Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **41** (4), 1-12 (2021).

Limit theorems for homogeneous branching processes with migration

Khrystyna M. Yakymyshyn ⋆ · **Iryna B. Bazylevych** ·
Soltan A. Aliyev

Received: 14.04.2021 / Revised: 10.06.2021 / Accepted: 07.07.2021

Abstract. The homogeneous branching process with migration and continuous time is studied. The generating function of the process is obtained. Limit theorem for subcritical branching process with migration is found.

Keywords. branching process · migration · continuous time · generating function · subcritical branching process

Mathematics Subject Classification (2010): 60J80

1 Introduction

Branching processes are mathematical models of many physical, chemical, biological, genetic, demographic, and other processes. Since third-party factors often exist, there is a need to study different modifications of this process. Among them are branching processes with immigration, emigration, or a combination of two processes, namely processes with migration for the case of discrete and continuous time.

An important feature of the branching process is the generating function. In the classical case, for processes with continuous time, it is obtained from the differential equation.

In the case of the branching process with immigration, the derivation of the differential equation and finding its solution is shown in [8], where the process is defined as a process with two types of particles.

In the case of the process of emigration, Formanov Sh. K. and Kaverin S. V. found the form of a differential equation, and the solution of this equation without detailed inference is shown in [4], [5].

The main results for branching processes with discrete time and different regimes of immigration and emigration are described in [13].

⋆ Corresponding author

K. M. Yakymyshyn
Lviv Polytechnic National University, Lviv, Ukraine
E-mail: yakymyshyn_hrystyna@ukr.net

I.B. Bazylevych
Ivan Franko National University of Lviv, Lviv, Ukraine
E-mail: i_bazylevych@yahoo.com

S.A. Aliyev
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan
Institute of Control Systems of NAS of Azerbaijan, Baku, Azerbaijan
E-mail: soltanaliyev@yahoo.com

The case of a branched process with migration and continuous time is considered in [3], [7], [10], [6], [1]. Chen A. Y. and Renshaw E. [3] have considered a case of the process which large immigration, i.e. the sum of immigration rates is infinite; excessively high population levels are avoided by allowing the carrying capacity of the system to be controlled by mass emigration. Also in 2000, Rahimov I. and Al-Sabah W.S. [7] considered a family of independent, equally distributed with continuous Markov branching processes. The migration was determined as follows: the particles first immigrate and stay in the population for some time, and then emigrate.

The limit distribution theorem for the classical branching process with continuous time is proved in [9].

In this article we consider a more general model of the branching processes with migration and continuous time [12]. Immigration, emigration, and evolution occur at random moments of time and are determined by the intensity of the transition probabilities. The form of a generating function for a branching process with migration and continuous time and the Kolmogorov system of equations held for the transition probabilities of the process are found in [12]. The limit theorems for the number of emigrated particles for a homogeneous branching process with continuous time, emigration one particle, and immigration are proved in [2].

Also, we obtain the form of a generating function and prove the limit distribution theorem for the classical branching process with continuous time and migration. Distribution of the number of emigrating particles and its limiting distribution has been found.

2 Description of a branching process model with migration and continuous time

Consider a Markov branching process with one type of particles and migration $\mu(t)$, $t \in [0, \infty)$. Let $\mu(t)$ denote the number of particles at the time $t \in [0, \infty)$.

We suppose, that at the time t = 0, the process starts with one particle, in the system

$$\mu(0) = 1. \tag{2.1}$$

The process $\mu(t)$, $t \in [0, \infty)$ then $\Delta t \to 0$ is given by transition probabilities

$$P\{\mu(t+\Delta t) = j | \mu(t) = i\} =$$

$$= \begin{cases} 1 + q_0 \Delta t + o(\Delta t), & i = j = 0; \\ q_j \Delta t + o(\Delta t), & i = 0, j = 1, 2, ...; \\ (p_0 + \sum\limits_{l=1}^{m} r_l)\Delta t + o(\Delta t), & i = 1, j = 0; \\ 1 + (q_0 + r_0 + p_1)\Delta t + o(\Delta t), & i = 1, j = 1; \\ (p_j + q_{j-1})\Delta t + o(\Delta t), & i = 1, j = 2, ...; \\ \sum\limits_{l=i}^{m} r_l \Delta t + o(\Delta t) & 1 < i \le m, j = 0; \\ (p_0 + r_1)\Delta t + o(\Delta t), & i = 2, 3, ..., j = i - 1; \\ r_{i-j}\Delta t + o(\Delta t), & i = 2, 3, ..., j < i - 1; \\ 1 + (q_0 + r_0 + ip_1)\Delta t + o(\Delta t), & i = 2, 3, ..., i = j; \\ (ip_{j-i+1} + q_{j-i})\Delta t + o(\Delta t), & i = 2, 3, ..., i < j; \\ o(\Delta t), & \text{in other cases,} \end{cases} \tag{2.2}$$

where m is a fixed integer, and p_k, q_k, r_n satisfy the conditions

$$p_k \ge 0,\ k \ne 1,\ p_1 < 0,\ \sum_{k=0}^{\infty} p_k = 0,$$

$$q_k \geq 0,\ k \neq 0,\ q_0 < 0,\ \sum_{k=0}^{\infty} q_k = 0,$$

$$r_n \geq 0,\ n = \overline{1, m},\ r_0 < 0,\ \sum_{k=0}^{m} r_k = 0.$$

Note, that p_k $(k = 0, 1, ...)$ is the intensity of reproduction particle, q_k $(k = 0, 1, ...)$ is the intensity of immigration, and r_n $(n = \overline{0, m})$ is the intensity of emigration.

We introduce the following notation,

$$F(t, s) = \sum_{n=0}^{\infty} P\{\mu(t) = n\} s^n,$$

$$f(s) = \sum_{n=0}^{\infty} p_n s^n,\ |s| \leq 1,\ s \in C,$$

$$g(s) = \sum_{n=0}^{\infty} q_n s^n,\ |s| \leq 1,\ s \in C,$$

$$r(s) = \sum_{n=0}^{m} r_n s^{-n},\ 0 < |s| \leq 1.$$

Let $\widehat{F}(t, s)$ is the generating function of a branching process with continuous time (without migration) ([8], page 24).

The process $\mu(t)$ has the following conditions.

Theorem 2.1 *[12] The generating function $\mu(t)$ satisfies the differential equation,*

$$\frac{\partial F(t, s)}{\partial t} = f(s) \frac{\partial F(t, s)}{\partial s} + g(s) F(t, s)$$

$$+ \sum_{n=0}^{m} P\{\mu(t) = n\} \left(s^n \sum_{k=0}^{n-1} r_k s^{-k} + \sum_{k=n}^{m} r_k \right) + \sum_{n=m+1}^{\infty} P\{\mu(t) = n\} s^n r(s), \qquad (2.3)$$

with the initial condition

$$F(0, s) = s. \qquad (2.4)$$

Theorem 2.2 *[12] The Kolmogorov system of equations holds for the process $\mu(t)$, $t \in [0, \infty)$*

$$\begin{cases} \frac{dP\{\mu(t)=0\}}{dt} = P\{\mu(t) = 0\} q_0 + P\{\mu(t) = 1\} p_0 + \sum\limits_{k=1}^{m} P\{\mu(t) = k\} \sum\limits_{j=k}^{m} r_j, \\ \frac{dP\{\mu(t)=n\}}{dt} = \sum\limits_{k=0}^{n} P\{\mu(t) = k\} q_{n-k} + \sum\limits_{k=1}^{n+1} k P\{\mu(t) = k\} p_{n+1-k} + \\ + \sum\limits_{k=n}^{n+m} P\{\mu(t) = k\} r_{k-n},\ n \geq 1. \end{cases} \qquad (2.5)$$

3 Generation function of a branching process with migration and continuous time

In this section, we find a generation function of the branching process with migration. The method of generating functions is widely used in the study of processes with continuous time, because in some cases it can be found in the form of its generating, and then calculate the corresponding probabilities of the process are calculated. The generating function of the process will uniquely determine the distribution of the process and the limiting behavior of the process.

Theorem 3.1 *The equation (2.3) with initial condition (2.4) has the solution*

$$F(t,s)=\widehat{F}(t,s)e^{\int\limits_0^t(g(\widehat{F}(u,s))+r(\widehat{F}(u,s)))du}$$

$$+\sum_{n=0}^{m}\int\limits_0^t P\{\mu(x)=n\}\sum_{k=n}^{m}r_k(1-\widehat{F}^{n-k}(t-x,s))e^{\int_0^{t-x}(g(\widehat{F}(u,s))+r(\widehat{F}(u,s)))du}dx, \quad (3.1)$$

which is unique in the class of continuous-differentiated functions in the interval $\Delta(\varepsilon)=\{s: s\in[\varepsilon,1], 0<\varepsilon<1\}$.

Proof. Consider the differential equation for the generation function of the process $\mu(t)$

$$\frac{\partial F(t,s)}{\partial t}=f(s)\frac{\partial F(t,s)}{\partial s}+g(s)F(t,s)$$

$$+\sum_{n=0}^{m}P\{\mu(t)=n\}\left(s^n\sum_{k=0}^{n-1}r_k s^{-k}+\sum_{k=n}^{m}r_k\right)+\sum_{n=m+1}^{\infty}P\{\mu(t)=n\}s^n r(s).$$

We represent it in the form,

$$\frac{\partial F(t,s)}{\partial t}=f(s)\frac{\partial F(t,s)}{\partial s}+g(s)F(t,s)+r(s)F(t,s)$$

$$+\sum_{n=0}^{m}P\{\mu(t)=n\}\sum_{k=n}^{m}r_k(1-s^{n-k}).$$

We get the equation of characteristics,

$$dt=-\frac{ds}{f(s)}=\frac{dF}{F(t,s)(g(s)+r(s))+\sum\limits_{n=0}^{m}P\{\mu(t)=n\}\sum\limits_{k=n}^{m}r_k(1-s^{n-k})}.$$

We find the first integrals of this equation.
Consider

$$dt=-\frac{ds}{f(s)}$$

and we get

$$t=-\int\limits_0^s\frac{du}{f(u)}+C_1. \quad (3.2)$$

Thus,

$$C_1 = t + \int_0^s \frac{du}{f(u)}.$$

The functions $P\{\mu(t) = n\}, n = 0, 1, ...$ can be determined from the Kolmogorov equation system (2.5) the only possible way $-$ satisfying the condition of regularity $P\{\mu(t) < \infty\} = 1, \forall t \in (0, +\infty)$, and Theorems 15, 16 ([11], page 71)

Consider the equation

$$\frac{\partial F(t,s)}{\partial t} = F(t,s)(g(s) + r(s)) + \sum_{n=0}^{m} P\{\mu(t) = n\} \sum_{k=n}^{m} r_k(1 - s^{n-k}).$$

Find the solution of the corresponding homogeneous equation,

$$\frac{\partial F(t,s)}{\partial t} = F(t,s)(g(s) + r(s)).$$

Since

$$\frac{\partial \widehat{F}(t,s)}{\partial t} = f(\widehat{F}(t,s))$$

and

$$\frac{d\widehat{F}(t,s)}{f(\widehat{F}(t,s))} = dt,$$

where $\widehat{F}(t,s)$ is the generating function branching process with continuous time (without migration), we get the general solution of the homogeneous equation

$$F(t,s) = C_2 e^{\int_0^t (g(\widehat{F}(u,s)) + r(\widehat{F}(u,s)))du}.$$

The method of variation of parameters is obtained

$$F(t,s) = \sum_{n=0}^{m} \int_0^t P\{\mu(x) = n\} \sum_{k=n}^{m} r_k(1 - \widehat{F}^{n-k}(t-x,s)) e^{\int_0^{t-x} (g(\widehat{F}(u,s)) + r(\widehat{F}(u,s)))du} dx.$$

Finally, the general solution of the nonhomogeneous equation is written, as

$$F(t,s) = C_2 e^{\int_0^t (g(\widehat{F}(u,s)) + r(\widehat{F}(u,s)))du} +$$

$$+ \sum_{n=0}^{m} \int_0^t P\{\mu(x) = n\} \sum_{k=n}^{m} r_k(1 - \widehat{F}^{n-k}(t-x,s)) e^{\int_0^{t-x} (g(\widehat{F}(u,s)) + r(\widehat{F}(u,s)))du} dx.$$

Thus, we get

$$C_2 = F(t,s) e^{-\int_0^t (g(\widehat{F}(u,s)) + r(\widehat{F}(u,s)))du} - \sum_{n=0}^{m} \int_0^t P\{\mu(x) = n\}$$

$$\times \sum_{k=n}^{m} r_k \left(1 - \widehat{F}^{n-k}(t-x,s)\right) e^{\int_0^{t-x} (g(\widehat{F}(u,s)) + r(\widehat{F}(u,s)))du} dx e^{-\int_0^t (g(\widehat{F}(u,s)) + r(\widehat{F}(u,s)))du}.$$

Thus, according to ([14] page 97), we obtain

$$V\left(t+\int\limits_0^s \frac{du}{f(u)}\right) = F(t,s)e^{-\int\limits_0^t (g(\widehat{F}(u,s))+r(\widehat{F}(u,s)))du} - \sum_{n=0}^{m}\int\limits_0^t P\{\mu(x)=n\}\sum_{k=n}^{m} r_k$$

$$\times\left(1-\widehat{F}^{n-k}(x,s)\right)e^{\int\limits_0^{t-x}(g(\widehat{F}(u,s))+r(\widehat{F}(u,s)))du} dx\, e^{-\int\limits_0^t (g(\widehat{F}(u,s))+r(\widehat{F}(u,s)))du},$$

where $V(\cdot)$ is any continuously differentiable function.
Therefore, the generation function of the process $\mu(t)$ becomes

$$F(t,s) = V\left(t+\int\limits_0^s \frac{du}{f(u)}\right)e^{\int\limits_0^t (g(\widehat{F}(u,s))+r(\widehat{F}(u,s)))du}$$

$$+\sum_{n=0}^{m}\int\limits_0^t P\{\mu(x)=n\}\sum_{k=n}^{m} r_k\left(1-\widehat{F}^{n-k}(t-x,s)\right)e^{\int\limits_0^{t-x}(g(\widehat{F}(u,s))+r(\widehat{F}(u,s)))du} dx,$$

where $V(\cdot)$ is any continuously differentiable function.
Since it is the initial condition, we obtain

$$V\left(\int\limits_0^s \frac{du}{f(u)}\right) = s.$$

If $s=1$, we get

$$F(t,1) = V\left(t+\int\limits_0^1 \frac{du}{f(u)}\right)e^{\int\limits_0^t (g(\widehat{F}(u,1))+r(\widehat{F}(u,1)))du} + \sum_{n=0}^{m}\int\limits_0^t P\{\mu(x)=n\}$$

$$\times\sum_{k=n}^{m} r_k\left(1-\widehat{F}^{n-k}(t-x,1)\right)e^{\int\limits_0^{t-x}(g(\widehat{F}(u,1))+r(\widehat{F}(u,1)))du} dx = V\left(t+\int\limits_0^1 \frac{du}{f(u)}\right) = 1.$$

Thus,

$$\begin{cases} V\left(\int\limits_0^s \frac{du}{f(u)}\right) = s \\ V\left(t+\int\limits_0^1 \frac{du}{f(u)}\right) = 1. \end{cases} \tag{3.3}$$

Suppose, that immigration and emigration don't occur, so $g(s)=0$ and $r(s)=0$ then we get

$$V\left(t+\int\limits_0^s \frac{du}{f(u)}\right) = \widehat{F}(t,s).$$

Clearly, that the function $\widehat{F}(t,s)$ also satisfies (3.3). Hence (3.1) follows.
Prove uniqueness. We suppose, that two solutions $F_1(t,s)$ and $F_2(t,s)$ exist. Consider their difference

$$F_0(t,s) = F_1(t,s) - F_2(t,s).$$

Since the function $\widehat{F}(t,s)$ is known and independent of the process $\mu(t)$, the functions $g(s)$ and $r(s)$ are defined, and the probabilities $P\{\mu(t) = n\}$, $n = 0, 1, ...$ are uniquely determined from the Kolmogorov equation (2.5). Hence, we get that $|F_0(t,s)| = 0$.

The theorem proved.

4 Limiting theorem for a subcritical branching process with continuous time and migration

In this section, we consider the subcritical branching process and find limiting distribution.

Theorem 4.1 *If* $a_0 = f'(1) < 0$, $a_1 = g'(1) < \infty$, $a_2 = r'(1) < \infty$ *and* $\int\limits_0^\infty M\mu(x)dx < \infty$*, then limiting distribution* $\mu(t)$ *exists*

$$\lim_{t\to\infty} P\{\mu(t) = n\} = P_n^*.$$

Proof. Taking into account ([8], page 222).

We know that a generating function of a branching process with migration has the form,

$$F(t,s) = \widehat{F}(t,s)e^{\int\limits_0^t (g(\widehat{F}(u,s))+r(\widehat{F}(u,s)))du}$$

$$+\sum_{n=0}^{m}\int\limits_0^t P\{\mu(x)=n\}\sum_{k=n}^{m} r_k(1-\widehat{F}^{n-k}(t-x,s))e^{\int\limits_0^{t-x}(g(\widehat{F}(u,s))+r(\widehat{F}(u,s)))du}dx.$$

In the case of a subcritical process $a_0 < 0,\ a_1 < \infty,\ a_2 < \infty$.

Prove that limit exists

$$\lim_{t\to\infty} F(t,s) = exp\left\{\int\limits_0^\infty (g(\widehat{F}(u,s)) + r(\widehat{F}(u,s)))du\right\}$$

$$+\sum_{n=0}^{m}\int\limits_0^\infty P\{\mu(x)=n\}\sum_{k=n}^{m} r_k(1-\widehat{F}^{n-k}(t-x,s))e^{\int\limits_0^{t-x}(g(\widehat{F}(u,s))+r(\widehat{F}(u,s)))du}dx,$$

moreover, the convergence is uniform on $|s| \le 1$.

We need to show that improper integral is the limit

$$\int\limits_0^\infty (g(\widehat{F}(u,s)) + r(\widehat{F}(u,s)))du, \tag{4.1}$$

$$\sum_{n=0}^{m}\int\limits_0^\infty P\{\mu(x)=n\}\sum_{k=n}^{m} r_k(1-\widehat{F}^{n-k}(t-x,s))e^{\int\limits_0^{t-x}(g(\widehat{F}(u,s))+r(\widehat{F}(u,s)))du}dx \tag{4.2}$$

uniform convergence on $|s| \le 1$, $s \ne 0$.

Since $|g(s)| \leq a_1|s-1|$, $|r(s)| \leq a_2|s-1|$ and $|\widehat{F}(u,s)-1| \leq e^{a_0 u}|s-1|$, then

$$|g(\widehat{F}(u,s)) + r(\widehat{F}(u,s))| \leq (a_1+a_2)e^{a_0 u}|s-1| \leq 2(a_1+a_2)e^{a_0 u}.$$

Thus, there is uniform convergence of the integral (4.1) on s in $|s| \leq 1$, $s \neq 0$.
Thus, there is a limiting distribution with the generation function

$$F(s) = \sum_{k=0}^{\infty} P_k s^k = exp\{\int_0^{\infty} (g(\widehat{F}(u,s)) + r(\widehat{F}(u,s)))du\}$$

$$+\sum_{n=0}^{m} \int_0^{\infty} P\{\mu(x)=n\} \sum_{k=n}^{m} r_k (1-\widehat{F}^{n-k}(t-x,s)) e^{\int_0^{t-x} (g(\widehat{F}(u,s))+r(\widehat{F}(u,s)))du} dx.$$

We differentiate the improper integrals by s

$$\frac{\partial}{\partial s} \int_0^{\infty} (g(\widehat{F}(u,s)) + r(\widehat{F}(u,s)))du$$

$$= \int_0^{\infty} \left(\frac{dg(\widehat{F}(u,s))}{d\widehat{F}(u,s)} + \frac{dr(\widehat{F}(u,s))}{d\widehat{F}(u,s)} \right) \frac{\partial \widehat{F}(u,s)}{\partial s} du.$$

From the equation for the generation function $\frac{\partial \widehat{F}(u,s)}{\partial t} = f(s) \frac{\partial \widehat{F}(u,s)}{\partial s}$, we get

$$\int_0^{\infty} \left(\frac{dg(\widehat{F}(u,s))}{d\widehat{F}(u,s)} + \frac{dr(\widehat{F}(u,s))}{d\widehat{F}(u,s)} \right) \frac{\partial \widehat{F}(u,s)}{\partial u} \frac{du}{f(s)}$$

$$= \frac{g(\widehat{F}(\infty,s)) - g(\widehat{F}(0,s))}{f(s)} + \frac{r(\widehat{F}(\infty,s)) - r(\widehat{F}(0,s))}{f(s)} = -\frac{g(s)+r(s)}{f(s)}.$$

Clearly, that

$$\lim_{t\to\infty} g(\widehat{F}(t,s)) = 0, \quad \lim_{t\to\infty} r(\widehat{F}(t,s)) = 0,$$

uniformly on $|s| \leq 1$ and $\widehat{F}(0,s) = s$ [8].

Convergence is also uniform across by $|s| \leq 1$, $s \neq 0$ in

$$\lim_{t\to\infty} \frac{g(\widehat{F}(t,s)) + r(\widehat{F}(t,s))}{f(s)} = 0,$$

therefore differentiation by the parameter integral (4.1) is legal.

Note, that the limit distribution of stationary, that is, a generation function satisfies the partial differential equation if we substitute $\frac{\partial F(t,s)}{\partial t} = 0$.

Let's now consider (4.2)

$$\lim_{t\to\infty} \sum_{n=0}^{m} \int_0^{t} P\{\mu(x)=n\} \sum_{k=n}^{m} r_k (1-\widehat{F}^{n-k}(t-x,s)) e^{\int_0^{t-x} (g(\widehat{F}(u,s))+r(\widehat{F}(u,s)))du} dx$$

$$< \lim_{t\to\infty} \sum_{n=0}^{m} \int_0^t nP\{\mu(x) = n\} \sum_{k=n}^{m} |r_0|(1 - \widehat{F}^{n-k}(t - x, s))e^{\frac{a_1+a_2}{|a_0|}|s-1|}dx.$$

Let $P\{\xi(t) > 0\} > 0$. For the subcritical process $\widehat{F}(t, s)$ increases by s, so $\widehat{F}(t, s) > s$ where $\forall s \in (0, 1)$, so where $s \in [s_0, 1)$

$$\frac{1}{\widehat{F}(t, s)} < \frac{1}{s} < \frac{1}{s_0}.$$

Thus, we get

$$\lim_{t\to\infty} \sum_{n=0}^{m} \int_0^t nP\{\mu(x) = n\} \sum_{k=n}^{m} |r_0|(1 - \widehat{F}^{n-k}(t - x, s))e^{2\frac{a_1+a_2}{|a_0|}}dx$$

$$< \lim_{t\to\infty} \int_0^t \sum_{n=0}^{m} nP\{\mu(x) = n\} \sum_{k=n}^{m} |r_0|\frac{2}{|s_0|^m}e^{2\frac{a_1+a_2}{|a_0|}}dx$$

$$< \lim_{t\to\infty} \int_0^t \sum_{n=0}^{m} M\mu(x)(m - n - 1)|r_0|\frac{2}{|s_0|^m}e^{2\frac{a_1+a_2}{|a_0|}}dx$$

$$< m(m + 1)|r_0|\frac{2}{|s_0|^m}e^{2\frac{a_1+a_2}{|a_0|}} \int_0^\infty M\mu(x)dx < \infty$$

The theorem is proved.

5 Distribution of the number of emigrating particles

In this section, we find the distribution of the number of emigrating particles and its limiting distribution.

Let τ_1, τ_2, τ_3, ... is independent, identically distributed random variables that determine the intervals between particle transformations in the system. Therefore, the first transformation of particles in the system took place at the moment τ_1, the second transformation of particles in the system is $\tau_1 + \tau_2$ etc.

We also introduce random variables θ_1, θ_2, θ_3, ... that are defined as follows

$$\theta_0 = 0,\ \theta_1 = \tau_1,\ \theta_2 = \tau_1 + \tau_2,\ ...,\ \theta_n = \tau_1 + ... + \tau_n, ...$$

Let $\rho(t)$ is determines the number of transformations in the system up to the time t. Note that $\rho(t)$ is a random process.

Let the random process $\nu(t)$ determine the number of particles $\mu(t)$ that have emigrated before the time t.

Given the previous notation it is easy to see that

$$\nu(t) = \nu_0 + \nu_1 + \nu_2 + ... + \nu_{\rho(t)}, \tag{5.1}$$

where ν_k $(k = 1, ..., \rho(t))$ $-$ the number of particles that emigrated during the k-th transformation in the system. Since at the initial moment in the process $\mu(t)$ there can be no emigration of particles, so the initial distribution of the process $\nu(t)$

$$\nu(0) = \nu_0 = 0.$$

Define the distribution of the process $\nu(t)$.

Note that at the first distribution of random variables should be determined as ν_k ($k = 1, ..., \rho(t)$). Also ν_k ($k = 1, ..., \rho(t)$) $-$ are independent of each other and depend only on the value of $\mu(\theta_{k-1})$.

Let $\mu(\theta_{k-1})$, then ν_k ($k = 1, ..., \rho(t)$) has the following distribution

- in case $m \leq n$
  - 0 particles with probability $r_0 \Delta t + o(\Delta t)$,
  - 1 particles with probability $r_1 \Delta t + o(\Delta t)$,
  - 2 particles with probability $r_2 \Delta t + o(\Delta t)$,
  - ...
  - $n-1$ particles with probability $r_{n-1} \Delta t + o(\Delta t)$,
  - n particles with probability $\sum_{l=n}^{m} r_l \Delta t + o(\Delta t)$.
- in case $m \leq n$
  - 0 particles with probability $r_0 \Delta t + o(\Delta t)$,
  - 1 particles with probability $r_1 \Delta t + o(\Delta t)$,
  - 2 particles with probability $r_2 \Delta t + o(\Delta t)$,
  - ...
  - $m-1$ particles with probability $r_{m-1} \Delta t + o(\Delta t)$,
  - m particles with probability $r_m \Delta t + o(\Delta t)$.

Thus, the distribution is given for the process $\nu(t)$ by the following transient probabilities,

$$P\{\nu(t+\Delta t) = j \mid \nu(t) = i,\ \mu(t) = n\} =$$

$$= \begin{cases} \begin{cases} r_0 \Delta t + o(\Delta t), & i = j; \\ r_{j-i} \Delta t + o(\Delta t), & i < j < i+n; \\ \sum_{l=n}^{m} r_l \Delta t + o(\Delta t), & j = i+n; \\ o(\Delta t), & \text{in other cases;} \end{cases} & m > n; \\ \begin{cases} r_{j-i} \Delta t + o(\Delta t), & i \leq j \leq i+m; \\ o(\Delta t), & \text{in other cases;} \end{cases} & m \leq n. \end{cases}$$

Theorem 5.1 *Let $\nu(t)$ is the number of particles that emigrated during the time period $[0, t]$ from the process $\mu(t)$, then at $t \to \infty$*

$$\lim_{t \to \infty} P\left\{ \frac{\nu(t) - M\nu(t)}{\sqrt{D\nu(t)}} < x | \mu(t) > 0 \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt.$$

Proof. Let θ_k $-$ is the moment of the k-th jump, and then the probability of emigration is equal to

$$P\{\nu_{j+1} = j | \mu(t) = n, \theta_j = t\} = \tilde{r_k} = \begin{cases} r_j \Delta t + o(\Delta t), & n \geq m; \\ \sum\limits_{l=n}^{m} r_l \Delta t + o(\Delta t), & n > m. \end{cases}$$

Let B_{jk} is the event when j particles are emigrating during the k-th transformation. Then

$$P\{B_{jk}\} = \frac{\tilde{r_j}}{-r_0 - q_0 - p_1}.$$

First, we assume that the number of particles in the system is not zero. Let $t \to \infty$.

It shows that the number of transformations also approaches ∞. This obvious because the probability that an arbitrary particle will not have been transformed by the time t is equal to $e^{p_1 t}$. The probability that all particles will not have been transformed by the time t is equal to $e^{n p_1 t}$ assuming that there were n particles in the system. In this case, there should be no immigration and emigration of particles, so the corresponding probability is equal to $e^{(np_1+q_0+r_0)t}$. For $t \to \infty$ get that $e^{(np_1+q_0+r_0)t} \to 0$.

Consider the case when the number of particles in the system is zero. Consider the case when the number of particles in the system is zero. Since it is possible to get out of the zero state due to the immigration of particles, the probability that the transformation will not take place is $e^{q_0 t}$. In case $t \to \infty$ it is obtained that $e^{q_0 t} \to 0$.

It follows that the number of transformations in the system $\rho(t)$ for $t \to 0$ approaches ∞.

Show that the conditions of Lyapunov's theorem are hold for ν_k $(k = 1, ..., \rho(t))$ using the representations $\nu(t)$ (5.1) is hold:

$$1.\ M\nu_k < \infty, \quad k = 1, 2, 3, ...$$

$$2.\ \lim_{n\to\infty} \frac{\sum\limits_{k=1}^{m} M|\nu_k - M\nu_k|^3}{B_n^3} = 0,$$

where $B_n = \sum\limits_{k=1}^{n} D\nu_k$. Note that ν_k $(k = 1, ..., \rho(t))$ are independent with a known distribution.

Consider the mathematical expectation ν_k, $k = 1, 2, ...$

$$M\nu_k = \sum_{j=0}^{n} jP\{\nu_k = j\} = \begin{cases} \sum\limits_{j=0}^{n} jr_j\Delta t + o(\Delta t), & n > m; \\ \sum\limits_{j=0}^{n-1} jr_j\Delta t + n\sum_{j=n}^{m} r_j\Delta t + o(\Delta t), & n \le m. \end{cases}$$

Since the number of particles that emigrated during the k-th transformation in the system can increase from 0 to m and accordingly, after k transformations can emigrate from 0 to mk particles, then $M\nu_k < mk$. Condition 1 holds.

Show that condition 2 also holds.

Obviously,

$$M\nu_k \le m.$$

Here it is obtained that

$$|\nu_k - M\nu_k| < m \ \Rightarrow\ |\nu_k - M\nu_k|^3 < m^3 \ \Rightarrow\ M|\nu_k - M\nu_k|^3 \le m^3.$$

Therefore,

$$\sum_{k=1}^{n} M|\nu_k - M\nu_k|^3 \le nm^3.$$

Let $\sigma^2 = \min\limits_k D\nu_k >$ ‘0.

Then,

$$B_n^3 = \left(\sqrt{\sum_{k=1}^{n} M|\nu_k - M\nu_k|^2}\right)^3 \ge (\sqrt{n\sigma^2})^3 = (\sqrt{n})^3\sigma^3.$$

So,

$$\lim_{n\to\infty} \frac{nm^3}{(\sqrt{n})^3\sigma^3} = 0.$$

The theorem is proved.

Conclusions.

This article investigates a more general model of the process than in [3], [7], [10], [6], [1]. The form of the generating function has been found. The limit theorem for the subcritical branching process with migration has been proved. Distribution of the number of emigrating particles and its limiting distribution has been found.

Acknowledgements.

The research of S. Aliyev was supported by the Science Development Foundation under the President of the Republic of Azerbaijan (Agreement Number No. EIF-ETL-2020-2(36)-16/05/1-M-05).

References

1. Balabaev, I.S.: *Limit theorems for a critical Markov branching process with continuous time and migration*, Uzbek mathematical journal, **2**, 12-15 (1994) (In Russian)
2. Bazylevysh, I. B., Yakymyshyn, Kh. M., Pylypchuk, Kh. V.: *Limit theorems for number of particles which emigrated from the system*, Precarpathian Bulletin of the Shevchenko Scientific Society. Number, **1**(15), 45-55 (2018) (in Ukraine)
3. Chen, A. Y., Renshaw, E.: *Markov branching processes regulated by emigration and large immigration*, Stochastic Processes and their Applications, **57**, 339–359 (1995)
4. Formanov, Sh.K., Kaverin S. V.: *Markov branching processes with emigration. I*, Izvestiya of the Academy of Sciences of the USSR. Avg. Physical motif. Sciences, **5**, 23–28 (1986). (In Russian)
5. Formanov, Sh.K., Kaverin, S. V.: *Markov branching processes with emigration. II*, Izvestiya of the Academy of Sciences of the USSR. Avg. Physical motif. Sciences, **3**, 36–41 (1987). (In Russian)
6. Pakes, A.G.: *Some properties of a branching process whith group immigration and emigration*, Adv. Appl. Prob., **18**, 628-645 (1986).
7. Rahimov, I., Al-Sabah, W.S.: *Branching processes with decreasing immigration and tribal emigration*, Arab J. Math. Sci., **6**(2), 81–97 (2000).
8. Sevastyanov, B.A.: *Branching processes*, Izdat. Nauka, Moscow, 436 pp. (1971) (In Russian)
9. Sevastyanov, B. A.: *Limit theorems for branching stochastic processes of special form*, Theory Probab. Appl., **2**(3), 339–348 (1957). (In Russian)
10. Srivastava, O. P., Gupta S. C.: *On a countinuous-time branching process with migration*, **XLIX** (4), 547–552 (1989)
11. Vatutin, V.A.: *Branching processes and their applications*, Lecture courses NOC / Mathematical Institute, Steklov VS, Russian Academy of Sciences (MIAN), Moscow, 108 pp. (In Russian)
12. Yakymyshyn, K.: *Equation for generation function for branching processes with migration*, Visnyk of the Lviv Univ. Series Mech. Math., **84**, 119–125 (2017). (in Ukraine)
13. Yanev, G.P.: *Critical controlled branching processes and their relatives*, Pliska Studia Mathematica., **24**, 111–130 (2015).
14. Zaitsev, V.F., Polyanin, A.D.: *Handbook of First Order Partial Differential Equations*, Izdat. Fizmatlit, Moscow, 416 pp. (2003). (In Russian) or A.D. Polyanin, V.F. Zaitsev, A. Moussiaux, *Handbook of First Order Partial Differential Equations*, Taylor & Francis, London, 500 pp. (2002).