sum_{i=1}^{2}\left[|u_{in_t}(l,t)|^{r_i+1} + \frac{1}{n}|u_{in_t}(l,t)|^2\right]$$

$$= \theta\int_0^l |u_{1n}(x,t)+u_{2n}(x,t)|^{2p(x)}(u_{1n}(x,t)+u_{2n}(x,t)) \times (u_{1n_t}(x,t)+u_{2n_t}(x,t))\,dx$$

$$+\theta\int_0^l |u_{1n}(x,t)|^{p(x)-1}|u_{2n}(x,t)|^{p(x)+1}u_{1n}(x,t)u_{1n_t}(x,t)dx$$

$$+\theta\int_0^l |u_{1n}(x,t)|^{p(x)+1}|u_{2n}(x,t)|^{p(x)-1}u_{2n}(x,t)u_{2n_t}(x,t)dx$$

$$+\mu|u_{1n}(l,t)+u_{2n}(l,t)|^{2q}(u_{1n}(l,t)+u_{2n}(l,t))(u_{1n_t}(l,t)+u_{2n_t}(l,t))$$

$$+\mu|u_{1n}(l,t)|^{q-1}|u_{2n}(l,t)|^{q+1}u_{1n}(l,t)u_{1n_t}(l,t)$$

$$+\mu|u_{1n}(l,t)|^{q+1}|u_{2n}(l,t)|^{q-1}u_{2n}(l,t)u_{2n_t}(l,t) = \sum_{i=1}^{6} J_K. \qquad (3.32)$$

Using Lemma 3.1 we obtain the following estimate:

$$|J_1| \le |\theta|\int_0^l |u_{1n}(x,t)+u_{2n}(x,t)|^{2(2p(x)+1)}dx + |\theta|\sum_{i=1}^{2}\int_0^l |u_{in_t}(x,t)|^2 dx$$

$$\le l\,|\theta|\max\left\{|u_{1n}(\cdot,t)+u_{2n}(\cdot,t)|_{C[0,l]}^{2(2p_1+1)}, |u_{1n}(\cdot,t)+u_{2n}(\cdot,t)|_{C[0,l]}^{2(2p_2+1)}\right\}$$

$$+ |\theta|\sum_{i=1}^{2}\int_0^l |u_{in_t}(x,t)|^2 dx.$$

So there is a constant $c_1 > 0$ such that

$$|J_1| \le c_1\sum_{k=1}^{2}\sum_{i=1}^{2}\|u_{in}(\cdot,t)\|_{{}_0H^1}^{2(2p_k+1)} + |\theta|\sum_{i=1}^{2}\int_0^l |u_{in_t}(x,t)|^2 dx. \qquad (3.33)$$

Similarly, taking Lemma 3.1 into account, we have the following estimates:

$$|J_2| + |J_3| \le c_2\,|\theta|\sum_{k=1}^{2}\sum_{i=1}^{2}[\|u_{in}(\cdot,t)\|_{{}_0H^1}^{4p_k}$$

$$+ \|u_{in}(\cdot,t)\|_{{}_0H^1}^{4(p_k+1)}] + |\theta|\sum_{i=1}^{2}\int_0^l |u_{in_t}(x,t)|^2 dx; \qquad (3.34)$$

$$|J_4| \le \frac{c_3}{\varepsilon}\sum_{k=1}^{2}\sum_{i=1}^{2}|u_{in}(l,t)|^{\frac{r_i(2q+1)}{r_i-1}} + \varepsilon\sum_{i=1}^{2}|u_{in_t}(l,t)|^{r_i}$$

$$\leq \frac{c_4}{\varepsilon}\sum_{k=1}^{2}\sum_{i=1}^{2}\|u_{in}(\cdot,t)\|_{{}_0H^1}^{\frac{r_i(2q+1)}{r_i-1}}+\varepsilon\sum_{i=1}^{2}|u_{in_t}(l,t)|^{r_i}; \tag{3.35}$$

$$|J_5|\leq \frac{c_5}{\varepsilon}\left\{\|u_{1n}(\cdot,t)\|_{{}_0H^1}^{\frac{2qr_1}{r_1-1}}+\|u_{2n}(\cdot,t)\|_{{}_0H^1}^{\frac{2(q+1)r_1}{r_1-1}}\right\}+\varepsilon\sum_{i=1}^{2}|u_{in_t}(l,t)|^{r_i}; \tag{3.36}$$

$$|J_6|\leq \frac{c_6}{\varepsilon}\left\{\|u_{1n}(\cdot,t)\|_{{}_0H^1}^{\frac{2(q+1)r_2}{r_2-1}}+\|u_{2n}(\cdot,t)\|_{{}_0H^1}^{\frac{2qr_2}{r_2-1}}\right\}+\varepsilon\sum_{i=1}^{2}|u_{in_t}(l,t)|^{r_i}. \tag{3.37}$$

Taking into account (3.33) - (3.37) in (3.32) we find that

$$E'_{n0}(t)+(1-3\varepsilon)\sum_{i=1}^{2}\left[|u_{in_t}(l,t)|^{r_i+1}+\frac{1}{n}|u_{in_t}(l,t)|^2\right]$$

$$\leq c_7\left\{\sum_{k=1}^{2}\sum_{i=1}^{2}\left[\|u_{in}(\cdot,t)\|_{{}_0H^1}^{4p_k}+\|u_{in}(\cdot,t)\|_{{}_0H^1}^{4p_k+2}+\|u_{in}(\cdot,t)\|_{{}_0H^1}^{4p_k+4}\right]\right.$$

$$+\sum_{k=1}^{2}\sum_{i=1}^{2}\|u_{in}(\cdot,t)\|_{{}_0H^1}^{\frac{r_i(2q+1)}{r_i-1}}+\|u_{1n}(\cdot,t)\|_{{}_0H^1}^{\frac{2qr_1}{r_1-1}}+\|u_{2n}(\cdot,t)\|_{{}_0H^1}^{\frac{2(q+1)r_1}{r_1-1}}$$

$$\left.+\|u_{1n}(\cdot,t)\|_{{}_0H^1}^{\frac{2(q+1)r_2}{r_2-1}}+\|u_{2n}(\cdot,t)\|_{{}_0H^1}^{\frac{2qr_2}{r_2-1}}\right\}.$$

Hence

$$E'_{n0}(t)+(1-3\varepsilon)\sum_{i=1}^{2}\left[|u_{in_t}(l,t)|^{r_i+1}+\frac{1}{n}|u_{in_t}(l,t)|^2\right]$$

$$\leq c_8\sum_{k=1}^{2}\left[[E_{n0}(t)]^{4p_k}+[E_{n0}(t)]^{4p_k+2}+[E_{n0}(t)]^{4p_k+4}\right]$$

$$+c_8\left\{\sum_{k=1}^{2}E_{n0}^{\frac{r_i(2q+1)}{r_i-1}}(t)+E_{n0}^{\frac{2qr_1}{r_1-1}}(t)+E_{n0}^{\frac{2(q+1)r_1}{r_1-1}}(t)+E_{n0}^{\frac{2(q+1)r_2}{r_2-1}}(t)+E_{n0}^{\frac{2qr_2}{r_2-1}}(t)\right\}.$$

Therefore, for $y=y(t)=E_{n0}(t)+1$ we have the inequality

$$y'\leq c_9y^P, \tag{3.38}$$

where $y(0)=y_{0n}=E_{n0}(0)+1$, $P=\max\left\{4p_2+4,\frac{2(q+1)r_1}{r_1-1},\frac{2(q+1)r_2}{r_2-1}\right\}$.

From (3.38) we get that $y(t)\leq 2^{\frac{1}{P-1}}y_{0n},\quad 0\leq t\leq \tilde{T}_n$, where $\tilde{T}_n=\min\left\{T_n,\frac{1}{2c_9y_{0n}^{P-1}(P-1)}\right\}$, i.e.

$$E_{n0}(t)\leq 2^{\frac{1}{P-1}}\left[E_{n0}(0)+1\right],\quad 0\leq t\leq \tilde{T}_n. \tag{3.39}$$

It follows from (3.30) that

$$\lim_{n\to\infty}E_{n0}(0)=E_0(0)=\frac{1}{2}\sum_{i=1}^{2}\left[\|u_1\|_{L_2(0,l)}^2+\left\|\sqrt{a_i(\cdot)}u_{i0_x}\right\|_{L_2(0,l)}^2\right].$$

For this reason, there exists a natural number N, such that for any $n \geq N$ the inequality $E_{n0}(0) \leq 2E_0(0)$. Taking this, from (3.39) we find that

$$E_{n0}(t) \leq 2^{\frac{1}{P-1}} \left[2E_0(0)+1\right], \quad 0 \leq t \leq T_0, \tag{3.40}$$

where $T_0 = \min\left\{\frac{1}{\omega+c_F(K)} \ln \frac{K}{\sqrt{E_0(0)}}, \frac{1}{2c_9[2E_0(0)+1]^{P-1}(P+1)}\right\}$.

It follows from (3.31), (3.40) that

$$\int_0^t |u_{in_\tau}(l,\tau)|^{r_i+1} d\tau \leq c_{10}, \quad 0 \leq t \leq T_0, \quad i=1,2 \tag{3.41}$$

and

$$\frac{1}{n}\int_0^t |u_{in_\tau}(l,\tau)|^2 d\tau \leq c_{10}, \quad 0 \leq t \leq T_0, \quad i=1,2. \tag{3.42}$$

From (3.40) - (3.42) we see that there exists a pair of functions $(u_1(t), u_2(t))$ and a subsequence $\{(u_{1n_m}(t), u_{2n_m}(t))\}$ of the sequences $\{(u_{1n}(t), u_{2n}(t))\}$ (which we still denote by $\{(u_{1n}(t), u_{2n}(t))\}$), where as $n \to \infty$ (see [27])

$$u_{in}(\cdot) \to u_i(\cdot), \text{ weakly star in } L_\infty(0,T_0; {}_0H^1), \quad i=1,2, \tag{3.43}$$

$$u_{in_t}(\cdot) \to u_{i_t}(\cdot), \text{ weakly star in } L_\infty(0,T_0; L_2(0,l)), \quad i=1,2, \tag{3.44}$$

$$|u_{in_t}(l,\cdot)|^{r_i-1} u_{in_t}(l,\cdot) \to \theta_i(\cdot) \text{ weakly in } L_{\frac{r_i}{r_i-1}}(0,T_0), \quad i=1,2, \tag{3.45}$$

$$\frac{1}{n} u_{in_t}(l,\cdot) \to 0 \text{ weakly in } L_2(0,T_0), \quad i=1,2. \tag{3.46}$$

Taking into account the monotonicity $g_i(v) = |v|^{r_i-1} v$, $L_{r_i}(0,T_0) \to L_{\frac{r_i}{r_i-1}}(0,T_0)$, $i=1,2$ we have

$$|u_{in_t}(\cdot)|^{r_i-1} u_{in_t}(\cdot) \to |u_{i_t}(l,t)|^{r_i-1} u_{i_t}(l,t) \text{ weakly in } L_{r_i}(0,T_0), i=1,2. \tag{3.47}$$

Substituting into (3.24) - (3.27) instead of (u_{1n}, u_{2n}) we pass this subsequence to the limit at $n \to \infty$. Then, taking into account (3.43) - (3.47), we get that (u_1, u_2), is a weak solution of problem (2.1) - (2.4) in the domain $(0,l) \times (0,T_0)$.

4 Global solvability

Proof Theorem 2.2. Let $\{(u_1(t,\cdot), u_2(t,\cdot))\}$ be a weak solution to the initial -boundary value problem (2.1) - (2.4) defined on $[0,T) \times \Omega$ as furnished by Theorem 2.1. Then we easily see from (2.12) that

$$\frac{1}{2}\sum_{i=1}^{2}\left[\|u_{it}(\cdot,t)\|_2^2 + \left\|\sqrt{a_i(\cdot)}u_{ix}(\cdot,t)\right\|_2^2\right] - F(u_1(\cdot,t),u_2(\cdot,t)) - G(u_1(l,t),u_2(l,t))$$

$$+\sum_{i=1}^{2}\int_0^t |u_{i\tau}(l,\tau)|^{r_i+1} d\tau = \frac{1}{2}\sum_{i=1}^{2}\left[\|u_{i1}(\cdot)\|_2^2 + \left\|\sqrt{a_i(\cdot)}u_{i0}(\cdot)\right\|_2^2\right]$$

$$-F(u_{10}(\cdot),u_{20}(\cdot)) - G(u_{10}(l),u_{20}(l)). \tag{4.1}$$

If $\lambda \leq 0,\ \mu \leq 0$, then from (4.1) we obtain that

$$\sum_{i=1}^{2}\left[\|u_{it}(\cdot,t)\|_2^2+\left\|\sqrt{a_i(\cdot)}u_{ix}(\cdot,t)\right\|_2^2\right]\leq c,$$

where $c>0$ does not depend on t. Then, by virtue of Note 3.2, this solution is global.

Proof Theorem 2.3. Let conditions (2.5) - (2.10) be satisfied. Suppose that $\lambda \leq 0,\ \mu > 0$.

From (4.1) it follows that

$$\sum_{i=1}^{2}\left[\|u_{it}(\cdot,t)\|_2^2+\left\|\sqrt{a_i(\cdot)}u_{ix}(\cdot,t)\right\|_2^2\right]+F(u_1(\cdot,t),u_2(\cdot,t))+G(u_1(l,t),u_2(l,t))$$

$$+\sum_{i=1}^{2}\int_0^t|u_{i\tau}(l,\tau)|^{r_i+1}\,d\tau\leq 2G(u_1(l,t),u_2(l,t))$$

$$+\frac{1}{2}\sum_{i=1}^{2}\left[\|u_{i1}(\cdot)\|_2^2+\left\|\sqrt{a_i(\cdot)}u_{i0}(\cdot)\right\|_2^2\right]+F(u_{10}(\cdot),u_{20}(\cdot))+G(u_{10}(l),u_{20}(l)). \quad (4.2)$$

On the other hand

$$G(u_1(l,t),u_2(l,t))=\int_0^t\frac{\partial}{\partial t}G(u_1(l,\tau),u_2(l,\tau))d\tau+G(u_{10}(l),u_{20}(l))$$

$$=(2q+1)\int_0^t\mu\,|u_1(l,\tau)+u_2(l,\tau)|^{2q}\,(u_1(l,\tau)+u_2(l,\tau))(u_{1\tau}(l,\tau)+u_{2\tau}(l,\tau))d\tau$$

$$+\int_0^t\mu(q+1)\,|u_1(l,\tau)u_2(l,\tau)|^{q-1}\,[u_1(l,\tau)u_{1\tau}(l,\tau)\,|u_2(l,\tau)|^2$$

$$+u_2(l,\tau)u_{2\tau}(l,\tau)\,|u_1(l,\tau)|^2\,d\tau+G(u_{10}(l),u_{20}(l))$$

$$\leq\ c_{11}\int_0^t[|u_1(l,\tau)|^{2q+1}+|u_2(l,\tau)|^{2q+1}][|u_{1\tau}(l,\tau)|+|u_{2\tau}(l,\tau)|]d\tau$$

$$+c_{11}\int_0^t|u_1(l,\tau)|^q\,|u_2(l,\tau)|^{q+1}\,|u_{1\tau}(l,\tau)|\,d\tau$$

$$+c_{11}\int_0^t|u_1(l,\tau)|^{q+1}\,|u_2(l,\tau)|^q\,|u_{2\tau}(l,\tau)|\,d\tau. \quad (4.3)$$

Taking into account (2.14) and applying the inequalities of Hölder and Young, we obtain

$$\int_0^t|u_i(l,\tau)|^{2q+1}\,|u_{i\tau}(l,\tau)|\,d\tau$$

$$\leq\left(\int_0^t|u_i(l,\tau)|^{(2q+1)\frac{r_i+1}{r_i}}\,d\tau\right)^{\frac{r_i}{r_i+1}}\left(\int_0^t|u_{i\tau}(l,\tau)|^{r_i+1}\,d\tau\right)^{\frac{1}{r_i+1}}$$

$$\leq\frac{r_i}{\varepsilon^{\frac{r_i+1}{r_i}}(r_i+1)}\int_0^t|u_i(l,\tau)|^{(2q+1)\frac{r_i+1}{r_i}}\,d\tau+\frac{\varepsilon^{r_i+1}}{r_i+1}\int_0^t|u_{i\tau}(l,\tau)|^{r_i+1}\,d\tau$$

$$\le \frac{2(q+1)r_i-(2q+1)(r_i+1)}{2(q+1)(r_i+1)\varepsilon^{\frac{r_i+1}{r_i}}}t+\frac{(2q+1)(r_i+1)}{2(q+1)(r_i+1)\varepsilon^{\frac{r_i+1}{r_i}}}\int_0^t |u_i(l,\tau)|^{2(q+1)}\,d\tau$$

$$+\frac{\varepsilon^{r_i+1}}{r_i+1}\int_0^t |u_{i\tau}(l,\tau)|^{r_i+1}\,d\tau \tag{4.4}$$

and

$$\int_0^t |u_i(l,\tau)|^{q+1}\,|u_i(l,\tau)|^q\,|u_{i\tau}(l,\tau)|\,d\tau$$

$$\le \left(\int_0^t |u_i(l,\tau)|^{(q+1)\frac{r_i+1}{r_i}}\,|u_i(l,\tau)|^{q\frac{r_i+1}{r_i}}\right)^{\frac{r_i}{r_i+1}}\left(\int_0^t |u_{i\tau}(l,\tau)|^{r_i+1}\,d\tau\right)^{\frac{1}{r_i+1}}$$

$$\le \frac{r_i(q+1)}{\varepsilon^{\frac{r_i+1}{r_i}}(r_i+1)(2q+1)}\int_0^t |u_i(l,\tau)|^{(2q+1)\frac{r_i+1}{r_i}}\,d\tau$$

$$+\frac{r_iq}{\varepsilon^{\frac{r_i+1}{r_i}}(r_i+1)(2q+1)}\int_0^t |u_i(l,\tau)|^{(2q+1)\frac{r_i+1}{r_i}}\,d\tau$$

$$+\frac{\varepsilon^{r_i+1}}{r_i+1}\int_0^t |u_{i\tau}(l,\tau)|^{r_i+1}\,d\tau \le \frac{(r_i-2q-1)t}{\varepsilon^{\frac{r_i+1}{r_i}}(r_i+1)(2q+1)}$$

$$+\frac{r_i+1}{\varepsilon^{\frac{r_i+1}{r_i}}(r_i+1)}\left\{\int_0^t |u_i(l,\tau)|^{2(q+1)}\,d\tau+\int_0^t |u_i(l,\tau)|^{2(q+1)}\,d\tau\right\}$$

$$+\frac{\varepsilon^{r_i+1}}{r_i+1}\int_0^t |u_{i\tau}(l,\tau)|^{r_i+1}\,d\tau. \tag{4.5}$$

Lemma 4.1 *There exist constants $0<c_{12}\le c_{13}$ such that for all $\xi_1,\xi_2\in R$ the inequality*

$$c_{12}(|\xi_1|^{2(q+1)}+|\xi_2|^{2(q+1)})\le G(\xi_1,\xi_2)\le c_{13}(|\xi_1|^{2(q+1)}+|\xi_2|^{2(q+1)}). \tag{4.6}$$

From (4.2) - (4.6) we have

$$\sum_{i=1}^{2}\left[\|u_{it}(\cdot,t)\|_2^2+\left\|\sqrt{a_i(\cdot)}u_{ix}(\cdot,t)\right\|_2^2\right]$$

$$+\left(1-\frac{6\varepsilon^{r_i+1}}{r_1+1}\right)\sum_{i=1}^{2}\int_0^t |u_{i\tau}(l,\tau)|^{r_i+1}\,d\tau$$

$$+c_{14}\sum_{i=1}^{2}|u_i(l,t)|^{2(q+1)}\le c_{15}+c_{16}t+c_{17}\int_0^t |u_i(l,\tau)|^{2(q+1)}\,d\tau.$$

Using Gronuall's inequality and taking into account condition (2.5), from this we obtain that

$$\sum_{i=1}^{2}\left[\|u_{it}(\cdot,t)\|_2^2+\|u_{ix}(\cdot,t)\|_2^2\right]\le c_{18}.$$

It follows that the local solution to problem (2.1) - (2.4) defined by Theorem 2.1 is global.

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