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# On some nonlinear fractional integro-differential equation and its connection with the Riemann-Liouville fractional integral in Lebesgue spaces

**Rovshan A. Bandaliyev, Ilhama A. Alekperova**

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**Abstract.** *In this work we study the connection of the Riemann-Liouville fractional integral with the nonlinear fractional integro-differential equation. In particular, we proved the boundedness of Riemann-Liouville fractional integral on Lebesgue spaces and give the interval for the best constant.*

**Keywords.** Nonlinear fractional differential equation, Riemann-Liouville fractional integral, Caputo fractional derivatives, Lebesgue spaces.

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## 1 Introduction

Fractional differential equations have attracted much attention and have been the focus of many studies due mainly to their varied applications in many fields of science and engineering. In other words, fractional differential equations are widely used to describe many important phenomena in various fields such as physics, biophysics, chemistry, biology, control theory, economy and so on; see [8]. For an extensive literature in the study of fractional differential equations, we refer the reader to [4]. However, it should be noted that in recent years, there have been many works related to fractional integro-differential equations, see [1], [2] and the references therein.

In this work we give a characterization for boundedness of Riemann-Liouville fractional integral by nonlinear ordinary fractional differential equation on Lebesgue spaces. The main contribution in this paper is the characterization of best possible constant by specially quantity. Similar problems for classical Hardy operator were studied in [5]-[7], [9], [10], [12], [13] and e.t.c.

The paper is structured as follows. Section 2 contains some preliminaries along with the standard ingredients used in the proofs. The main results are stated and proved in Section 3. Namely, in Section 3 we found the interval for the best possible constant for boundedness of Riemann-Liouville fractional integral on Lebesgue spaces.

R.A. Bandaliyev
Azerbaijan University of Architecture and Construction, Baku, Azerbaijan
E-mail: bandaliyevr@gmail.com

I.T. Alekperova
Azerbaijan State University of Economics (UNEC), Baku, Azerbaijan
E-mail: meryem-elekberova@mail.ru

## 2 Preliminaries

For convenience, in this section we recall some basic definitions and properties of the fractional calculus theory and auxiliary lemmas which will be used throughout this work, see [11].

Let $1 \leq p < \infty$ and let $p'$ be Hölder conjugate of $p$ defined by $p' = \dfrac{p}{p-1}$. We denote by $L_p(0,1)$ the space of Lebesgue measurable functions $f$ on $(0,1)$ such that

$$\|f\|_{L_p(0,1)} = \|f\|_p = \left(\int_0^1 |f(t)|^p \, dt\right)^{\frac{1}{p}} < \infty.$$

It is well known that the space $L_p(0,1)$ is a Banach spaces.

The set of all absolutely continuous functions on $(0,1)$ is denoted by $AC(0,1)$.

In this section, we present a review of some definitions and preliminary facts which are particularly relevant for the results of the book [11].

**Definition 2.1** *Let $f \in L_1(0,1)$. For almost all $t \in (0,1)$ and $\alpha > 0$, the left and right Riemann-Liouville fractional integrals of order $\alpha$ are defined by*

$$I_{0+}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau,$$

*and*

$$I_{0-}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_t^1 (\tau-t)^{\alpha-1} f(\tau) d\tau,$$

*respectively, where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} \, dt$ is the Euler gamma function.*

**Definition 2.2** *Let $f \in AC(0,1)$. For almost all $t \in (0,1)$ and $0 < \alpha < 1$, the left and right Caputo fractional derivatives of order $\alpha$ are defined by*

$${}^{C}D_{0+}^{\alpha} f(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} f'(\tau) d\tau,$$

*and*

$${}^{C}D_{0-}^{\alpha} f(t) := -\frac{1}{\Gamma(1-\alpha)} \int_t^1 (\tau-t)^{-\alpha} f'(\tau) d\tau,$$

*respectively.*

It is obvious that the Caputo fractional derivative of a constant is equal to zero.

**Theorem 2.1** *Let $0 < \alpha < 1$ and let $f \in C^1(0,1)$. Then,*

$${}^{C}D_{0+}^{\alpha} I_{0+}^{\alpha} f(t) = f(t), \;\; {}^{C}D_{0-}^{\alpha} I_{0-}^{\alpha} f(t) = f(t),$$

*and*

$$I_{0+}^{\alpha} \, {}^{C}D_{0+}^{\alpha} f(t) = f(t) - f(0), \;\; I_{0-}^{\alpha} \, {}^{C}D_{0-}^{\alpha} f(t) = f(t) - f(1).$$

**3 Main results.**

Suppose that $\lambda$ is a positive number. Let us consider the nonlinear fractional integro-differential equation
$$\lambda \left({}^C D^{\alpha}_{0+} y(t)\right)^{p-1} - I^{\alpha}_{0-} y^{p-1}(t) = 0, \tag{3.1}$$
where
$$y(t) > 0, \ \ {}^C D^{\alpha}_{0+} y(t) > 0, \ \ {}^C D^{\alpha}_{0+} y(t) \in AC(0,1), \ \ 0 < t < 1. \tag{3.2}$$

We say that $y$ is a solution of the problem (3.1)-(3.2), if $y$ satisfies the nonlinear fractional integro-differential equation (3.1) almost everywhere on $(0,1)$ and the condition (3.2). We set $y(0) = \lim\limits_{t \to +0} y(t)$.

First we prove the following theorem.

**Theorem 3.1** *Let $1 < p < \infty$ and let $\lambda$ be a positive number given in (3.1). Suppose that $u$ is an absolutely continuous function on $(0,1)$ satisfies condition $u(0) = u(+0) = 0$. If the problem (3.1)-(3.2) has a solution, then*
$$\|u\|_p \le \lambda^{\frac{1}{p}} \left\|{}^C D^{\alpha}_{0+} u\right\|_p .$$

**Proof.** By Theorem 2.1 for any absolutely continuous function the integral representation
$$u(x) = u(0) + I^{\alpha}_{0+} {}^C D^{\alpha}_{0+} u(x) = u(0) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \ {}^C D^{\alpha}_{0+} u(t)\, dt$$
holds. Since $u(0) = 0$, it follows that
$$u(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \ {}^C D^{\alpha}_{0+} u(t)\, dt.$$

Let a function $y$ be a solution of problem (3.1)-(3.2). Then using Hölder inequality, we have
$$|u(x)| \le \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \ \left|{}^C D^{\alpha}_{0+} u(t)\right| \, dt$$
$$= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\frac{\alpha-1}{p}} (x-t)^{\frac{\alpha-1}{p'}} \left|{}^C D^{\alpha}_{0+} u(t)\right| \, \left[{}^C D^{\alpha}_{0+} y(t)\right]^{-\frac{1}{p'}} \, \left[{}^C D^{\alpha}_{0+} y(t)\right]^{\frac{1}{p'}} dt$$
$$\le \left(\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \ {}^C D^{\alpha}_{0+} y(t)\, dt\right)^{\frac{1}{p'}} \left(\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \left|{}^C D^{\alpha}_{0+} u(t)\right|^p \, \left[{}^C D^{\alpha}_{0+} y(t)\right]^{-\frac{p}{p'}} \, dt\right)^{\frac{1}{p}}$$
$$= \left(I^{\alpha}_{0+} {}^C D^{\alpha}_{0+} y(x)\right)^{\frac{1}{p'}} \left(\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \left|{}^C D^{\alpha}_{0+} u(t)\right|^p \, \left[{}^C D^{\alpha}_{0+} y(t)\right]^{1-p} \, dt\right)^{\frac{1}{p}}$$
$$= (y(x) - y(0))^{\frac{1}{p'}} \left(\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \left|{}^C D^{\alpha}_{0+} u(t)\right|^p \, \left[{}^C D^{\alpha}_{0+} y(t)\right]^{1-p} \, dt\right)^{\frac{1}{p}}$$
$$\le (y(x))^{\frac{1}{p'}} \left(\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \left|{}^C D^{\alpha}_{0+} u(t)\right|^p \, \left[{}^C D^{\alpha}_{0+} y(t)\right]^{1-p} \, dt\right)^{\frac{1}{p}}$$
$$= \left(\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \left|{}^C D^{\alpha}_{0+} u(t)\right|^p \, \left[{}^C D^{\alpha}_{0+} y(t)\right]^{1-p} \, (y(x))^{p-1} \, dt\right)^{\frac{1}{p}}.$$

Thus, we get

$$\left(\int_0^1 |u(x)|^p dx\right)^{\frac{1}{p}}$$

$$\leq \left(\int_0^1 \left(\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\left|{}^C D_{0+}^{\alpha} u(t)\right|^p \left[{}^C D_{0+}^{\alpha} y(t)\right]^{1-p} (y(x))^{p-1}\, dt\right) dx\right)^{\frac{1}{p}}$$

$$= \left(\frac{1}{\Gamma(\alpha)}\int_0^1\int_0^1 (x-t)^{\alpha-1}\left|{}^C D_{0+}^{\alpha} u(t)\right|^p \left[{}^C D_{0+}^{\alpha} y(t)\right]^{1-p} (y(x))^{p-1}\, \chi_{(0,x)}(t)\, dt\, dx\right)^{\frac{1}{p}}$$

$$= \left(\int_0^1 \left|{}^C D_{0+}^{\alpha} u(t)\right|^p \left[{}^C D_{0+}^{\alpha} y(t)\right]^{1-p}\left(\frac{1}{\Gamma(\alpha)}\int_t^1 (x-t)^{\alpha-1}(y(x))^{p-1}\, dx\right) dt\right)^{\frac{1}{p}}$$

$$= \left(\int_0^1 \left|{}^C D_{0+}^{\alpha} u(t)\right|^p \left[{}^C D_{0+}^{\alpha} y(t)\right]^{1-p} I_{0-}^{\alpha} y^{p-1}(t)\, dt\right)^{\frac{1}{p}} = \lambda^{\frac{1}{p}}\left(\int_0^1 \left|{}^C D_{0+}^{\alpha} u(t)\right|^p\, dt\right)^{\frac{1}{p}}.$$

This completes the proof.

We need the following Theorem.

**Theorem 3.2** *Let $1 < p < \infty$ and let $0 < \alpha < 1$. Then the inequality*

$$\left\|I_{0+}^{\alpha} f\right\|_p \leq C\, \|f\|_p \tag{3.3}$$

*holds if and only if $\alpha > \dfrac{1}{p'}$.*

*Besides, if $C > 0$ is the best constant in (3.3), then*

$$\frac{\left(1-\frac{1}{\alpha p'}\right)^{\frac{1}{p}}}{2^{\frac{1}{p'}}\,((\alpha-1)p+1)^{\frac{1}{p}}\,(\alpha p')^{\frac{1}{p'((\alpha-1)p+1)}}\,\Gamma(\alpha)} \leq C \leq \frac{1}{((\alpha-1)p+1)^{\frac{1}{p}}\,(\alpha p')^{\frac{1}{p'}}\,\Gamma(\alpha)}$$

**Proof.** Sufficiency. By Minkowski's inequality,we have

$$\left\|I_{0+}^{\alpha} f\right\|_p \leq \frac{1}{\Gamma(\alpha)}\left(\int_0^1\left(\int_0^x (x-t)^{\alpha-1}|f(t)|\, dt\right)^p dx\right)^{\frac{1}{p}}$$

$$= \frac{1}{\Gamma(\alpha)}\left(\int_0^1\left(\int_0^1 (x-t)^{\alpha-1}|f(t)|\,\chi_{(0,x)}(t)\, dt\right)^p dx\right)^{\frac{1}{p}}$$

$$\leq \frac{1}{\Gamma(\alpha)}\left(\int_0^1\left(\int_0^x (x-t)^{(\alpha-1)p}|f(t)|^p\, dx\right)^{\frac{1}{p}} dt\right)$$

$$= \frac{1}{\Gamma(\alpha)}\left(\int_0^1 |f(t)|\left(\int_0^x (x-t)^{(\alpha-1)p}\, dx\right)^{\frac{1}{p}} dt\right)$$

$$= \frac{1}{((\alpha-1)p+1)^{\frac{1}{p}}\,\Gamma(\alpha)}\left(\int_0^1 |f(t)|\,(1-t)^{\alpha-\frac{1}{p'}}\, dt\right).$$

Applying Hölder's inequality in the last integral, we get

$$\left\|I_{0+}^{\alpha} f\right\|_p \leq \frac{1}{((\alpha-1)p+1)^{\frac{1}{p}}\,\Gamma(\alpha)}\left(\int_0^1 (1-t)^{\alpha p'-1}\, dt\right)^{\frac{1}{p'}}\|f\|_p$$

$$= \frac{1}{((\alpha-1)p+1)^{\frac{1}{p}}\ (\alpha p')^{\frac{1}{p'}}\ \Gamma(\alpha)}\ \|f\|_p.$$

*Necessity.* Let us suppose that $I^\alpha_{0+} : L_p(0,1) \to L_p(0,1)$ is bounded and (3.3) holds. Let $t > 0$ be fixed. We define the test function $f$ as $f_t(y) = \chi_{\left(0,\frac{t}{2}\right)}(y)$. We have

$$\left\|I^\alpha_{0+} f_t\right\|_p = \frac{1}{\Gamma(\alpha)} \left(\int_0^1 \left(\int_0^x (x-y)^{\alpha-1}\, f_t(y)\, dy\right)^p dx\right)^{\frac{1}{p}}$$

$$\ge \frac{1}{\Gamma(\alpha)} \left(\int_t^1 \left(\int_0^{\frac{t}{2}} (x-y)^{\alpha-1}\, dy\right)^p dx\right)^{\frac{1}{p}} \ge \frac{1}{2\,\Gamma(\alpha)} \left(\int_t^1 x^{(\alpha-1)p}\, dx\right)^{\frac{1}{p}}\ t$$

$$= \frac{1}{2\,\Gamma(\alpha)} \left(\frac{1-t^{(\alpha-1)p+1}}{(\alpha-1)p+1}\right)^{\frac{1}{p}}\ t.$$

We set $A_\alpha = \{\alpha :\ (\alpha-1)p+1 > 0\}$ and let $A^\star_\alpha = \{\alpha :\ (\alpha-1)p+1 < 0\}$ . Then, we have

$$\left(\frac{1-t^{(\alpha-1)p+1}}{(\alpha-1)p+1}\right)^{\frac{1}{p}} = \frac{\left(1-t^{(\alpha-1)p+1}\right)^{\frac{1}{p}}}{((\alpha-1)p+1)^{\frac{1}{p}}}\ \chi_{A_\alpha}(\alpha) + \frac{\left(t^{(\alpha-1)p+1}-1\right)^{\frac{1}{p}}}{\left[-\left((\alpha-1)p+1\right)\right]^{\frac{1}{p}}}\ \chi_{A^\star_\alpha}(\alpha)$$

$$\ge \frac{\left(1-t^{(\alpha-1)p+1}\right)^{\frac{1}{p}}}{((\alpha-1)p+1)^{\frac{1}{p}}}.$$

By (3.3), we have

$$\left\|I^\alpha_{0+} f_t\right\|_p \ge \frac{1}{2\ ((\alpha-1)p+1)^{\frac{1}{p}}\ \Gamma(\alpha)}\ \left(1-t^{(\alpha-1)p+1}\right)^{\frac{1}{p}}\ t.$$

On the other hand, $\|f_t\|_p = \left(\frac{t}{2}\right)^{\frac{1}{p}}$. So, by (3.3), one has

$$\frac{1}{2^{\frac{1}{p'}}\ ((\alpha-1)p+1)^{\frac{1}{p}}\ \Gamma(\alpha)}\ \sup_{0<t<1} \left(1-t^{(\alpha-1)p+1}\right)^{\frac{1}{p}}\ t^{\frac{1}{p'}} \le C.$$

It is obvious that

$$\sup_{0<t<1} \left(1-t^{(\alpha-1)p+1}\right)^{\frac{1}{p}}\ t^{\frac{1}{p'}} = \left(1-\frac{1}{\alpha\, p'}\right)^{\frac{1}{p}} \left(\frac{1}{\alpha\, p'}\right)^{\frac{1}{p'((\alpha-1)p+1)}}.$$

Thus, we get

$$\frac{\left(1-\frac{1}{\alpha\, p'}\right)^{\frac{1}{p}}}{2^{\frac{1}{p'}}\ ((\alpha-1)p+1)^{\frac{1}{p}}\ (\alpha\, p')^{\frac{1}{p'((\alpha-1)p+1)}}\ \Gamma(\alpha)} \le C.$$

This completes the proof.

**Theorem 3.3** *Let* $1 < p < \infty$ *and let* $\frac{1}{p'} < \alpha < 1$. *Suppose that* $u$ *is an absolutely continuous function on* $(0,1)$ *satisfies condition* $u(0) = 0$. *Let* $\lambda > 0$ *is a possible best constant such that*

$$\|u\|_p \leq \lambda^{\frac{1}{p}} \left\|{}^C D_{0+}^{\alpha} u\right\|_p.$$

*Then*

$$\frac{1 - \frac{1}{\alpha p'}}{2^{p-1} \left((\alpha-1)p+1\right) \left(\alpha p'\right)^{\frac{p-1}{(\alpha-1)p+1}} \Gamma^{1/p}(\alpha)} \leq \lambda \leq \frac{1}{\left((\alpha-1)p+1\right) \left(\alpha p'\right)^{p-1} \Gamma^{1/p}(\alpha)}.$$

**Proof.** Substituting $f$ with ${}^C D_{0+}^{\alpha} u$ and take into account $u(0) = 0$ in inequality (3.3), we prove Theorem 3.3.

Now we reduce integro-differential equation which corresponds to (3.1).

**Theorem 3.4** *Let* $0 < \alpha < 1$. *Then every solution to the nonlinear integral equation*

$$\lambda I_{0+}^{\alpha} \left({}^C D_{0+}^{\alpha} y\right)^{p-1}(t) - \int_0^1 K_{\alpha}(t,u) y^{p-1}(u) du = 0 \tag{3.4}$$

*is a solution to nonlinear fractional differential equation*

$$\lambda\, {}^C D_{0-}^{\alpha} \left({}^C D_{0+}^{\alpha} y\right)^{p-1}(t) = y^{p-1}(t),$$

*where*

$$K_{\alpha}(t,u) = \frac{1}{\Gamma^2(\alpha)} \int_0^t \frac{q(u,\tau)}{(t-\tau)^{1-\alpha}} d\tau,$$

$$q(u,\tau) = \begin{cases} (u-\tau)^{\alpha-1}, & if\ \ 0 < \tau < u < 1 \\ 0, & if\ \ 0 < u \leq \tau < 1. \end{cases}$$

**Proof.** We have

$$\int_0^1 K_{\alpha}(t,u) y^{p-1}(u) du = \frac{1}{\Gamma^2(\alpha)} \int_0^1 y^{p-1}(u) \int_0^t \frac{q(u,\tau)}{(t-\tau)^{1-\alpha}} d\tau du$$

$$= \frac{1}{\Gamma^2(\alpha)} \int_0^t \frac{d\tau}{(t-\tau)^{1-\alpha}} \int_0^1 y^{p-1}(u) q(u,\tau) du$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^t \frac{d\tau}{(t-\tau)^{1-\alpha}} \frac{1}{\Gamma(\alpha)} \int_{\tau}^1 \frac{y^{p-1}(u)}{(u-\tau)^{1-\alpha}} du = I_{0+}^{\alpha} \left(I_{0-}^{\alpha} y^{p-1}\right)(t).$$

Therefore, we can rewrite the equation (3.4) in the form

$$\lambda I_{0+}^{\alpha} \left({}^C D_{0+}^{\alpha} y\right)^{p-1}(t) - I_{0+}^{\alpha} \left(I_{0-}^{\alpha} y^{p-1}\right)(t) = 0. \tag{3.5}$$

By Theorem 2.1, we can prove that ${}^C D_{0-}^{\alpha} \circ {}^C D_{0+}^{\alpha} \circ \left(I_{0+}^{\alpha} \left(I_{0-}^{\alpha} y^{p-1}\right)\right) = y^{p-1}$. Thus, by applying operator ${}^C D_{0-}^{\alpha} \circ {}^C D_{0+}^{\alpha}$ to (3.5) we prove Theorem 3.4.

**Remark 3.1** Let $p = 2$. Then equation (3.1) is a linear fractional integro-differential equation

$$\lambda\, {}^C D^{\alpha}_{0+} y(t) - I^{\alpha}_{0-} y(t) = 0.$$

It is obvious that the last equation can be reduced to a differential equation

$$\lambda\, {}^C D^{\alpha}_{0-} \left({}^C D^{\alpha}_{0+} y\right)(t) - y(t) = 0,$$

which contains the left and right fractional Caputo derivatives. Similar differential equations were studied in [3].

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