Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **43** (1), 1-11 (2023).

On fredholm property of a periodic type boundary value problem

Gulsum A. Agayeva

Received: 11.01.2023 / Revised: 12.05.2023 / Accepted: 26.06.2023

Abstract. *In the paper we study Fredholm property of a boundary value problem in a finite domain of a class of second-order differential equation of elliptic type in a seperable Hilbert space. Sufficient conditions that provide regular and Fredholm solvability of the given problem, are found. These conditions are expressed only by the coefficients. The paper shows how the regular and Fredholm solvability of the boundary value problem are related with the norms of the intermediate derivative operators. Furthermore, the property of the internal compactness of the homogeneous equation is proved.*

Keywords. Hilbert space · linear operator · izomorfism · self adjoint operator · regular solvability · Fredholm solvability.

Mathematics Subject Classification (2010): 35125, 47A10, 58140

1 Introduction

Solvability of operator-differential equations and related problems originate from the works of E. Hille, K. Iosido, T. Kato and others. These authors have mainly studied the Cauchy problem. Further, boundary value problems and related problems have been studied by many authors. Some of these results have found their reflection in the books of A.A. Dezin [6], V.I. Gorbachuk and M.L. Gorbachuk [11], S.Ya. Yakubov [21] and others. In an infinite domain, boundary value problems have been studied in the important papers of Yu.A. Dubinsky [7], M.G. Gasymov [8], S.S. Mirzoev [19], A.A. Shkalikov [20], A.R. Aliyev [4, 5], G.M. Gasymova [9, 10], S.S. Mirzoev, A.R. Aliyev, L.M. Rustamov [17, 18], S.S. Mirzoev, A.R. Aliyev, G. M. Gasymova [16] and others. In an finite domain, boundary value problems with variable coefficients have been studied very little. We can note the works of S.S. Mirzoev with G.A.Agayeva [14, 15], G.A.Agayeva [1, 2, 3].

Let H be a separable Hilbert space. Assume that C is a self-adjoint operator with domain of definition $D(C)$. Then for all $\gamma \geq 0$ the domain of definition of the operator C^γ will be a Hilbert space $H_\gamma (\gamma \geq 0)$ with a scalar product $(x, y)_\gamma = (C^\gamma x, C^\gamma y)$. For $\gamma = 0$ we assume $H_0 = H$ and $(x, y)_0 = (x, y)$.

G.A. Agayeva
Baku State University, AZ 1148, Baku, Azerbaijan
E-mail: Gulsum_agayeva@mail.ru

Denote by $L_2((0,1):H)$ a Hilbert space of vector –functions determined almost everywhere in $(0,1)$ for which

$$\|f\|_{L_2((0,1):H)} = \left(\int_0^1 \|f(t)\|^2 \, dt\right)^{1/2}.$$

Following the book [13], we determine the Hilbert space $W_2^2((0,1):H) = \{u : u'' \in L_2((0,1):H),\ C^2u \in L_2((0,1):H)\}$ with the norm

$$\|u\|_{W_2^2((0,1):H)} = \left(\|u''\|^2_{L_2((0,1):H)} + \ \|C^2u\|^2_{L_2((0,1):H)}\right)^{1/2}.$$

We determine the subspace $W^2_{2,\psi}((0,1):H)$ as follows

$$W^2_{2,\psi}((0,1):H) = \{u : u'' \in W_2^2((0,1):H),$$

$$u(0) = e^{i\psi}u(1), u'(0) = e^{i\psi}u'(1),\ \psi \in \mathbb{R} = (-\infty,\infty)\}.$$

From the trace theorem it follows that $W^2_{2,\psi}$ is a complete Hilbert space [13, p.41]. Note that for $\psi = 2\pi\ k\ (k = 0,1,2,...)$ we obtain a subspace of periodic functions, while for $\psi = \pi(2k+1)\ (k = 0,1,2,...)$ we obtain a space of anti periodic functions.

Consider in H the boundary value problem

$$L(d/\,dt)u(t) = -u''(t) + \rho(t)A^2u(t) + (A_1 + K_1)u'(t)$$

$$+(A_2 + K_2)u(t) = f(t),\ t \in (0,1), \tag{1.1}$$

$$u(0) = e^{i\psi}u(1), \quad u'(0) = e^{i\psi}u'(1), \tag{1.2}$$

where the operator coefficients of the equation (1.1) satisfy the conditions:

1) is a normal operator with a completely continuous operator in H, whose set of spectra is contained in the angular sector

$$S_\varepsilon = \{\lambda : |\arg\lambda < \varepsilon,\ 0 \le \varepsilon \le \pi/4\};$$

2) $\rho(t)$ is a scalar function defined in $(0,1)$, measurable and bounded, moreover $0 < \alpha \le \rho(t) \le \beta < \infty$, where $\alpha, \beta \in \mathbb{R}$;

3) The operators $B_1 = A_1A^{-1}$ and $B_2 = A_2A^{-2}$ are bounded in H;

4) The operators $T_1 = K_1A^{-1}$ and $T_2 = K_2A^{-2}$ are completely continuous operators in H.

Note that subject to the conditions 1), the operator A has an orthonormal basis system in H, i.e. $Ae_k = \lambda_k e_k \quad (k = 1,2,...)\ 0 < |\lambda_1| \le |\lambda_2| \le ... \le |\lambda_k| < ...$, moreover

$$(e_k, e_j) = \delta_{k,j} = \begin{cases} 0, k \ne j \\ 1, \ \ k = j \end{cases} \text{ and } \lambda_k = |\lambda_k| e^{i\varphi_k}, \varphi_k \in S_\varepsilon,\ k = 1,2,...,$$

$$A(\cdot) = \sum_{k=1}^{\infty} \lambda_k(\cdot, e_k)e_k,$$

$$C(\cdot) = \sum_{k=1}^{\infty} |\lambda_k|(\cdot, e_k)e_k,$$

$$U(\cdot) = \sum_{k=1}^{\infty} e^{i\varphi_k}(\cdot, e_k)e_k, \qquad \varphi_k \in S_\varepsilon,\ k = 1,2,....$$

In what follows, we will use the theorems on intermediate derivatives and from the trace theorem [13 , pp. 31, 41], i.e.

1. if $u \in W_2^2((0,1):H)$, then $Cu' \in L_2((0,1):H)$ and

$$\|Cu'\| \leq const\|u\|_{W_2^2((0,1):H)},$$

2. if $u \in W_2^2((0,1):H)$, then for any $t_0 \in [0,1]$ $u(t_0)$ and $u'(t_0)$ these exists $u(t_0) \in H_{3/2}$, $u'(t_0) \in H_{1/2}$ and we have the inequality

$$\|u(t_0)\|_{3/2} \leq const\,\|u\|_{W_2^2((0,1):H)},\ 0 \leq t_0 \leq 1$$

and

$$\|u'(t_0)\|_{1/2} \leq const\,\|u\|_{W_2^2((0,1):H)}.$$

Definition 1.1 *If for $f \in L_2((0,1):H)$ there exist $u \in W_2^2((0,1):H)$, then $u(t)$ is called a regular solution of the equation* (1.1).

Definition 1.2 *If for any $f \in L_2((0,1):H)$ there exists a regular solution to the equation* (1.1) *satisfying the boundary conditions* (1.2) *in the sense of convergence*

$$\lim_{t\to 0}\|u(t) - e^{i\psi}u(1-t)\|_{3/2} = 0\ ,\ \lim_{t\to +0}\|u'(t) - e^{i\psi}u'(1-t)\|_{1/2} = 0$$

and we have the estimates

$$\|u(t)\|_{W_2^2((0,1):H)}\ \leq\ const\|f\|_{L_{2((0,1):H)}}$$

then the problem (1.1) –(1.2) *is called regularly solvable.*

Denote by

$$Lu = \mathrm{P}u + Ku,\ u \in W_{2,\psi}^2((0,1):H), \tag{1.3}$$

where

$$Pu = -u'' + \rho(t)A^2u + A_1u' + A_2u,\ \ u \in W_{2,\psi}^2((0,1):H) \tag{1.4}$$

and

$$Ku = K_1u' + K_2u,\ \ u \in W_{2,\psi}^2((0,1):H). \tag{1.5}$$

Definition 1.3 *If the operator L mapping $u \in W_{2,\psi}^2((0,1):H)$ to $L_2((0,1):H)$ is Fredholm, we say that the problem* (1.1), (1.2) *is Fredholm solvable.*

2 Some results

Thus, the set of spectra of the operator A is contained in the spectrum of S_ε, i.e. there exists bounded spectra $e^{-At}(t \geq 0)$ generated by the operator A.

We have.

Lemma 2.1 *For $\varphi \in H_{3/2}$ the inequality*

$$\|e^{-tA}\varphi\|_{W_2^2((0,1):H)} \leq \frac{1}{\sqrt{\cos\varepsilon}}\|\varphi\|_{3/2} \tag{2.1}$$

holds.

Proof. Since $\varphi \in H_{3/2}$, then $C^{3/2}\varphi = x \in H$. Then

$$\|e^{-tA}\varphi\|^2_{W_2^2((0,1):H)} = \|A^2 e^{-tA}\varphi\|^2_{L_2((0,1):H)} + \|C^2 e^{-tA}\varphi\|^2_{L_2((0,1):H)}.$$

Since for $y \in D(A^2)$, $|A^2 x\| = \|C^2 x\|$, then

$$\|e^{-tA}\varphi\|^2_{W_2^2((0,1):H)} = 2\|C^2 e^{-tA}\varphi\|^2_{L_2((0,1):H)} = 2\|C^{1/2} e^{-tA}x\|^2_{L_2((0,1):H)}.$$

Using spectral expansion of the operators A and C , we have:

$$\|C^{1/2}e^{-tA}x\|^2_{L_2((0,1):H)} = \int_0^1 (C^{1/2}e^{-tA}x \ , \ C^{1/2}e^{-tA}x)dt = \int_0^1 (Ce^{-t(A+A^*)}x, x)dt =$$

$$= \int_0^1 \sum_{k=1}^{\infty} |\lambda_k| e^{-2t|\lambda_k| Re\varphi_k} |(x, e_k)|^2 dt = \sum_{k=1}^{\infty} |\lambda_k| \ |(x, e_k)|^2 \frac{1}{2|\lambda_k| Re\varphi_k} e^{-2|\lambda_k| \cos\varphi_k t}\Big|_0^1 \le$$

$$\le \sum_{k=1}^{\infty} |(x, e_k)|^2 \frac{1}{2\cos\varepsilon}(1 - e^{-2|\lambda_1|\cos\varepsilon}) \le \frac{1}{2\cos\varepsilon}\|x\|^2 = \frac{1}{2\cos\varepsilon}\|C^{3/2}\varphi\|^2 = \frac{1}{2\cos\varepsilon}\|\varphi\|^2_{3/2}.$$

Then

$$\|e^{-tA}\varphi\|^2_{W_2^2((0,1):H)} \le \frac{2}{2\cos\varepsilon}\|\varphi\|^2_{3/2} = \frac{1}{\cos\varepsilon}\|\varphi\|^2_{3/2}.$$

The lemma is proved .

Lemma 2.2 *Let $x \in D(A)$, then*

$$Re(A^* x, Ax) \ge \cos 2\varepsilon \|Cx\|^2. \tag{2.2}$$

Proof. From spectral expansion of the operator A it follows that

$$(A^*x, Ax) = (\sum_{k=1}^{\infty} \bar{\lambda}_k (x, e_k) e_k, \sum_{p=1}^{\infty} \lambda_p (x, e_p) e_p) = \sum_{k=1}^{\infty} \bar{\lambda}_k (x, e_k)(e_k, \lambda_k (x, e_k) e_k)$$

$$= \sum_{k=1}^{\infty} \bar{\lambda}_k^2 |(x, e_k)|^2 = \sum_{k=1}^{\infty} |\lambda_k|^2 e^{-2\varphi_k} |(x, e_k)|^2 \quad , \quad \varphi_k \in S_\varepsilon$$

then $Re(A^*x, Ax) = \sum\limits_{k=1}^{\infty} |\lambda_k|^2 Re\, e^{-2\varphi_k} |x, e_k|^2 \ge \cos 2\varepsilon \|Cx\|^2$.

The lemma is proved.

The operator $P : W^2_{2,\psi}((0,1) : H) \to L_2(R_+; H)$ determined by the equality (1.4) are represented in the form

$$\mathrm{P} = \mathrm{P}_0 + \mathrm{P}_1,$$

where

$$P_0 u = -u''(t) + \rho(t) A^2 u(t) \, , \ \ u \in W^2_{2,\psi}((0,1) : H) \tag{2.3}$$

$$P_1 u = A_1 u' + A_2 u, \ \ u \in W_2^2((0,1) : H). \tag{2.4}$$

We have

Theorem 2.1 *Let conditions 1) and 2) be fulfilled. Then for any* $u \in W^2_{2,\psi}((0,1):H)$ *we have the inequality*

$$\left\|Au'\right\|_{L_2((0,1):H)} \leq d_1(\varepsilon)\left\|P_0u\right\|_{L_2((0,1):H)} \tag{2.5}$$

and

$$\left\|A^2u\right\|_{L_2((0,1):H)} \leq d_0(\varepsilon)\left\|P_0u\right\|_{L_2((0,1):H)}, \tag{2.6}$$

where

$$d_1(\varepsilon) = \frac{1}{2\sqrt{\alpha}}\ \frac{1}{\cos\varepsilon},\ \ d_0 = \frac{1}{\alpha}.$$

Proof. Obviously,

$$\|\rho^{-1/2}P_0u\|^2_{L_2((0,1):H)} = \|-\rho^{-1/2}u'' + \rho^{1/2}u\|^2_{L_2((0,1):H)} = \|-\rho^{-1/2}u''\|^2_{L_2((0,1):H)}$$
$$+\|\rho^{1/2}A^2u\|^2_{L_2((0,1):H)} - 2Re(\rho^{-1/2}u'', \rho^{1/2}A^2u) = \|\rho^{-1/2}u''\|^2_{L_2((0,1):H)}$$
$$+\|\rho^{1/2}A^2u\|^2_{L_2((0,1):H)} - Re(u'', A^2u)_{L_2((0,1):H)}. \tag{2.7}$$

On the other hand, integrating by parts and considering $u \in W^2_{2,\psi}((0,1):H)$, $u(0) = e^{i\psi}u(1),\ u'(0) = e^{i\psi}u'$ we obtain

$$(u'', A^2u)_{L_2((0,1):H)} = \int_0^1 (u''(t), A^2u(t))_H dt = \int_0^1 (u''(t), U^2C^2u(t))_H dt$$
$$= (C^{1/2}u'(t), U^2C^{3/}u(1))|_0^1 - \int_0^1 U^*Cu'(t), UCu'(t))dt = (C^{1/2}u'(1), U^2C^{3/2}u(1))$$
$$-(C^{1/2}u'(0), U^2C^{3/2}u(0)) - (A^*u'(t), Au'(t))dt. \tag{2.8}$$

Since $u(0) = e^{i\psi}u(1),\ u'(0) = e^{i\psi}u'(1)$, then

$$C^{1/2}u'(1), U^2C^{3/2}u(1)) - (C^{1/2}e^{i\psi}u'(1), U^2C^{3/2}e^{-i\psi}u(1)) = 0.$$

Then it follows from the equality (2.8) that

$$-(Reu'', A^2u) = (A^*u'(t), Au(t)).$$

Applying the inequality (2.2) from Lemma 2.2 from the equality (2.7) we obtain

$$\|\rho^{-1/2}\mathrm{P}_0u\|^2_{L_2((0,1):H)} \geq \|\rho^{-1/2}u''\|^2_{L_2((0,1):H)}$$
$$+\|\rho^{1/2}A^2u\|^2_{L_2((0,1):H)} + 2\cos 2\varepsilon\|Cu'\|^2_{L_2(R_+:H)}. \tag{2.9}$$

It follows from inequality (2.9) that

$$\|\rho^{1/2}A^2u\|^2_{L_2((0,1):H)} \leq \|\rho^{-1/2}P_0u\|^2_{L_2((0,1):H)},$$

then

$$\|A^2u\|^2_{L_2((0,1):H)} = \|\rho^{1/2}\rho^{-1/2}A^2u\|^2_{L_2((0,1):H)} \leq \frac{1}{\alpha}\|\rho^{-1/2}P_0u\|^2 = \frac{1}{\alpha^2}\|P_0u\|^2_{L_2((0,1):H)},$$

i.e.

$$\|A^2u\|_{L_2((0,1):H)} \leq \frac{1}{\alpha}\|P_0u\||_{L_2((0,1):H)}.$$

Inequality (2.6) is proved. We now prove inequality (2.5).

Obviously, for $u \in W^2_{2,\psi}((0,1):H)$ integrating by parts, we obtain

$$\|Au'\|^2_{L_2((0,1):H)} = \|Cu'\|^2_{L_2((0,1):H)} = (Cu', Cu')_{L_2((0,1):H)}$$

$$= -(u'', C^2u)_{L_2((0,1):H)} \leq -(\rho^{-1/2}u'', \rho^{1/2}C^2u)_{L_2((0,1):H)}$$

$$\leq \frac{1}{2}\big(\|\rho^{-1/2}u''\|^2_{L_2((0,1):H)} + \|\rho^{1/2}C^2u\|^2_{L_2((0,1):H)}\big). \tag{2.10}$$

Using inequality (2.9) in inequality (2.10), we get

$$\|Au'\|^2_{L_2((0,1):H)} \leq \frac{1}{2}\big(\|\rho^{-1/2}P_0u\|^2_{L_2((0,1):H)} - 2\cos 2\varepsilon\|Au'\|^2_{L_2((0,1):H)}\big)$$

or

$$(1+\cos 2\varepsilon)\,\|Au'\|^2_{L_2((0,1):H)} \leq \frac{1}{2}\|\rho^{-1/2}P_0u\|^2_{L_2((0,1):H)},$$

i.e.

$$2\cos^2\varepsilon\|Au'\|^2_{L_2((0,1):H)} \leq \frac{1}{2}(\|\rho^{-1/2}\mathrm{P}_0u\|^2_{L_2((0,1):H)}.$$

Hence we obtain

$$\|Au'\|^2_{L_2((0,1):H)} \leq \frac{1}{4\cos^2\varepsilon}\|\rho^{-1/2}\mathrm{P}_0u\|^2_{L_2((0,1):H)}.$$

Hence we obtain

$$\|Au'\|_{L_2((0,1):H)} \leq \frac{1}{2\cos\varepsilon}\|\rho^{-1/2}\mathrm{P}_0u\|_{L_2((0,1):H)} \tag{2.11}$$

or

$$\|Au'\|_{L_2((0,1):H)} \leq \frac{1}{2\cos\varepsilon}\frac{1}{\alpha}\|\mathrm{P}_0u\|_{L_2((0,1):H)}. \tag{2.12}$$

Inequality (2.5) is also proved.

Considering the operator P_0 in $L_2((0,1):H$ with domain of definition $D(\mathrm{P}_0) = W^2_{2,\psi}((0,1):H)$ we obtain that the adjoint

$${\mathrm{P}_0}^*u = -u'' + \rho(t)A^*u$$

has the domain of definition $W^2_{2,\psi}((0,1):H)$ and A and A^* have the same properties, we obtain the following corollary.

Corollary 2.1 *For $u \in W^2_{2,\psi}((0,1):H)$ we have the inequalities*

$$\left\|A^*u'\right\|_{L_2((0,1):H)} \leq d_1(\varepsilon)\left\|L^*u\right\|_{L_2((0,1):H)} \tag{2.13}$$

and

$$\left\|A^{*2}u\right\|_{L_2((0,1):H)} \leq d_0(\varepsilon)\left\|L^*u\right\|_{L_2((0,1):H)}. \tag{2.14}$$

3 Main results

At first we prove the following theorem.

Theorem 3.1 *The operator L_0 isomorphically maps the space $W^2_{2,\psi}((0,1):H)$ onto the space $L_2((0,1):H$.*

Proof. From the inequalities (2.5) and from (2.13) it follows $Ker\mathrm{P}_0=\{0\}$ and $Ker\mathrm{P}_0^*=\{0\}$.

Indeed, if $Pu=0$, then from inequality (2.5) it follows that $A^2u=0$, i.e. $u=0$ since $KerP_0^*=\{0\}$, then ImP is everywhere dense in $L_2(R:H)$. On the other hand, for $u\in D(P)$

$$\|P_0u\|_{L_2((0,1):H)}=\|\rho^{1/2}P_0u\|_{L_2((0,1):H)}$$
$$\le\beta^{1/2}P_0u\|^2_{L_2((0,1):H)}\le\beta^{1/2}2(\|\rho^{-1/2}u''\|^2_{L_2((0,1):H)})$$
$$+\|\rho^{1/2}A^2u\|^2_{L_2((0,1):H)})\le const\|u\|^2_{W_2^2((0,1):H)},$$

i.e. P_0 is a continuous operator . On the other hand

$$\|\rho^{1/2}P_0u\|^2_{L_2((0,1):H)}\ge\|\rho^{1/2}u''\|^2_{L_2((0,1):H)}+\|\rho^{1/2}A^2u\|^2_{L_2((0,1):H)}$$
$$\ge const\|u\|^2_{W_2^2((0,1):H)}\ge const\|u\|^2_{L_2((0,1):H)},$$

then

$$\|\mathrm{P}_0u\|^2_{L_2((0,1):H)}=\|\rho^{1/2}\rho^{-1/2}P_0u\|_{L_2((0,1):H)}$$
$$\ge\alpha\|\rho^{-1/2}P_0u\|^2_{L_2((0,1):H)})\ge const\|u|_{L_2((0,1):H)}$$

Thus, there exists P_0^{-1} and it is bounded.

The theorem is proved.

Theorem 3.2 *Let conditions 1)-4) be fulfilled and we have the inequality*

$$q=\frac{1}{2\sqrt{\alpha}}\frac{1}{\cos\varepsilon}\|B_1+T_1\|\frac{1}{\alpha}\|B_2+T_2\|<1. \tag{3.1}$$

Then the problem (1.1)-(1.2) *is regularly solvable.*

Proof. Let's write the problem (1.1)-(1.2) as an equation $Lu=f$, where $u\in W^2_{2,\psi}((0,1):H)$,$f\in L_2((0,1):H$ while $Lu=\mathrm{P}_0u+\mathrm{P}_1u+Ku$, where $P_0u=-u''+\rho(t)u,\ P_1u=A_1u'+A_2u,\ Ku=K_1u'+K_2u.$

Since the operator implements isomorphism between the spaces $W^2_{2,\psi}((0,1):H)$ and $L_2((0,1):H$, then for any $w\in L_2((0,1):H$ there exists $u\in W^2_{2,\psi}((0,1):H)$, where $Lu=w$. Then from the equation $Lu=f$ we obtain the equation

$$w+(P_1P_0^{-1}+KP_0^{-1})w=f.$$

In the space $L_2((0,1):H$, we estimate the norms of the operator $\mathrm{P}_1\mathrm{P}_0^{-1}+K\mathrm{P}_0^{-1}$

$$\|\mathrm{P}_1\mathrm{P}_0^{-1}+K\mathrm{P}_0^{-1}\|_{L_2((0,1):H)}=\|\mathrm{P}_1u+Ku\|_{L_2((0,1):H)}$$
$$\le\|(A_1+K_1)A^{-1}Au'\|_{L_2((0,1):H)}+\|(A_2+K_2)A^{-2}A^2u\|_{L_2((0,1):H)}$$
$$\le\|B_1+T_1\|\,\|Au'\|_{L_2((0,1):H)}+\|B_2+T_{21}\|\,\|A^2u\|_{L_2((0,1):H)}. \tag{3.2}$$

Taking into account the inequalities (2.11) and (2.12) in the inequality (3.2), we get

$$\|\mathrm{P}_1\mathrm{P}_0^{-1}+K\mathrm{P}_0^{-1}w\|_{L_2((0,1):H)}\leq\|(\frac{1}{2\sqrt{\alpha}}\frac{1}{\cos\varepsilon}\|B_1$$

$$+T_1\|\frac{1}{\alpha}\|B_2+T_2\|)\|\mathrm{P}_0u\|_{L_2((0,1):H)}=q\|w\|_{L_2((0,1):H)}.$$

Since $q<1$, the operator $E+(P_1+K)P_0^{-1}$ is invertible in the space $L_2((0,1):H$, then

$$w=(E+(P_1+K)\mathrm{P}_0^{-1})^{-1}f$$

while

$$u=P_0^{-1}(E+(P_1+K)P_0^{-1})f.$$

Hence it follows that

$$\|u\|_{W_2^2((0,1):H)}\leq cons\|f\|_{L_2((0,1):H)}.$$

The theorem is proved.

Note that in proving the theorem we did not use complete continuity of the operators $T_1=K_1A^{-1}$ and $T_2'=K_2A^{-2}$, we used their boundedness in H.

Corollary 3.1 *If the conditions 1)-3) are fulfilled, and*

$$q_1=\frac{1}{2\sqrt{\alpha}}\frac{1}{\cos\varepsilon}\|B_1\|\frac{1}{\alpha}\|B_2\|<1 \tag{3.3}$$

where $B_j=A_jA^{-j}\quad(j=0,2)$*, the problem* (1.1), (1.2) *is regularly solvable for* $T_1=0,\ T_2=0$.

Let us now prove a theorem on the Fredholm solvability of problem (1.1)-(1.2).

Theorem 3.3 *Let the conditions 1)-4) and inequality* (3.3) *be fulfilled. Then problem* (1.1)-(1.2) *is Fredholm solvable.*

Proof. It suffices to prove the operator $L=P+K$ is a Fredholm operator, where the operators P and K are determined from the equalities (1.4), (1.5).

Corollary 3.1 yields that the operator P isomorphically maps the space $W_{2,\psi}^2((0,1):H)$ onto the space $L_2((0,1):H$. At first we show that for rather small $\varepsilon>0$ the following inequality is fulfilled.

$$\|Ku\|_{L_2((0,1):H)}\leq\varepsilon\|u\|_{W_2^2((0,1):H)}+\eta(\varepsilon)\|u\|_{L_2((0,1):H)}. \tag{3.4}$$

Since

$$Ku=K_1u'+K_2u=K_1A^{-1}u'+K_2A^{-2}u=T_1Au'+T_2A^2u,$$

where T_1 8 T_2 are completely continuous operators in H. Therefore, they can be represented in the form of a finite-dimensional operator of the poles of the operators with rather small norms: i.e.

$$K_1=S_1+F_1,\ K_2=S_2+F_2$$

moreover $S_1(\cdot)=\sum\limits_{k=1}^{m}(\cdot,\varphi_k^{(1)})\,\psi_k^1$, $S_2(\cdot)=\sum\limits_{j=1}^{p}(\cdot,\varphi_k^{(1)}\psi_k^{(1)},\ \varphi_k^1,\psi_k^1,\varphi_j^2,\psi_j^2\in H,\ (k=1,...,m,\ \ j=1,..,p)$, a $\|F_1\|<\varepsilon$ and $\|F_2\|<\varepsilon$. Then obviously it follows from the theorem on intermediate derivatives, that

$$\|F_1(Cu')\|_{L_2((0,1):H)}\leq\varepsilon\|Cu'\|_{L_2((0,1):H}\leq\varepsilon\|u\|_{W_2^2((0,1):H)}^2,$$

$$\|F_2(C^2u)\|_{L_2((0,1):H)} \leq \varepsilon\|C^2u\|_{L_2((0,1):H} \leq \varepsilon\|u\|^2_{W_2^2((0,1):H)}.$$

Therefore, we must prove the inequality (3.4) for the operators S_1 and S_2. Since S_1 and S_2 is the sum of the finite number of finite -dimensional operators of the form $T_0\ (\cdot)\ = (\cdot,\ \varphi)\psi, \varphi, \psi \in H$, we prove inequality (3.4) for the operators T_0.
Since

$$\varphi = \sum_{k=1}^{\infty}(\varphi, e_k)e_k = \sum_{k=1}^{N}(\cdot\ ,\ \varphi)e_k + \sum_{N+1}^{\infty}(\cdot\ ,\ \varphi)e_k$$

we choose N rather large so that $\|\tilde{\varphi}\| = \sum_{k=1}^{\infty}(\cdot\,,\varphi)e_k\| < \varepsilon$. Thus

$$\varphi = \sum_{k=1}^{N}(\varphi, e_k) + \tilde{\varphi}\ ,\ \ \|\tilde{\varphi}\| < \varepsilon.$$

Then, obviously it follows from the theorem on intermediate derivatives that

$$\|Au', \tilde{\varphi})\psi\|_{L_2((0,1):H)} \leq \|Au'\|_{L_2((0,1):H)}\|\tilde{\varphi}\|\ \|\psi\|$$
$$\leq \|Cu'\|\ \|\tilde{\varphi}\|\ \ \|\psi\| \leq \varepsilon_1\|u\|_{W_2^2((0,1):H)}. \tag{3.5}$$

In a similar way, we have

$$\|A^2u, \tilde{\varphi})\psi\|_{L_2((0,1):H)} \leq \varepsilon_1\|u\|_{W_2^2((0,1):H)}.$$

On the other hand,

$$\|S_1(Au')\|_{L_2((0,1):H)} = \sum_{k=1}^{h}((Au', e_k)e_k, e_k)\psi)_{L_2((0,1):H)}$$

$$= \|\sum_{k=1}^{h}((u', \bar{\lambda}_k e_k)e_k, e_k)\psi)_{L_2((0,1):H)} \leq |\lambda_N| \sum_{k=1}^{h}\|u'\|_{L_2((0,1):H)}\|\psi\|.$$

Since A^{-1} is a completely continuous operator, then the imbedding $W_2^2((0,1)\ :\ H) \rightarrow W_2^1((0,1)\ :\ H) \rightarrow L_2((0,1)\ :\ H)$ is compact in finite interval (0,1), applying theorem (16.4 p. 126 from the book [13] we obtain

$$\|u\|_{W_2^1((0,1):H)} \leq \varepsilon\|u\|_{W_2^2((0,1):H)} + \eta(\varepsilon)\|u\|_{L_2((0,1):H)}.$$

Hence it follows that

$$\|u'\|_{L_2((0,1):H)} \leq \varepsilon\|u\|_{W_2^2((0,1):H)} + \eta(\varepsilon)\|u\|_{L_2((0,1):H)},$$

i.e.

$$\|S_1(Au')\|_{L_2((0,1):H)} \leq \varepsilon\|u\|_{W_2^2((0,1):H)} + \eta(\varepsilon)\|u\|_{L_2((0,1):H)}. \tag{3.6}$$

In a similar way we have

$$\|S_2(A^2u)\|_{L_2((0,1):H)} \leq \varepsilon\|u\|_{W_2^2((0,1):H)} + \eta(\varepsilon)\|u\|_{L_2((0,1):H)}. \tag{3.7}$$

It follows from the inequality (3.7) that for $\varepsilon > 0$ the inequality (3.4) is valid. We now prove that the operator T is a compact operator acting from $W_2^2((0,1)\ :\ H)$ to $L_2((0,1)\ :\ H)$. Let $M > 0$ while

$$Q_M = \left\{u:\ \ u \in W_{2,\psi}^2((0,1):H),\ \|u\|_{W_2^2((0,1):H)}\ \leq M\right\}.$$

Since the imbeddings $W_2^2((0,1):H) \to L_2((0,1):H)$, then there exists such a sequence $u_n \in Q_M$ $(\|u\|_{W_2^2((0,1):H)} \leq M)$ that u_n converges in $L_2((0,1):H)$. Then, using the inequality (3.4), we have

$$\|Ku_n - Ku_m\|_{L_2((0,1):H)} \leq \varepsilon\|u_n - u_m\|_{W_2^2((0,1):H)} + \eta(\varepsilon)\|u_n - u_m\|\|_{L_2((0,1):H)}$$

$$\leq \varepsilon(\|u_n\|\|_{W_2^2((0,1):H)} + \|u_m\|\|_{W_2^2((0,1):H)} + \eta(\varepsilon))\|u_n - u_m\|\|_{L_2((0,1):H)}$$

$$\leq |2\varepsilon M + \eta(\varepsilon)\|u_n - u_m\|_{L_2((0,1):H)}.$$

Now, choosing rather large n and m, we obtain

$$\|Ku_n - Ku_m\|_{L_2((0,1):H)} \leq \delta,$$

where δ is a rather small number. Thus, the operator K is a compact operator acting from $W_2^2((0,1):H)$ to $L_2((0,1):H)$.

On the other hand,

$$Lu = Pu + Ku = P(E + P^{-1}K)u$$

the operator $E+P^{-1}K$ is Fredholm, the operator P isomorphic, the operator L is Fredholm, and the solution of the equation $Lu = f$ satisfies the estimates

$$\|u\|_{W_2^2((0,1):H)} \leq const\|f\|_{L_2((0,1):H)}.$$

The theorem is proved.

Acknowledgements: The author expresses her gratitude to Prof. S.S. Mirzoev for valuable discussions.

References

1. Agayeva G.A.: *On a boundary value problem for operator-differential equations of the second order*. Proceedings of the Pedagogical University, Section of the Natural Sciences. 9-17 (2017).
2. Agayeva G.A.:*On the existence and uniqueness of the generalized solution of a boundary value problem for second order operator-differential equations*. Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **34**(4), 3-8 (2014).
3. Agayeva G.A.: *On the solvability of a boundary value problem for elliptic operator-differential equations with an operator coefficient in the boundary condition*. Bulletin of Baku University, **1**, 37-42 (2015).
4. Aliyev A.R.: *Boundary value problems for a class of high-order operator-differential equations with variable coefficients*. Mat. Zametki **74**(6), 803-814 (2003).
5. Aliyev A.R.: *On the solvability of a class of boundary value problems for second-order operator-differential equations with a discontinuous coefficient in a weighted space*. Differ. Uravn. **43**(10), 1423-1426 (2007).
6. Dezin A.A.: General issues of theory of boundary value problem Ì : Nauka, p.207, (1980).
7. Dubinsky Yu.A.: *On some arbitrary order differential operator equations*. Matemat. sbornik, **90**(1) (132), 3-22 (1972).
8. Gasimov M.G.: *On solvabilety of boundary value problem for a class operator-differential equations*. Dokl. Akad. Nauk SSSR **235**(3), 505-508 (1977).
9. Gasimova G.M.: *On solvability conditions of a boundary value problem with an operator in the boundary condition for a second order elliptic operator-differential equations*. Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **40**, 172-177 (2014).

10. Gasimova G.M.: *On well-defined solvability of a boundary value problem for an elliptic differential equation in Hilbert space.* Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **35**(1), 31-36 (2015).
11. Gorbacuk V.I.,Gorbacuk M.L.: *Boundary value problem for differential-operator equations*. Kiev, Nauova Dumka, 284 pp. (1984).
12. Krein S.G.: *Linear differential equation in Banach space*. M, Nauka, 464 pp. (1967).
13. Lions-J.L., Magenes E.: *Inhomogeneous boundary value problems and their applications*. Moscow: Mir, 371 pp. (1977).
14. Mirzoyev S.S., Aghayeva G.A.: *On the solvability conditions of one boundary value problem for the second order differential equations with operator coefficients*. Int. J. Math. Anal. **8**(1-4), 149-156, (2014).
15. Mirzoyev S.S., Agayeva G.A.: *On correct solvability of one boundary value problems for the differential equations of the second order in Hilbert space*. Appl. Math. Sci. **7**(77-80), 3935-3945 (2013).
16. Mirzoev S.S., Aliyev A.R., Gasimova G.M.: *Solvability conditions for a boundary value problem with operator coefficients and estimates for the norms of the intermediate derivative operators related to them*. Dokl. Akad. Nauk **470**(5), 511-513 (2016).
17. Mirzoyev S.S., Aliyev A.R., Rustamova L.A.: *On the boundary value problem with the operator in boundary conditions for the operator-differential equation of second order with discontinuous coefficients.* J. Math. Phys. Anal. Geom. **9**(2), 207-226 (2013).
18. Mirzoyev S.S., Aliyev A.R., Rustamova L.A.: *Solvability conditions for boundary-value problems for elliptic operator-differential equations with discontinuous coefficient*, Math. Notes **92**(5-6), 722-726 (2012).
19. Mirzoyev S.S.: *Conditions for the correct solvability of boundary value problems for operator-differential equations*. Dokl. Akad. Nauk SSSR **273**(2), 292-295 (1983).
20. Skalikov A.A.: *Elliptic equations in Hilbert space and related problems*. Trudy Seminara im. I.G. Petrovskogo, **14**, 140-224 (1988).
21. Yakubov S.Ya.: *Linear-differential equations and their applications*. Baku, Elm, 220 pp. (1985).