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Weak solvability of the first boundary value problem for nonuniformly and strongly degenerate second-order elliptic-parabolic equations in divergent form

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Abstract. *The paper considers the first boundary value problem for a non-uniformly and strongly degenerate second-order elliptic-parabolic equation in divergent form. A Friedrichs-type inequality is proved and conditions are found under which this problem is uniquely generalized solvable in a weighted anisotropic Sobolev space.*

Keywords. elliptic-parabolic equation, non-uniformly and strongly degenerate, Sobolev space.

Mathematics Subject Classification (2010): 35D30, 35J15, 35J70

1 Introduction

Let $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ be Euclidean spaces of points $x = (x_1, ..., x_n)$ and $(x,t) = (x_1, ..., x_n, t)$, respectively, $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary $\partial\Omega \in C^2$, $0 \in \overline{\Omega}, Q_T$ is a cylinder $\Omega \times (-T, 0)$, where , $n \geq 1$ and $T > 0$ is a constant. Denote

$$Q_0 = \left\{(x,t) : x \in \overline{\Omega}, t = -T\right\}, S_T = \partial\Omega \times [-T, 0] \text{ and } \Gamma(Q_T) = Q_0 \cup S_T.$$

Consider in Q_T the first boundary value problem

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right) + \frac{\partial}{\partial t}\left(\varphi(-t)\frac{\partial u}{\partial t}\right) - \frac{\partial u}{\partial t} = f(x,t), \tag{1.1}$$

$$u|_{\Gamma(Q_T)} = 0 \tag{1.2}$$

assuming that $f(x,t) \in L_2(Q_T)$, $\|a_{ij}(x,t)\| -$ is a real symmetric matrix with measurable elements in Q_T, and for all $(x,t) \in Q_T$ and $\xi \in E_n$ the condition

$$\gamma\sum_{i=1}^{n}\lambda_i(x,t)\xi_i^2 \leq \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \leq \gamma^{-1}\sum_{i=1}^{n}\lambda_i(x,t)\xi_i^2, \tag{1.3}$$

is fulfilled, and $\varphi(z)$ is a continuous non-negative non-decreasing function on $[-T, 0]$ and for sufficiently small $z > 0$

$$\varphi(0) = 0, \varphi(z) \geq 0,\ \varphi^{'}(z) \geq 0, \varphi'(0) = 0, \varphi^{''}(z) \geq 0,\ \varphi^{'}(z) \geq \varphi(z)\varphi^{''}(z). \tag{1.4}$$

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Here $\gamma \in (0,1]$ is a constant, and the functions $\lambda_i(x,t), i=1,...,n$ are finite almost everywhere in Q_T and are positive.

Let $\delta>0$ be a constant. We impose the following conditions on the functions $\lambda_i(x,t),\ i=1,...,n$:

$$\lambda_i(x,t)\in L_1(Q_T),\ \ \lambda_i^{-1}(x,t)\in L_{n/2}(\Omega),\ \text{if}\ \ n\geq 2; \tag{1.5}$$

$$\lambda_1^{-1}(x_1,t)\in L_{1+\delta}(\Omega),\ \ \text{if}\ \ n=1. \tag{1.6}$$

The aim of this paper is to find conditions on the functions $f(x,t),\varphi(z)$ and $\lambda_i(x,t),\ i=1,...,n$ for which problem (1.1)-(1.2) is uniquely generalized solvable in the corresponding Sobolev space. We find conditions on the function $\varphi(z)$ under which the properties of solutions of problem (1.1)-(1.2) are identical to the properties of solutions of this problem for non-uniformly degenerate second-order parabolic equations (for $\varphi\equiv 0$) (see e.g. [24]).

Initially, the theory of degenerate elliptic-parabolic equations was studied in the classical work of Keldysh [1], in which, in the case of one space variable and a power type of the function $\varphi(z)$, the correct formulations of boundary value problems for second-order elliptic-parabolic equations were indicated. The results of Keldysh found their development in the work of Fichera [2], in which the weak solvability of the first boundary value problem for second-order elliptic-parabolic equations of a non-divergence structure with smooth coefficients was studied. Let us note the works of Petrushko [3–7], who studied the problems of weak solvability of boundary value problems and the behavior on the boundary of solutions of second-order elliptic-parabolic equations with a divergent structure. As for similar questions for elliptic-parabolic equations of non-divergence structure with smooth coefficients, we point out the works [8–12]. We also note the works [13–18], where the existence and uniqueness of the solution of the first boundary value problem for second-order elliptic and parabolic equations with discontinuous coefficients and Cordes-type conditions are proved. A more complete survey of results on the solvability of boundary value problems for elliptic-parabolic equations can be found in [19–23].

Let us accept some notation and definitions. We will say that $u(x,t)\in A(Q_T)$, if there exists a compact $\overline{K}_u\subset\Omega$ such that $\operatorname{supp} u(x,t)\subset K_u\times[-T,0],\ \ u(x,t)\in C^\infty(\overline{Q}_T)$, $u|_{t=-T}=0$. Denote by $W^{1,1}_{2,\lambda}(Q_T)$, $W^{1,1}_{2,\lambda,\varphi}(Q_T)$ and $W^{2,2}_{2,\lambda,\varphi}(Q_T)$ Banach spaces of measurable functions defined on Q_T, for which the norms

$$\|u\|_{W^{1,1}_{2,\lambda}(Q_T)}=\Big(\int_{Q_T}\left(u^2+\sum_{i=1}^n\lambda_i(x,t)\left(\frac{\partial u}{\partial x_i}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2\right)dxdt\Big)^{1/2},$$

$$\|u\|_{W^{1,1}_{2,\lambda,\varphi}(Q_T)}=\Big(\int_\Omega u^2(x,0)dx+\int_{Q_T}\Big(\sum_{i=1}^n\lambda_i(x,t)\left(\frac{\partial u}{\partial x_i}\right)^2+\varphi(-t)\left(\frac{\partial u}{\partial t}\right)^2\Big)dxdt\Big)^{1/2},$$

and

$$\|u\|_{W^{2,2}_{2,\lambda,\varphi}(Q_T)}=\Big(\int\limits_{Q_T}\Big(u^2+\sum_{i=1}^n\lambda_i(x,t)\Big(\frac{\partial u}{\partial x_i}\Big)^2+\sum_{i,j=1}^n\lambda_i(x,t)\lambda_j(x,t)\Big(\frac{\partial^2 u}{\partial x_i\partial x_j}\Big)^2$$

$$+\left(\frac{\partial u}{\partial t}\right)^2+\varphi^2(-t)\left(\frac{\partial^2 u}{\partial t^2}\right)^2+\varphi(-t)\sum_{i=1}^n\lambda_i(x,t)\left(\frac{\partial^2 u}{\partial x_i\partial t}\right)^2\Big)dxdt\Big)^{\frac{1}{2}},$$

are finite, respectively, where $\lambda=(\lambda_1(x,t),...,\lambda_n(x,t))$. Let $\overset{\circ}{W}{}^{1,1}_{2,\lambda}(Q_T)$, $\overset{\circ}{W}{}^{1,1}_{2,\lambda,\varphi}(Q_T)$ and $\overset{\circ}{W}{}^{2,2}_{2,\lambda,\varphi}(Q_T)$ subspaces of $W^{1,1}_{2,\lambda}(Q_T)$, $W^{1,1}_{2,\lambda,\varphi}(Q_T)$ and $W^{2,2}_{2,\lambda,\varphi}(Q_T)$ are completion of the set of all functions $u(x,t)\in A(Q_T)$ with respect to the norm of the space $W^{1,1}_{2,\lambda}(Q_T)$, $W^{1,1}_{2,\lambda,\varphi}(Q_T)$ and $W^{2,2}_{2,\lambda,\varphi}(Q_T)$, respectively.

The function $u(x,t)\in \overset{\circ}{W}{}^{1,1}_{2,\lambda,\varphi}(Q_T)$ is called a weak solution to problem (1.1)- (1.2) if for the function $v(x,t)\in \overset{\circ}{W}{}^{1,1}_{2,\lambda}(Q_T)$ and $t_1\in(-T,0]$ the integral identity

$$\int_{Q_{t_1}}\Big(\sum_{i,j=1}^{n}a_{ij}(x,t)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}+\varphi(-t)\frac{\partial u}{\partial t}\frac{\partial v}{\partial t}-u\frac{\partial v}{\partial t}\Big)dxdt$$

$$+\int_{\Omega}u(x,t_1)v(x,t_1)dx=-\int_{Q_{t_1}}fvdxdt, \tag{1.7}$$

is valid, where $Q_{t_1}=\Omega\times(-T,t_1)$. Throughout what follows, the notation $C(\cdots)$ means that the positive constant C depends only on the contents of the brackets.

Theorem 1.1 *Let conditions* (1.5) *and* (1.6) *be satisfied. Then for any function* $u(x,t)\in \overset{\circ}{W}{}^{1,1}_{2,\lambda}(Q_{t_1})$ *and* $t_1\in(-T,0]$ *the following inequality*

$$\int\limits_{Q_{t_1}}u^2(x,t)dxdt\le C_{1.1}(\lambda,n,\Omega)\int\limits_{Q_{t_1}}\sum_{i=1}^{n}\lambda_i(x,t)\left(\frac{\partial u}{\partial x_i}\right)^2dxdt. \tag{1.8}$$

holds.

Proof. Let $n\ge 2$. Obviously, it suffices to prove (1.8) for the function $u\in A(Q_T)$. We will use the following classical embedding theorem (see e.g. [21]): for any function $u(x,t)\in C_0^\infty(\Omega)$ for $1\le p<n$ the inequality

$$\|u\|_{L_{\frac{pn}{n-p}}(\Omega)}\le C_{1.2}(n,p,\Omega)\|\nabla u\|_{L_p(\Omega)}, \tag{1.9}$$

holds. Setting $p=\frac{2n}{n+2}$ in (1.9), we obtain

$$\|u\|_{L_2(\Omega)}\le C_{1.2}(n,\Omega)\|\nabla u\|_{L_{\frac{2n}{n+2}}(\Omega)}. \tag{1.10}$$

But on the other hand

$$\|\nabla u\|_{L_{\frac{2n}{n+2}}(\Omega)}=\Big(\int_{\Omega}\sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^{\frac{2n}{n+2}}\Big)^{\frac{2+n}{2n}}$$

$$=\Big(\int_{\Omega}\sum_{i=1}^{n}\lambda_i^{-q}(x,t)\lambda_i^{q}(x,t)\left|\frac{\partial u}{\partial x_i}\right|^{\frac{2n}{n+2}}dx\Big)^{\frac{n+2}{2n}}$$

$$\le\Big(\sum_{i=1}^{n}\left(\int_{\Omega}\lambda_i^{qS}(x,t)\left|\frac{\partial u}{\partial x_i}\right|^{\frac{2nS}{n+2}}dx\right)^{\frac{1}{S}}\left(\int_{\Omega}\lambda_i^{-qS'}(x,t)dx\right)^{\frac{1}{S'}}\Big)^{\frac{n+2}{2n}},$$

where $q>0$ and $S>1$ are arbitrary numbers and, $S'=\frac{S}{S-1}$. Let us now set $S=\frac{n+2}{n}$, $q=\frac{n}{n+2}$. Then $S'=\frac{n+2}{2}$ and therefore

$$\|\nabla u\|_{L_{\frac{2n}{n+2}}(\Omega)}\le\Big(\sum_{i=1}^{n}\Big(\int_{\Omega}\lambda_i(x,t)\left|\frac{\partial u}{\partial x_i}\right|^2dx\Big)^{\frac{n}{n+2}}\Big(\int_{\Omega}\lambda_i^{-n/2}(x,t)dx\Big)^{2/(n+2)}\Big)^{\frac{n+2}{2n}}. \tag{1.11}$$

By virtue of condition (1.5), we have

$$\Big(\int_\Omega \lambda_i^{-n/2}(x,t)dx\Big)^{1/n} \le C_{1.3}(\lambda,n,\Omega), i=1,...,n.$$

Thus, from (1.11) we conclude that

$$\|\nabla u\|_{L_{\frac{2n}{n+2}}(\Omega)} \le C_{1.4}(\lambda,n,\Omega)\Big(\sum_{i=1}^n\int_\Omega \lambda_i(x,t)\left(\frac{\partial u}{\partial x_i}\right)^2 dx\Big)^{1/2}. \quad (1.12)$$

Then from (1.10) and (1.12) it follows

$$\Big(\int_\Omega u^2(x,t)dx\Big)^{1/2} \le C_{1.2}\cdot C_{1.4}\Big(\sum_{i=1}^n\int_\Omega \lambda_i(x,t)\left(\frac{\partial u}{\partial x_i}\right)^2 dx\Big)^{1/2}.$$

We integrate the last inequality with respect to t from $-T$ to t_1. Thus, the required estimate (1.8) follows from this expression if $n \ge 2$.

Let, now $n = 1$. We will use the following embedding theorem (see e.g. [21]):for any function $u(x,t) \in C_0^\infty(\Omega)$ for $1 < p < 2$ the inequality

$$\sup_\Omega |u(x_1,t)| \le C_{1.5}(p,\Omega)\left\|\frac{\partial u}{\partial x_1}\right\|_{L_p(\Omega)},$$

holds.

Then

$$\Big(\int_\Omega u^2(x_1,t)dx_1\Big)^{1/2} \le \sup_\Omega |u(x_1,t)| \le C_{1.5}\Big(\int_\Omega \lambda_1^{-p/2}(x_1,t)\lambda_1^{p/2}\left|\frac{\partial u(x_1,t)}{\partial x_1}\right|^p dx_1\Big)^{1/p}$$

$$\le C_{1.5}\Big(\int_\Omega \lambda_1(x_1,t)\left(\frac{\partial u}{\partial x_1}\right)^2 dx_1\Big)^{1/2}\Big(\int_\Omega \lambda_1^{-p/(2-p)}(x_1,t)dx_1\Big)^{\frac{2-p}{2p}}.$$

Let $\frac{p}{2-p} = 1+\delta$, then $p = \frac{1+\delta}{1+\delta/2}$ and if $n = 1$ then the required estimate (1.8) is proved. Theorem 1.1 is proved.

Theorem 1.2 *Let the coefficients of the operator L satisfying conditions* (1.3)-(1.6) *be defined in the cylindrical region $Q_T \subset \mathbb{R}^{n+1}$. Then the first boundary value problem* (1.1)-(1.2) *is uniquely generalized solvable in the space $\overset{\circ}{W}{}^{1,1}_{2,\lambda,\varphi}(Q_T)$ for any $f(x,t) \in L_2(Q_T)$. Moreover, for the solution $u(x,t)$ the following estimate is true:*

$$\|u\|_{W^{1,1}_{2,\lambda,\varphi}(Q_T)} \le C_{1.6}(\gamma,\lambda,n,\Omega)\,\|f\|_{L_2(Q_T)}. \quad (1.13)$$

Proof. Suppose $\partial\Omega \in C^2$. Let us introduce the following notation for natural numbers $m, (x,t) \in Q_T$ and $i = 1,...,n$:

$$\lambda_i^m(x,t) = \begin{cases} \frac{1}{m}, \text{ if } \lambda_i(x,t) < \frac{1}{m}; \\ \lambda_i(x,t), \text{ if } \frac{1}{m} \le \lambda_i(x,t) \le m, \\ m, \text{if } \lambda_i(x,t) > m. \end{cases}$$

Let $\left\|a_{ij}^m(x,t)\right\|$ be a real symmetric matrix with measurable elements in Q_T and for $i,j = 1,...,n$ as $m\to\infty$ in Q_T $a_{ij}^m(x,t) \to a_{ij}(x,t)$, and for $(x,t) \in Q_T$ and $\xi \in E_n$

$$\gamma\sum_{i=1}^n \lambda_i^m(x,t)\xi_i^2 \le \sum_{i,j=1}^n a_{ij}^m(x,t)\xi_i\xi_j \le \gamma^{-1}\sum_{i=1}^n \lambda_i^m(x,t)\xi_i^2.$$

Denote by $(a_{ij})_h$ the Friedrichs averaging of the function $a_{ij}^m(x,t)$ with the parameter $h>0$. Further, by $\lambda_i^h(x,t)$ and $u^h(x,t)$ we denote the Friedrichs averaging of the function $\lambda_i^m(x,t)$ and $u^m(x,t)$ with parameter $h>0$, respectively.

Consider for $h>0$ the family of the following first boundary value problems

$$L^h u^h=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left((a_{ij})_h\frac{\partial u^h}{\partial x_j}\right)+\frac{\partial}{\partial t}\left(\varphi(-t)\frac{\partial u^h}{\partial t}\right)-\frac{\partial u^h}{\partial t}=f(x,t), \tag{1.14}$$

$$u^h\Big|_{\Gamma(Q_T)}=0, \tag{1.15}$$

where φ satisfies conditions (1.4).

It is clear that $(a_{ij})_h\in C^\infty(\overline{Q}_T)$, and for all $h>0$ with respect to $(a_{ij})_h$ a condition of type (1.3) with constant γ is satisfied. Then, according to [23], there exists a uniquely strong solution $u^h(x,t)\in \overset{\circ}{W}{}^{2,2}_{2,\lambda,\varphi}(Q_T)$ of problem (1.14)-(1.15). It is obvious that $u^h(x,t)\in \overset{\circ}{W}{}^{1,1}_{2,\lambda}(Q_T)$.

We multiply both sides of equation (1.14) by the functions $\upsilon(x,t)\in \overset{\circ}{W}{}^{1,1}_{2,\lambda}(Q_T)$, and then integrate it over the domain Q_T:

$$\int_{Q_T}L^h u^h\upsilon dxdt=\int_{Q_T}f\upsilon dxdt. \tag{1.16}$$

Since $u^h\in \overset{\circ}{W}{}^{1,1}_{2,\lambda}(Q_T)$, we can substitute $\upsilon=u^h$ in (1.16). Then we have

$$\int\limits_{Q_T}\sum_{i,j=1}^{n}(a_{ij})_h\frac{\partial u^h}{\partial x_i}\frac{\partial u^h}{\partial x_j}dxdt-\int\limits_{Q_T}u^h\frac{\partial u^h}{\partial t}dxdt$$

$$+\int\limits_{\Omega}(u^h(x,0))^2dx+\int\limits_{Q_T}\varphi(-t)\left(\frac{\partial u^h}{\partial t}\right)^2dxdt=-\int\limits_{Q_T}fu^hdxdt. \tag{1.17}$$

On the other hand, it follows from (1.3) that

$$\gamma\int\limits_{Q_T}\sum_{i=1}^{n}\lambda_i^h(x,t)\left(\frac{\partial u^h}{\partial x_i}\right)^2dxdt\le\int\limits_{Q_T}\sum_{i,j=1}^{n}(a_{ij})_h\frac{\partial u^h}{\partial x_i}\frac{\partial u^h}{\partial x_j}dxdt.$$

Let us represent the second term on the left-hand side of equality (1.17) as follows

$$\int_{Q_T}u^h\cdot\frac{\partial u^h}{\partial t}dxdt=\frac{1}{2}\int_{\Omega}\Big(u^h(x,t)\Big)^2dx\Big|_{t=-T}^{t=0}=\frac{1}{2}\int_{\Omega}\Big(u^h(x,0)\Big)^2dx.$$

As a result we have the following inequality

$$\gamma\int_{Q_T}\sum_{i=1}^{n}\lambda_i^h(x,t)\left(\frac{\partial u^h}{\partial x_i}\right)^2dxdt+\frac{1}{2}\int_{\Omega}\Big(u^h(x,0)\Big)^2dx$$

$$+\int_{Q_T}\varphi(-t)\left(\frac{\partial u^h}{\partial t}\right)^2dxdt\le\frac{\sigma}{2}\int_{Q_T}\Big(u^h\Big)^2dxdt+\frac{1}{2\sigma}\int_{Q_T}f^2dxdt,$$

where $\sigma > 0$ will be chosen later.

By inequality (1.8), we have

$$\int_{Q_T} (u^h)^2 dxdt \le C_{1.7}(\lambda, n, \Omega) \int_{Q_T} \sum_{i=1}^{n} \lambda_i^h(x,t) \left(\frac{\partial u^h}{\partial x_i}\right)^2 dxdt.$$

Thus, the number σ can be chosen so small that the inequality

$$\left\| u^h \right\|_{W^{1,1}_{2,\lambda,\varphi}(Q_T)} \le C_{1.8}(\lambda, n, \Omega) \left\| f \right\|_{L_2(Q_T)}, \tag{1.18}$$

is fulfilled. It follows from (1.18) that the sequence $\left\{u^h(x,t)\right\}$ is strongly bounded in $\overset{\circ}{W}{}^{1,1}_{2,\lambda,\varphi}(Q_T)$. Thus, this sequence is weakly compact in $\overset{\circ}{W}{}^{1,1}_{2,\lambda,\varphi}(Q_T)$. In other words, there is a subsequence $\left\{u^{h_k}(x,t)\right\}$, $h_k \to 0$ for $k \to \infty$ and the function $u(x,t) \in \overset{\circ}{W}{}^{1,1}_{2,\lambda,\varphi}(Q_T)$ such that for any $\psi(x,t) \in C_0^\infty(\overline{Q_T})$

$$\lim_{k\to\infty} \left(Lu^{h_k}, \psi\right) = (Lu, \psi). \tag{1.19}$$

Moreover, the function $u(x,t)$ satisfies the estimate

$$\|u\|_{W^{1,1}_{2,\lambda,\varphi}(Q_T)} \le C_{1.8} \|f\|_{L_2(Q_T)}.$$

Let us now show that the function $u(x,t)$ satisfies equality (1.7) for any $\upsilon(x,t) \in \overset{\circ}{W}{}^{1,1}_{2,\lambda}(Q_T)$. Since the function $u^{h_k} \in \overset{\circ}{W}{}^{2,2}_{2,\lambda,\varphi}(Q_T)$ is a weak solution of equation (1.14) (see [22]), then for any $\upsilon(x,t) \in \overset{\circ}{W}{}^{1,1}_{2,\lambda}(Q_T)$ and $t_1 \in (-T, 0]$ the following equality holds

$$\int_{Q_{t_1}} \Big(\sum_{i,j=1}^{n} (a_{i,j})_{h_k} \frac{\partial u^{h_k}}{\partial x_j} \frac{\partial \upsilon}{\partial x_i} + \varphi(-t) \frac{\partial u^{h_k}}{\partial t} \frac{\partial \upsilon}{\partial t} - u^{h_k} \frac{\partial \upsilon}{\partial t}\Big) dxdt$$

$$+ \int_{\Omega} u^{h_k}(x, t_1) \upsilon(x, t_1) dx = - \int\limits_{Q_{t_1}} f \upsilon dxdt. \tag{1.20}$$

Hence if we pass to the limit as $k \to \infty$, then by virtue of (1.19) it remains to prove that

$$\int_{Q_{t_1}} \sum_{i,j=1}^{n} (a_{ij})_{h_k} \frac{\partial u^{h_k}}{\partial x_j} \frac{\partial \upsilon}{\partial x_i} dxdt \to \int_{Q_{t_1}} \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \upsilon}{\partial x_i} dxdt,$$

for $k \to \infty$. We have

$$\int_{Q_{t_1}} \sum_{i,j=1}^{n} (a_{ij})_{h_k} \frac{\partial u^{h_k}}{\partial x_j} \frac{\partial \upsilon}{\partial x_i} dxdt = \int_{Q_{t_1}} \sum_{i,j=1}^{n} ((a_{ij})_{h_k} - a_{ij}) \frac{\partial u^{h_k}}{\partial x_j} \frac{\partial \upsilon}{\partial x_i} dxdt$$

$$+ \int_{Q_{t_1}} \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial u^{h_k}}{\partial x_j} \frac{\partial \upsilon}{\partial x_i} dxdt. \tag{1.21}$$

The first term on the right-hand side of equality (1.21) tends to zero as $k \to \infty$. Indeed

$$\Big| \int_{Q_{t_1}} \sum_{i,j=1}^{n} \left((a_{ij})_{h_k} - a_{ij}\right) \frac{\partial u^{h_k}}{\partial x_j} \frac{\partial \upsilon}{\partial x_i} dxdt \Big|$$

$$\le \int_{Q_{t_1}} \sum_{i,j=1}^{n} \left| ((a_{ij})_{h_k} - a_{ij}) \frac{\partial u^{h_k}}{\partial x_j} \frac{\partial \upsilon}{\partial x_i} \right| \sqrt{\lambda_j(x,t)} \sqrt{\lambda_j^{-1}(x,t)} dxdt$$

$$\le \sum_{i,j=1}^{n} \sup_{Q_{t_1}} |(a_{ij})_{h_k} - a_{ij}| \cdot \Big(\int_{Q_{t_1}} \lambda_i^{-1}(x,t) \left(\frac{\partial \upsilon}{\partial x_i} \right)^2 dxdt \Big)^{\frac{1}{2}}$$

$$\times \Big(\int_{Q_{t_1}} \lambda_j(x,t) \left(\frac{\partial u^{h_k}}{\partial x_j} \right)^2 dxdt \Big)^{\frac{1}{2}} \to 0, k \to \infty$$

due to estimate (1.18).

The second term on the right-hand side of equality (1.21) can be represented as

$$\int_{Q_{t_1}} \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial u^{h_k}}{\partial x_j} \frac{\partial \upsilon}{\partial x_i} dxdt = \int_{Q_{t_1}} \sum_{i,j=1}^{n} a_{ij}(x,t) \left(\frac{\partial u^{h_k}}{\partial x_j} - \frac{\partial u}{\partial x_j} \right) \frac{\partial \upsilon}{\partial x_i} dxdt$$

$$+ \int_{Q_{t_1}} \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial u}{\partial x_j} \frac{\partial \upsilon}{\partial x_i} dxdt.$$

We have

$$\int_{Q_{t_1}} \sum_{i,j=1}^{n} a_{ij}(x,t) \left(\frac{\partial u^{h_k}}{\partial x_j} - \frac{\partial u}{\partial x_j} \right) \frac{\partial \upsilon}{\partial x_i} dxdt$$

$$= \int_{Q_{t_1}} \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial}{\partial x_j} (u^{h_k} - u) \frac{\partial \upsilon}{\partial x_i} dxdt \to 0,\ k \to \infty$$

due to the weak convergence of the sequence $\{u^{h_k}(x,t)\}$ to the function $u(x,t)$ in space $W^{1,1}_{2,\lambda,\varphi}(Q_T)$.

Consequently

$$\int_{Q_{t_1}} \sum_{i,j=1}^{n} (a_{ij}(x,t))_{h_k} \frac{\partial u^{h_k}}{\partial x_j} \frac{\partial \upsilon}{\partial x_i} dxdt \to \int_{Q_{t_1}} \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial u}{\partial x_j} \frac{\partial \upsilon}{\partial x_i} dxdt,\ k \to \infty.$$

Thus, the existence of a weak solution to problem (1.1)-(1.2) for $\partial\Omega \in C^2$ is proved.

Now let $\partial\Omega \overline{\in} C^2$. Consider a sequence of domains $\Omega_m, m = 1, 2, ...,$ for which $\partial\Omega_m \in C^2$; $\overline{\Omega}_m \subset \Omega_{m+1} \subset \overline{\Omega}_{m+1} \subset \Omega,\ \lim\limits_{m\to\infty} \Omega_m = \Omega$. Assume $Q^m_T = \Omega_m \times (-T, 0)$.

Let u^m- be the solution of the boundary value problem

$$Lu^m = f(x,t), (x,t) \in Q^m_T; \quad u^m|_{\Gamma(Q^m_T)} = 0.$$

By what was proved above, for every natural number m such a solution exists, and

$$\|u^m\|_{W^{1,1}_{2,\lambda,\varphi}(Q^m_T)} \le C_{1.10} \|f\|_{L_2(Q^m_T)},$$

holds, where the constant $C_{1.10}$ is independent of m.

Let us extend the function u^m by zero in $Q_T \backslash Q^m_T$ and denote the extended function again by u^m. It is clear that $u^m \in \overset{\circ}{W}{}^{1,1}_{2,\lambda,\varphi}(Q_T)$ and

$$\|u^m\|_{W^{1,1}_{2,\lambda,\varphi}(Q_T)} \le C_{1.10} \|f\|_{L_2(Q_T)}.$$

Thus the sequence $\{u^m\}$ is strongly bounded in $\overset{\circ}{W}{}^{1,1}_{2,\lambda,\varphi}(Q_T)$ and therefore, it is weakly compact in the same space, i.e., there is a function $u(x,t) \in \overset{\circ}{W}{}^{1,1}_{2,\lambda,\varphi}(Q_T)$ and a sequence $\{m_k\}, m_k \to 0$ as $k \to \infty$ such that the corresponding sequence $\{u^{m_k}(x,t)\}$ weakly converges to the function $u(x,t)$ in $\overset{\circ}{W}{}^{1,1}_{2,\lambda,\varphi}(Q_T)$ as $k \to \infty$. It remains to show that $u(x,t)$ is a solution of the equation $Lu = f$. This is done quite similarly to the previous one.

Let us now prove the uniqueness of the solution of the problem (1.1)-(1.2). To do this, it suffices to prove that the homogeneous boundary value problem $Lu = 0$, $u|_{\Gamma(Q_T)} = 0$ has only the zero solution.

In equality (1.7) we set $f = 0$, and then as $\upsilon(x,t)$ we take the function

$$\upsilon_{(\overline{h})}(x,t) = \frac{1}{h}\int_{t-h}^{t} \upsilon(x,\tau)d\tau, \tag{1.22}$$

where $\upsilon(x,t)$ is an arbitrary element of $\overset{\circ}{W}{}^{1,1}_{2,\lambda,\varphi}(Q_T^{-h})$, equal to zero for $t \geq -h$ and for $t \leq -T$ (see [21]), and fix $h > 0$. Here $Q_T^{-h} = \Omega \times (-h, 0)$. Therefore, we have

$$\int_{Q_{-h}} \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial u}{\partial x_j}\frac{\partial(\upsilon_{(\overline{h})})}{\partial x_i}dxdt - \int_{Q_{-h}} u\frac{\partial(\upsilon_{(\overline{h})})}{\partial t}dxdt$$
$$+\int_{Q_{-h}} \varphi(-t)\frac{\partial u}{\partial t}\frac{\partial(\upsilon_{(\overline{h})})}{\partial t}dxdt = 0. \tag{1.23}$$

In all terms of equality (1.23), we transfer the averages $(\cdot)_{\overline{h}}$ from υ by the factors in front of it, in addition, in the second term we will integrate by parts over t. Then we obtain

$$\int_{Q_{-h}} \sum_{i,j=1}^{n} \left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right)_{(h)}\frac{\partial \upsilon}{\partial x_i}dxdt + \int_{Q_{-h}} \frac{\partial(u_{(h)})\upsilon}{\partial t}dxdt$$
$$+\int_{Q_{-h}} \left(\varphi(-t)\frac{\partial u}{\partial t}\right)_{(h)}\frac{\partial \upsilon}{\partial t}dxdt = 0, \tag{1.24}$$

where

$$u_{(h)}(x,t) = \frac{1}{h}\int_{t}^{t+h} u(x,\tau)d\tau.$$

We have

$$\frac{\partial(u_{(h)})}{\partial t} = \frac{\partial}{\partial t}\Big(\frac{1}{h}\int_{t}^{t+h} u(x,\tau)d\tau\Big) = \frac{1}{h}(u(x,t+h) - u(x,t)).$$

Consequently, $u_{(h)} \in \overset{\circ}{W}{}^{1,1}_{2,\lambda}(Q_T)$. Therefore, in equality (1.24), instead of υ we can take the function $u_{(h)}$. Then

$$\int\limits_{Q_{-h}} \sum_{i,j=1}^{n} \left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right)_{(h)}\frac{\partial(u_{(h)})}{\partial x_i}dxdt + \int\limits_{Q_{-h}} \frac{\partial(u_{(h)})u_{(h)}}{\partial t}dxdt$$
$$+\int\limits_{Q_{-h}} \left(\varphi(-t)\frac{\partial u}{\partial t}\right)_{(h)}\left(\frac{\partial u}{\partial t}\right)_{(h)}dxdt = 0.$$

Since
$$\int_{Q_{-h}} \frac{\partial (u_{(h)}) u_{(h)}}{\partial t} dxdt = \frac{1}{2} \int_{\Omega} \left(u_{(h)}(x,0) \right)^2 dx \geq 0,$$
then
$$\int\limits_{Q_{-h}} \sum_{i,j=1}^{n} \left(a_{ij} \frac{\partial u}{\partial x_j} \right)_{(h)} \frac{\partial (u_{(h)})}{\partial x_i} dxdt + \int\limits_{Q_{-h}} \left(\varphi(-t) \frac{\partial u}{\partial t} \right)_{(h)} \left(\frac{\partial u}{\partial t} \right)_{(h)} dxdt \leq 0.$$

Fix an arbitrary $h_0 \in (-T, 0)$. Then in the previous inequality the domain Q_{-h} can be replaced by the domain Q_{-h_0}, where $h \leq h_0$. Thus
$$\int_{Q_{-h_0}} \sum_{i,j=1}^{n} \left(a_{ij} \frac{\partial u}{\partial x_j} \right)_{(h)} \frac{\partial (u_h)}{\partial x_i} dxdt + \int_{Q_{-h_0}} \left(\varphi(-t) \frac{\partial u}{\partial t} \right)_{(h)} \left(\frac{\partial u}{\partial t} \right)_{(h)} dxdt \leq 0.$$
Hence as $h \to 0$, we have
$$\int_{Q_{-h_0}} \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} dxdt + \int_{Q_{-h_0}} \varphi(-t) \left(\frac{\partial u}{\partial t} \right)^2 dxdt \leq 0.$$
Taking into account condition (1.3), we have
$$\int_{Q_{-h_0}} \left(\gamma \sum_{i=1}^{n} \lambda_i(x,t) \left(\frac{\partial u}{\partial x_i} \right)^2 + \varphi(-t) \left(\frac{\partial u}{\partial t} \right)^2 \right) dxdt \leq 0. \qquad (1.25)$$
From (1.25) it follows that $\int\limits_{Q_{-h_0}} \sum\limits_{i=1}^{n} \lambda_i(x,t) \left(\frac{\partial u}{\partial x_i} \right)^2 dxdt = 0.$

On the other hand
$$\int\limits_{Q_{-h_0}} u^2 dxdt \leq C_{1.11} \int\limits_{Q_{-h_0}} \sum_{i=1}^{n} \lambda_i(x,t) \left(\frac{\partial u}{\partial x_i} \right)^2 dxdt = 0.$$
Thus, the function $u(x,t) = 0$ almost everywhere in Q_{-h_0}. Since h_0 is arbitrary, it follows that $u(x,t) = 0$ almost everywhere in Q_T. Theorem 1.2 is proved.

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References

1. Keldysh, M.V.: *On certain cases of degeneration of equations of elliptic type on the boundry of a domain*, (Russian) Doklady Akad. Nauk SSSR (N.S.) **77** (2), 181–183 (1951).
2. Fichera, G.: *On a unified theory of boundary value problems for elliptic-parabolic equations of second order*, Mathematica **7** (6), 99–122 (1963).
3. Petrushko, I.M.: *Boundary values in L_p, $p > 1$, of solutions of elliptic equations in domains with Lyapunov boundary*, (Russian) Mat. Sb. (N.S.) **120(162)** (4), 569-588 (1983).

4. Petrushko, I.M.: *Boundary values of solutions that degenerate on the boundary of the domain of elliptic equations*, (Russian) Mat. Sb. (N.S.) **136(178)** (2), 241259 (1988).
5. Petrushko, I.M.: *Boundary values of solutions of degenerate elliptic equations of Keldysh type with weak degeneracy*, (Russian) Non-classical equations and equations of mixed type, (Russian) 166-182, Akad. Nauk SSSR Sibirsk. Otdel., Inst. Mat., Novosibirsk (1990).
6. Petrushko, I.M.: *On the existence of boundary values of solutions of degenerate elliptic equations*, (Russian) Mat. Sb. **190** (7), 4172 (1999); translation in Sb. Math. **190** (7-8), 973-1004 (1999).
7. Petrushko, I.M.: *Boundary and initial conditions in L_p, $p > 1$, for solutions of parabolic equations*, (Russian) Mat. Sb. (N.S.), **125(167)** (4), 489521 (1984).
8. Gasanov, I.G.: *The behavior near the boundary of solutions of second-order elliptic-parabolic equations*, Izv. Akad. Nauk Azerbadzhan. SSR Ser. Fiz.-Tekhn. Mat. Nauk **7** (3), 31-40 (1986).
9. Gasanov, I.G.: *Generalized solvability of the first boundary value problem for second order nondivergent elliptic-parabolic equations*, Den. at VINITI, **314-B**, 28 pp. (1988).
10. Alvino, A., Trombetti, G.: *Second order elliptic equation whose coefficients have their first derivatives weakly- L^n*, Ann. Mat. Pura Appl **138** (4), 331-340 (1984).
11. Franciosi, M.: *On an elliptic-parabolic equation with discontinuous coefficients*, (Italian) Boll. Un. Mat. Ital. **2(6)** (1), 63-75 (1983).
12. Franciosi, M.: *An existence and uniqueness theorem for the solution of an elliptic-parabolic equation, with discontinuous coefficients, in nondivergence form*, (Italian) Boll. Un. Mat. Ital. **4(6)** (1), 253-263 (1985).
13. Alkhutov, Yu.A., Mamedov, I.T.: *Some properties of the solutions of the first boundary value problem for parabolic equations with discontinuous coefficients*, (Russian) Dokl. Akad. Nauk SSSR **284** (1) 11-16 (1985).
14. Alkhutov, Yu.A., Mamedov, I.T.: *The first boundary value problem for second-order nondivergence parabolic equations with discontinuous coefficients*, (Russian) Sb. Math. (N.S.) **131(173)** (4), 477500 (1986); translation in Math. USSR-Sb. **59** (2), 471495 (1988).
15. Fiorito, G.: *A contribution to the solution of the Cauchy-Dirichlet problem*, (Italian) Matematiche (Catania) **35** (1-2), 53-60 (1980).
16. Wen, G.C.: *Oblique derivative problems for second order equations of mixed type in multiply connected domains*, Proceedings of the Second ISAAC Congress, Vol. 2 (Fukuoka, 1999), 15391546, Int. Soc. Anal. Appl. Comput., 8, Kluwer Acad. Publ., Dordrecht, (2000).
17. Gadjiev, T.S., Zulfaliyeva, G.M., Mamedova, K.N.: *The solvability degenerate elliptic-parabolic problem with nonlinear boundary conditions*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **42** (1), Mathematics, 120126 (2022).
18. Mamedov, F.I., Amanova, N.R.: *On Harnack's inequality for positive solutions of linear elliptic equations with discontinuous coefficients*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **42** (4), Mathematics, 112124 (2022).
19. Glushko, V.P., Savchenko, Yu.B.: *High order degenerate elliptic equations: spaces, operators, boundary value problems*, (Russian) Mathematical analysis, Vol. 23, 125218, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, (1985).
20. Oleinik, O.A., Radkevich, E.V.: *Second order equations with nonnegative characteristic form*, (Russian) Mathematical analysis, 1969 (Russian), pp. 7252. (errata insert) Akad. Nauk SSSR Vsesojuzn. Inst. Naun. i Tehn. Informacii, Moscow, (1971).
21. Ladyzhenskaya, O.A., Solonnikov, V.A., Uraltseva, N.N.: Linear and quasilinear equations of parabolic type, (Russian) Translated from the Russian by S. Smith Transla-

tions of Mathematical Monographs, Vol. 23 American Mathematical Society, Providence, R.I. (1968).

22. Salmanova, Sh.Yu.: *Existence and uniqueness of a weak solution of the first boundary value problem for a class of degenerate elliptic-parabolic equations of the second order*, Bulletin of BSU, ser. Phys.-Math. Sciences, **1**, 106-112 (2002).
23. Mamedov, I.T.: *The first boundary value problem for second-order elliptic-parabolic equations with discontinuous coefficients*, J. Math. Sci. (N.Y.) **190** (1), 104134 (2013).
24. Mushtagov, F.M.: *Boundary properties of solutions of the second order parabolic equations with non-uniform degree degeneration*, Proc. Inst. Math. Mech. Acad. Sci. Azerb. **7**, 87-95 (1997).