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General decay estimate for a viscoelastic wave equation with Balakrishnan-Taylor damping and strong dissipation

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Abstract. *In this paper, an initial-boundary value problem for a class of viscoelastic wave equations with Balakrishnan-Taylor damping and strong dissipation is studied. The existence and uniqueness of solutions for the proposed problem are obtained by using the linear approximation and the Faedo-Galerkin method. Under several suitablely sufficient conditions on the initial data and the relaxation function, a general decay estimate of the solution is established by the perturbed energy method.*

Keywords. viscoelastic wave equation, Faedo-Galerkin method, general decay, Balakrishnan-Taylor damping.

Mathematics Subject Classification (2010): 35L20, 35L70, 35Q74, 37B25

1 Introduction

In this paper, we consider the following viscoelastic wave equation with Balakrishnan-Taylor damping and strong dissipation

$$\begin{aligned} &u_{tt} - \lambda u_{xxt} - \frac{\partial}{\partial x}\left[\mu\left(x,t,u\left(x,t\right),\langle u_x\left(t\right),u_{xt}\left(t\right)\rangle,\left\|u\left(t\right)\right\|^2,\left\|u_x\left(t\right)\right\|^2\right)u_x\right] \\ &\quad + \int_0^t h\left(t-s\right)u_{xx}\left(s\right)ds = f\left(x,t,u\left(x,t\right),u_x\left(x,t\right),u_t\left(x,t\right),u_{xt}\left(x,t\right)\right), \end{aligned} \tag{1.1}$$

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where $0 < x < 1,\ 0 < t < T$, associated with boundary conditions

$$u\left(0,t\right) = u\left(1,t\right) = 0, \tag{1.2}$$

and initial conditions

$$u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \tag{1.3}$$

where $\lambda > 0$ is a given constant and $\tilde{u}_0,\ \tilde{u}_1\ \mu,\ f,\ h$ are given functions satisfying some suitable conditions. In (1.1), the nonlinear quantity μ depends on the integrals $\langle u_x\left(t\right), u_{xt}\left(t\right)\rangle = \int_0^1 u_x\left(x,t\right) u_{xt}\left(x,t\right) dx,\ \|u(t)\|^2 = \int_0^1 u^2\left(x,t\right) dx$ and $\|u_x(t)\|^2 = \int_0^1 u_x^2\left(x,t\right) dx$, which are known as the Balakrishnan-Taylor damping, the Carrier term and the Kirchhoff term respectively.

It is clear that the equation (1.1) includes a complex structure of mathematical model, so there doesn't seem to be any actual model that fits it. However, we shall introduce and analyze below numerous related models that take a very important role in many fields of science such as physics, mechanics and engineering. Indeed, one of the most important mathematical models is of Kirchhoff [20] in order to describe the changes in length of the string produced by transverse vibrations

$$\rho h u_{tt} = \left(P_0 + \frac{Eh}{2L}\int_0^L \left|u_x\right|^2 dx\right) u_{xx}, \tag{1.4}$$

where u is the lateral deflection, L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension. Apparently, in this case, the equation (1.4) can be consider as a special form of the equation (1.1) with $\lambda = 0,\ \mu = \mu\left(\|u_x\left(t\right)\|^2\right),\ h = 0$ and $f = 0$. Note that, the equation (1.4) is a generalization for the well-known classical wave equation of D'Alembert describing free vibrations of elastic strings; and later has been also studied by Carrier [5] but with the model of vibrations of an elastic string when changes in tension are not small

$$\rho h u_{tt} - \left(1 + \frac{EA}{LT_0}\int_0^L u^2 dx\right) u_{xx} = 0, \tag{1.5}$$

where $u(x,t)$ is the x-derivative of the deformation, T_0 is the tension in the rest position, E is the Young modulus, A is the cross-section of a string, L is the length of a string, ρ is the density of a material. In this case, it is clear that the equation (1.5) also is a special form of the equation (1.1) with $\lambda = 0,\ \mu = \mu\left(\|u\left(t\right)\|^2\right),\ h = 0$ and $f = 0$. Thereafter, the equations in the forms of (1.4) or (1.5) were commonly called as Kirchhoff-Carrier type equations. Over a very long period of developments, there have been thousands of published works of Kirchhoff-Carrier type equations. The early one in those should be mentioned here was the studies of Medeiros [28] to the local existence of the mixed problem for the perturbed Kirchhoff-Carrier operator; the next interesting results were that of Canvalcanti et al. [6]-[8] to the existence, global existence, exponential or uniform decay rates, asymptotic behaviour for the various models of Kirchhoff-Carrier type, and later were that of [21], [26] and [36] to some more abstract models. Meanwhile, many authors concerned with the steady-state Kirchhoff equations modeling several physical and biological systems and also for describing the dynamics of an axially moving string Alves et al. [1] and [2], Ma and Rivera [27]. Recently, several studies of Kirchhoff equations related to optimal control problems have been considered, we refer to [12], [18] and the references therein.

It also notes that the equation (1.1) includes the nonlocal term $\int_0^1 u_x\left(x,t\right) u_{xt}\left(x,t\right) dx$, so it can be considered as an abstract form generalizing for a class of problems with

Balakrishnan-Taylor damping. For a physical interruption, the problems with Balakrishnan-Taylor damping have been arisen from the studies relating to the panel flutter equation and the spillover problem, and first introduced by Balakrishnan and Taylor [3], then also studied by Bass and Zes [4] in which the proposed one-dimensional model was in the generalized form as follow

$$u_{tt}+\lambda u_{xxxx}-2\varsigma\sqrt{\lambda}u_{xxt}-\gamma\left[P_0+\frac{Eh}{2L}\int_0^L|u_x|^2dx+\left(\int_0^L u_xu_{xt}dx\right)^{2(n+\beta)+1}\right]u_{xx}=f, \tag{1.6}$$

where $0<x<L,\ t>0,\ n\in\mathbb{N},\ 0\leq\beta<\frac{1}{2}$ and λ is the appropriate structure constant. Since its appearance, the equation (1.6) has been received much attention of interest, but mainly in multi-dimensional cases and given by the following model

$$\begin{aligned}u_{tt}+\alpha\Delta^2u-\lambda\Delta u_t+\mu\Delta^2u_t-\left[\beta+\gamma\|\nabla u\|_2^2+\sigma\left|\int_\Omega\nabla u\nabla u_tdx\right|^{q-2}\int_\Omega\nabla u\nabla u_tdx\right]\Delta u\\+\int_0^t h(t-s)\Delta u(s)ds+g(u_t)=f(u),\end{aligned} \tag{1.7}$$

where $\alpha>0$ is the elasticity coefficient, $\gamma>0$ is the extensibility coefficient, $\lambda\geq0$ is the viscous damping coefficient, $\sigma>0$ is the Balakrishnan-Taylor damping coefficient. There are so many results related to the equation (1.7) on the existence (local or global) and stability of solutions, but mainly in the case $q=2$. Indeed, in the absence of memory term and $f(u)=|u|^pu$, Zaraï and Tatar [40] proved the global existence and polynomial decay of energy in (1.7); and later they have considered (1.7) in the case $\alpha=0,\ \lambda=0,\ g=0$ and $f(u)=|u|^pu$, see [33], and also established the exponential decay and the blow up of solutions. At the same time, Emmrich and Thalhammer [13] considered (1.7) without the memory effect ($h=0$) and with the linear weak damping and the linear source, more precisely when $h=0,\ g=\kappa u_t$ and $f=h(x,t)-\xi u$. The authors proved the existence of a weak solution in either cases: in the presence of viscous and strong damping ($\lambda,\ \mu>0$) and $q\geq2$ or else in the absence of dampings ($\lambda=\ \mu=0$) and $q=2$; however they were not able to prove the existence in the case $\lambda=\ \mu=0$ and $q>2$. For other results of the existence and the stability to the solutions in (1.7) as $q=2$, we refer to [22], [31], [34], [37] and [40]. When $q>2$, one of initial studies on (1.6) was considered by You [39] in which the existence of global solutions and the existence of absorbing sets were proved by using the semigroup theory. Recently, Tavares et. al. [35] have proved the Hadamard well-posedness and the long-time behavior of solutions in (1.7) when $\lambda=\ \mu=0,\ h=0$ and $g=\kappa u_t$.

It is well known that time delay are arisen in many sciences such as physical, chemical, biological, thermal and economical phenomena. The presence of delay may be a source of instability. Hence, the problems with the Balakrishnan-Taylor damping in the presence of time delay effects have become one of very interesting topics in recent years, see for instance [9], [10], [11], [15], [16], [17], [19], [22], [23], [24], [38] and the reference therein. In most cases, the interest of this type was contained in studying the following equation

$$\begin{aligned}u_{tt}-\left[\beta+\gamma\|\nabla u\|_2^2+\sigma\left(\int_\Omega\nabla u\nabla u_tdx\right)\right]\Delta u\\+\int_0^t h(t-s)\Delta u(s)ds+g_1(u_t)+g_2(u_t(t-\tau(t)))=f(u).\end{aligned} \tag{1.8}$$

Actually, when $f = 0,\ g_1$ and g_2 are linear in (1.8), more precisely $g_1 = \mu_1 u_t$ and $g_2 = \mu_2 u_t(t-\tau(t))$, Lee [23] studied the asymptotic stability of the problem and established general energy decay result by suitable Lyapunov functionals. Extending the results given in [23], Kang et. al [19] considered (1.8) when $f = 0,\ g_1 = \mu_1 f_1(u_t)$ and the nonlinear time delay in the form $\mu_2 f_2(u_t(t-\tau(t)))$, then the authors proved a general stability result for the equation without the condition $\mu_2 > 0$ by establishing some Lyapunov functionals and using some properties of convex functions. Very recently, Li [24] has also studied (1.8) when $f = 0$ and the strong time-dependent delay $-\mu_2 \Delta u_t(t-\tau(t)$; where a generalized stability result has been established by suitable assumptions on the coefficients of the delay term.

As mentioned above, many authors have tried to study some problems with Balakrishnan-Taylor damping terms including more general forms, for example as in (1.6) with $2(n+\beta)+1 > 2$ or with $q > 2$ in (1.7). In these cases, the obtained models are much different from the original model, however, such problems also take some certain mathematical donations motivating us to take the solvability and solution properties of the problem (1.1)-(1.3) into consideration. Therefore, in this paper, we first confirm a local existence of the problem (1.1)-(1.3) by using the linear approximation and the Faedo-Galerkin method, in which the proof are not presented in details and can be adapted from our previous works. Next, in order to study the long-time behavior of the solution, we consider the equation (1.1) in a special case given by

$$\begin{aligned} &f = -\lambda_1 u_t + g(u) + F(x,t), \\ &\mu = \mu_1(t) + \mu_2\left(\|u_x(t)\|^2\right) + \mu_3\left(\langle u_x(t), u_{xt}(t)\rangle\right). \end{aligned}$$

Precisely, we consider the initial-boundary value problem as follows

$$\begin{cases} u_{tt} - \lambda u_{xxt} - \left[\mu_1(t) + \mu_2\left(\|u_x(t)\|^2\right) + \mu_3\left(\langle u_x(t), u_{xt}(t)\rangle\right)\right] u_{xx} \\ \qquad\qquad + \displaystyle\int_0^t h(t-s)\, u_{xx}(s)\, ds + \lambda_1 u_t = g(u) + F(x,t),\ 0 < x < 1,\ t > 0, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{1.9}$$

where $\lambda_1 > 0$ is a constant and $\mu_1,\ \mu_2,\ \mu_3,\ g,\ F$ satisfy some given conditions. Then, several suitablely sufficient conditions on the initial data and the relaxation function h, we shall show that any global solution of (1.9) is generally decayed in time. Clearly, the nonlinear quantities $\mu_2\left(\|u_x(t)\|^2\right)$ and $\mu_3\left(\langle u_x(t), u_{xt}(t)\rangle\right)$ in (1.9) are generalizations of the Kirchhoff term and the Balakrishnan-Taylor damping term in (1.6) or in (1.7) (in one-dimensional case) respectively. In this case, we are difficult to establish sufficient conditions on the nonlinear quantities μ_2, μ_3 and the relaxation function h in order to obtain a generally decayed property. Moreover, in our previous paper [29], we only obtained an exponentially decayed estimate for the proposed problem with the same nonlinear quantities in (1.9) but without the viscoelastic term ($h = 0$). Therefore, it can be said with much confidence that the obtained results in this paper can be considered as a generalization of [29] directly, and of [4] and [39] relatively. Here, we futher analyze that, in this paper, we have supposed that the problem (1.9) admits a global solution without a proof in details; then we show that the global solution is generally decayed in time. In our previous articles, see for example as in [30], we used some arguments of continuity to prove the global solution of the proposed problem; unfortunately these techniques can not be applied to the present paper. In addition, it seems that there are not many results of finite-time blow up of solutions for viscoelastic problems with Balakrishnan-Taylor damping term, see for example as in [32] and [33].

Thus, studies of global existence and finite-time blow up of solutions for problems, for example such as (1.9), are still open problems.

Motivated by the above papers, we study the problem (1.1)–(1.3) according to the following structure. In Section 2, some required preliminaries are introduced, then we confirm the local existence and uniqueness of solutions for the problem (1.1)-(1.3). In Section 3, we consider a special case of (1.1)-(1.3) provided by (1.9). Then, by establishing some sufficient conditions and using some energy estimates suitablely, we show that the solution of the problem (1.9) is generally decayed in time.

2 Local existence and uniqueness

Put $\Omega = (0,1)$. Throughout this paper, we denote the function spaces C^0, L^2 and H^m by $C^0(\overline{\Omega})$, $L^2(\Omega)$ and $H^m(\Omega)$ (m is a natural number), and the norms $\|\cdot\|_{C^0} = \|\cdot\|_{C^0(\overline{\Omega})}$, $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{H^m} = \|\cdot\|_{H^m(\Omega)}$ respectively. Also, let $\langle u, v\rangle = \int_0^1 u(x)v(x)dx$ be a scalar product in L^2 or a dual pair of a linear continuous functional and an element of a function space.

Denote $u(t) = u(x,t)$, $u'(t) = u_t(t) = \dot{u}(t) = \frac{\partial u}{\partial t}(x,t)$, $u''(t) = u_{tt}(t) = \ddot{u}(t) = \frac{\partial^2 u}{\partial t^2}(x,t)$, $u_x(t) = \frac{\partial u}{\partial x}(x,t)$, $u_{xx}(t) = \frac{\partial^2 u}{\partial x^2}(x,t)$.

With $f \in C^k([0,1]\times[0,T^*]\times\mathbb{R}^4)$, $f = f(x,t,y_1,\cdots,y_4)$, we put $D_1 f = \frac{\partial f}{\partial x}$, $D_2 f = \frac{\partial f}{\partial t}$, $D_{i+2} f = \frac{\partial f}{\partial y_i}$, with $i = 1,\cdots,4$ and $D^\alpha f = D_1^{\alpha_1}\cdots D_8^{\alpha_8} f$, $\alpha = (\alpha_1,\cdots,\alpha_6) \in \mathbb{Z}_+^6$, $|\alpha| = \alpha_1 + \cdots + \alpha_6 \le k$, $D^{(0,\cdots,0)} f = f$.

Similarly, with $\mu \in C^k([0,1]\times[0,T^*]\times\mathbb{R}^2\times\mathbb{R}_+^2)$, $\mu = \mu(x,t,y_1,\cdots,y_4)$, we put $D_1\mu = \frac{\partial \mu}{\partial x}$, $D_2 f = \frac{\partial f}{\partial t}$, $D_{i+2}\mu = \frac{\partial \mu}{\partial y_i}$, with $i = 1,\cdots,4$ and $D^\beta \mu = D_1^{\beta_1}\cdots D_6^{\beta_6}\mu$, $\beta = (\beta_1,\cdots,\beta_6) \in \mathbb{Z}_+^6$, $|\beta| = \beta_1 + \cdots + \beta_6 \le k$, $D^{(0,\cdots,0)}\mu = \mu$.

On $H^1 \equiv H^1(\Omega)$, we shall use the following norm

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2\right)^{\frac{1}{2}}.$$

It is well known that the imbedding $H^1 \hookrightarrow C^0$ is compact and

$$\|v\|_{C^0} \le \sqrt{2}\,\|v\|_{H^1}, \text{ for all } v \in H^1,$$

Furthermore, on $H_0^1 = \{v \in H^1 : v(0) = v(1) = 0\}$, two norms $v \longmapsto \|v\|_{H^1}$ and $v \longmapsto \|v_x\|$ are equivalent and

$$\|v\|_{C^0} \le \|v_x\| \text{ for all } v \in H_0^1.$$

In the next, we shall prove the existence and uniqueness of solutions for the problem (1.1)-(1.3). For this purpose, we consider $T^* > 0$ fixed, and make the following assumptions:

$(\mathbf{H}_1)$: $\tilde{u}_0, \tilde{u}_1 \in H_0^1 \cap H^2$,

$(\mathbf{H}_2)$: $\mu \in C^2\left([0,1]\times[0,T^*]\times\mathbb{R}^2\times\mathbb{R}_+^2\right)$, and there exists a constant $\mu_* > 0$ such that

$$\mu(z) \ge \mu_* > 0, \forall z \in [0,1]\times[0,T^*]\times\mathbb{R}^2\times\mathbb{R}_+^2,$$

$(\mathbf{H}_3)$: $h \in H^1(0,T^*)$,

$(\mathbf{H}_4)$: $f \in C^1\left([0,1]\times[0,T^*]\times\mathbb{R}^4\right)$.

Definition 2.1. *For every $T \in (0, T^*]$, u is a weak solution of the problem (1.1)-(1.3) if u is consisted of the set below*

$$S_T = \left\{ w \in L^\infty(0,T; H_0^1 \cap H^2) : w' \in L^\infty(0,T; H_0^1 \cap H^2), w'' \in L^2(0,T; H_0^1) \cap L^\infty(0,T; L^2) \right\},$$

and satisfies the following variational equation

$$\langle u''(t), v\rangle + \lambda \langle u_x'(t), v_x\rangle + \langle \mu[u](t) u_x(t), v_x\rangle - \int_0^t h(t-s) \langle u_x(s), v_x\rangle ds = \langle f[u](t), v\rangle, \tag{2.1}$$

for all $v \in H_0^1$ and a.e. $t \in (0,T)$, together with the initial conditions

$$u(0) = \tilde{u}_0, u'(0) = \tilde{u}_1, \tag{2.2}$$

where

$$\begin{cases} \mu[u](x,t) = \mu\left(x, t, u(x,t), \langle u_x(t), u_{xt}(t)\rangle, \|u(t)\|^2, \|u_x(t)\|^2\right), \\ f[u](x,t) = f(x,t,u(x,t), u_x(x,t), u_t(x,t), u_{xt}(x,t)). \end{cases} \tag{2.3}$$

For each $T \in (0, T^*]$, let

$$V_T = \left\{ z \in L^\infty\left(0,T; H_0^1 \cap H^2\right) : z' \in L^\infty(0,T; H_0^1 \cap H^2),\ z'' \in L^2\left(0,T; H_0^1\right) \right\}$$

be a Banach space with respect to the norm

$$\|z\|_{V_T} = \max\left\{ \|z\|_{L^\infty\left(0,T;H_0^1 \cap H^2\right)},\ \|z'\|_{L^\infty\left(0,T;H_0^1 \cap H^2\right)},\ \|z''\|_{L^2\left(0,T;H_0^1\right)} \right\}$$

and

$$W_1(T) = \left\{ z \in C^0\left([0,T]; H_0^1\right) \cap C^1\left([0,T]; L^2\right) : z' \in L^2\left(0,T; H_0^1\right) \right\},$$

be a Banach space with respect to the norm (see Lions [25])

$$\|z\|_{W_1(T)} = \|z\|_{C^0\left([0,T];H_0^1\right)} + \|z'\|_{C^0([0,T];L^2)} + \|z'\|_{L^2\left(0,T;H_0^1\right)}.$$

For every $M > 0$, we put

$$\mathcal{B}_T(M) = \left\{ v \in V_T : \|v\|_{V_T} \le M \right\},$$
$$\mathcal{W}(M,T) = \left\{ v \in \mathcal{B}_T(M) : v'' \in L^\infty\left(0,T; L^2\right) \right\}.$$

In what follows, we shall use the linear approximation method combined with the Faedo-Galerkin method and the weak compact method to prove the existence and uniqueness of weak solutions of the problem (1.1)-(1.3).

First, we establish the following recurrent sequence $\{u_m\}$ satisfying $u_0 \equiv 0$, and supposed that

$$u_{m-1} \in \mathcal{W}(M,T). \tag{2.4}$$

We associate the problem (1.1)-(1.3) with finding $u_m \in \mathcal{W}(M,T)$ $(m \ge 1)$ to be satisfied the linear variational problem

$$\langle u_m''(t), v\rangle + \lambda \langle u_{mx}'(t), v_x\rangle + \langle \mu_m(t) u_{mx}(t), v_x\rangle - \int_0^t h(t-s) \langle u_{mx}(s), v_x\rangle ds = \langle F_m(t), v\rangle \tag{2.5}$$

for all $v \in H_0^1$ and a.e., $t \in (0,T)$, together with the initial conditions

$$u_m(0) = \tilde{u}_0,\ u_m'(0) = \tilde{u}_1, \tag{2.6}$$

where

$$\begin{cases} \mu_m(x,t) = \mu[u_{m-1}] = \mu\left(x,t,u_{m-1},\langle\nabla u_{m-1}(t),\nabla u_{m-1}'(t)\rangle,\|u_{m-1}(t)\|^2,\|\nabla u_{m-1}(t)\|^2\right), \\ F_m(x,t) = f[u_{m-1}] = f\ \left(x,t,u_{m-1}(x,t),\nabla u_{m-1}(x,t),u_{m-1}'(x,t),\nabla u_{m-1}'(x,t)\right). \end{cases} \tag{2.7}$$

Then, the existence of $\{u_m\}$ and the local solution for the problem (1.1)-(1.3) can be similarly proved by the methods and the techniques given in [29] (see Theorem 1 and Theorem 2), and presented in the following theorem.

Theorem 2.2. *If $(H_1)-(H_4)$ hold, then there exist positive constants M and T such that*

(i) For $u_0 \equiv 0$, there exists a recurrent sequence $\{u_m\} \subset \mathcal{W}(M,T)$ defined by (2.4) - (2.7).

(ii) The sequence $\{u_m\}$ converges strongly in $W_1(T)$ to a function $u \in \mathcal{W}(M,T)$ to be a unique weak solution of the problem (1.1)-(1.3).

Furthermore, the following estimate is valid

$$\|u_m - u\|_{W_1(T)} \le C_T k_T^m, \text{ for all } m \in \mathbb{N}, \tag{2.8}$$

where $k_T \in [0,1)$ is a constant and C_T is a positive constant independ of m.

3 General decay of solutions

This section investigates the decay of the solution for the problem (1.1)-(1.3) corresponding to

$$\begin{aligned} f &= -\lambda_1 u_t + g(u) + F(x,t), \\ \mu &= \mu_1(t) + \mu_2\left(\|u_x(t)\|^2\right) + \mu_3\left(\langle u_x(t), u_{xt}(t)\rangle\right). \end{aligned}$$

More precisely, we consider the following problem

$$\begin{cases} u_{tt} - \lambda u_{xxt} - \left[\mu_1(t) + \mu_2\left(\|u_x(t)\|^2\right) + \mu_3\left(\langle u_x(t), u_{xt}(t)\rangle\right)\right] u_{xx} \\ \qquad\qquad + \int_0^t h(t-s)\, u_{xx}(s)\, ds + \lambda_1 u_t = g(u) + F(x,t),\ 0 < x < 1,\ t > 0, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x). \end{cases} \tag{3.1}$$

where $\lambda, \lambda_1 > 0$ are given constants and $\mu_i, \left(i = \overline{1,3}\right)$, $g,\ F, h,\ \tilde{u}_0,\ \tilde{u}_1$, are the given functions.

In order to present the main results of this section, we need the following assumptions

(A_1) : $\tilde{u}_0,\ \tilde{u}_1 \in H_0^1 \cap H^2$;

(A_2) : $\mu_1,\ \mu_2 \in C^1(\mathbb{R}_+)$, $\mu_3 \in C^1(\mathbb{R})$ and there exist the constants $\chi_* > 0$, $\mu_i^* > 0$, $\left(i = \overline{1,3}\right)$, $\mu_1^* + \mu_2^* > \mu_3^*$ such that

(i) $\mu_1(t) \ge \mu_1^* > 0,$ for all $t \ge 0$,

(ii) $\mu_1'(t) \le 0,$ for all $t \ge 0$,

(iii) $\mu_2(y) \ge \mu_2^* > 0,$ for all $y \in \mathbb{R}_+$,

(iv) $y\mu_2(y) \ge \chi_* \int_0^y \mu_2(z)dz,$ for all $y \in \mathbb{R}_+$,

(iii) $\mu_3(y) \ge -\mu_3^*,$ for all $y \in \mathbb{R}$,

(iv) $y\mu_3(y) \ge 0,$ for all $y \in \mathbb{R}$;

$(A_3):\ h \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ such that

(i) $l = \mu_1^* + \mu_2^* - \int_0^\infty h(s)\,ds > 0,\ h(0) > 0,$

(ii) there exists a function $\xi \in C^1(\mathbb{R}_+)$ such that

$$\xi'(t) \leq 0 < \xi(t),\ \text{for all } t \geq 0, \int_0^\infty \xi(s)\,ds = +\infty,$$
$$h'(t) \leq -\xi(t)\,h(t) < 0,\ \text{for all } t \geq 0;$$

$(A_4):\ g \in C^1(\mathbb{R})$ and there exist the constants $\alpha,\ \beta > 2;\ d,\ \bar{d} > 0$, such that

(i) $yg(y) \leq d\int_0^y g(z)dz$, for all $y \in \mathbb{R}$,

(ii) $\int_0^y g(z)dz \leq \bar{d}\left(|y|^\alpha + |y|^\beta\right)$, for all $y \in \mathbb{R}$;

$(A_5):\ F \in L^\infty\left(\mathbb{R}_+; L^2\right) \cap L^1\left(\mathbb{R}_+; L^2\right)$, and there exist two positive constants C_0, γ_0 such that $\|F(t)\|^2 \leq C_0 \exp(-\gamma_0 t)$, for all $t \geq 0$;

$(A_6):\ p > \max\left\{2, d, \dfrac{d}{\chi_*}\right\}.$

Remark. There are some examples of the nonlinear functions $\mu_2,\ \mu_3$ and g satisfying (A_2) and (A_4). We refer to the example given in [29], in which the nonlinear functions $B,\ \sigma,\ f$ are substituted with $\mu_2,\ \mu_3,\ g$ respectively.

By the same method used for the proof of Theorem 2.2, the problem (3.1) admits a weak solution $u(x,t)$ such that

$$\begin{aligned} u &\in C([0,T]; H_0^1 \cap H^2) \cap C^1([0,T]; H_0^1) \cap L^\infty(0,T; H_0^1 \cap H^2), \\ u' &\in C([0,T]; H_0^1) \cap L^\infty(0,T; H_0^1 \cap H^2), \\ u'' &\in L^\infty(0,T; L^2) \cap L^2\left(0,T; H_0^1\right) \end{aligned} \tag{3.2}$$

for $T > 0$ small enough.

In the following, we prove that if $\mu_1(0)\,\|\tilde{u}_{0x}\|^2 + \int_0^{\|\tilde{u}_{0x}\|^2} \mu_2(z)dz - p\int_0^1 dx \int_0^{\tilde{u}_0(x)} g(z)\,dz >$ 0 and if the initial energy $E(0)$ and $\|F(t)\|^2$ are small enough, then the solution of the problem (3.1) is decayed generally as $t \to +\infty$.

First, we construct the total energy functional by

$$\mathcal{L}(t) = E(t) + \delta\psi(t), \tag{3.3}$$

where $\delta > 0$ is chosen later and

$$\begin{aligned} E(t) = {} & \frac{1}{2}\left\|u'(t)\right\|^2 + \left(\frac{1}{2} - \frac{1}{p}\right)(h * u)(t) \\ & + \left(\frac{1}{2} - \frac{1}{p}\right)\left[\left(\mu_1(t) - \overline{h}(t)\right)\|u_x(t)\|^2 + \int_0^{\|u_x(t)\|^2} \mu_2(z)dz\right] + \frac{1}{p}I(t), \end{aligned} \tag{3.4}$$

$$I(t) = (h * u)(t) + \left(\mu_1(t) - \overline{h}(t)\right)\|u_x(t)\|^2 + \int_0^{\|u_x(t)\|^2} \mu_2(z)dz - p\int_0^1 dx \int_0^{u(x,t)} g(z)\,dz, \tag{3.5}$$

$$\psi(t) = \langle u'(t), u(t) + \frac{\lambda}{2}\|u_x(t)\|^2 + \frac{\lambda_1}{2}\|u(t)\|^2, \tag{3.6}$$

where $(h * u)(t) = \int_0^t h(t-s)\|u_x(s) - u_x(t)\|^2 ds$ and $\overline{h}(t) = \int_0^t h(s)\, ds$.

Then, we have the following estimates of $E'(t)$.

Lemma 3.1. *Let u be a solution of (3.1). We have*

(i) $E'(t) \le \frac{1}{2}\|F(t)\| + \frac{1}{2}\|F(t)\|\|u'(t)\|^2,$

(ii) $E'(t) \le -\lambda\|u_x'(t)\|^2 - \left(\lambda_1 - \frac{\varepsilon_1}{2}\right)\|u'(t)\|^2 - \frac{1}{2}\xi(t)(h*u)(t) + \frac{1}{2\varepsilon_1}\|F(t)\|^2$

for all $\varepsilon_1 > 0$.

Proof. Multiplying $(3.1)_1$ by $u'(x,t)$ and integrating over $[0,1]$, we get

$$\begin{aligned} E'(t) = &-\lambda\left\|u_x'(t)\right\|^2 - \lambda_1\left\|u'(t)\right\|^2 - \langle u_x(t), u_x'(t)\rangle\mu_3\left(\langle u_x(t), u_x'(t)\rangle\right) \\ &+ \frac{1}{2}\mu_1'(t)\|u_x(t)\|^2 + \frac{1}{2}\left(h' * u\right)(t) - \frac{1}{2}h(t)\|u_x(t)\|^2 + \langle F(t), u'(t)\rangle, \end{aligned} \tag{3.7}$$

where $(h' * u)(t) = \int_0^t h'(t-s)\|u_x(s) - u_x(t)\|^2 ds$.

By using Cauchy - Schwarz inequality, it is easy to prove that

$$\langle F(t), u'(t)\rangle \le \frac{1}{2}\|F(t)\| + \frac{1}{2}\|F(t)\|\left\|u'(t)\right\|^2, \tag{3.8}$$

$$\langle F(t), u'(t)\rangle \le \frac{1}{2\varepsilon_1}\|F(t)\|^2 + \frac{\varepsilon_1}{2}\left\|u'(t)\right\|^2, \ \forall\, \varepsilon_1 > 0. \tag{3.9}$$

By (A_3), we also have

$$\left(h' * u\right)(t) \le -\xi(t)(h*u)(t) \le 0, \forall t \ge 0. \tag{3.10}$$

Since $y\mu_3(y) \ge 0, \forall y$ and $g(t) > 0, \forall t \ge 0$, so we deduce from (3.7) and (3.8) that (i) hold.

Similarly, (ii) is inferred from (3.7), (3.9) and (3.10). □

Now, we shall use Lemma 3.1 (i) to prove the following lemma.

Lemma 3.2. *Assume that $(A_1) - (A_6)$ hold. Let $\tilde{u}_0 \in H_0^1 \cap H^2$* such that $I(0) > 0$ *and the initial energy $E(0)$ satisfy*

$$\eta^* = l - p\bar{d}\left(R_*^{\alpha-2} + R_*^{\beta-2}\right) > 0, \tag{3.11}$$

where $R_* = \sqrt{\frac{2pE_*}{(p-2)l}}$, $E_* = (E(0)+\rho)\exp(2\rho)$, $\rho = \frac{1}{2}\int_0^{+\infty}\|F(t)\|\, dt$. *Then* $I(t) > 0, \forall t \ge 0$.

Proof. By the continuity of $I(t)$ and $I(0) > 0$, there exists $\tilde{T} > 0$ such that

$$I(t) = I(u(t)) > 0,\ \forall t \in [0, \tilde{T}],$$

this implies

$$\begin{aligned} E(t) &\ge \frac{1}{2}\left\|u'(t)\right\|^2 + \left(\frac{1}{2} - \frac{1}{p}\right)\left[\left(\mu_1(t) - \overline{h}(t)\right)\|u_x(t)\|^2 + \int_0^{\|u_x(t)\|^2}\mu_2(z)dz\right] \\ &\ge \frac{1}{2}\left\|u'(t)\right\|^2 + \frac{(p-2)l}{2p}\|u_x(t)\|^2,\ \forall t \in [0, \tilde{T}]. \end{aligned} \tag{3.12}$$

Using (3.12), Lemma 3.1 (i) and Gronwall's inequality, we obtain

$$\|u_x(t)\|^2 \leq \frac{2p}{(p-2)l}E(t) \leq \frac{2pE_*}{(p-2)l} \equiv R_*^2,\ \forall t \in [0,\tilde{T}], \tag{3.13}$$
$$\left\|u'(t)\right\|^2 \leq 2E\,(t) \leq 2E_*, \forall t \in [0,\tilde{T}].$$

Moreover, it follows from (A_5) and (3.13) that

$$\begin{aligned} p\int_0^1 dx \int_0^{u(x,t)} g\,(z)\,dx &\leq p\bar{d}\left(\|u(t)\|_{L^\alpha}^\alpha + \|u(t)\|_{L^\beta}^\beta\right) \\ &\leq p\bar{d}\left(\|u_x(t)\|^\alpha + \|v_x(t)\|^\beta\right) \\ &\leq p\bar{d}\left(\|u_x(t)\|^{\alpha-2} + \|u_x(t)\|^{\beta-2}\right)\|u_x(t)\|^2 \\ &\leq p\bar{d}\left(R_*^{\alpha-2} + R_*^{\beta-2}\right)\|u_x(t)\|^2. \end{aligned}$$

Thus,

$$\begin{aligned} I\,(t) &= (h*u)\,(t) + \left(\mu_1\,(t) - \overline{h}\,(t)\right)\|u_x(t)\|^2 + \int_0^{\|u_x(t)\|^2} \mu_2(z)dz - p\int_0^1 dx \int_0^{u(x,t)} g\,(z)\,dz \\ &\geq (h*u)\,(t) + l\,\|u_x(t)\|^2 - p\bar{d}\left(R_*^{\alpha-2} + R_*^{\beta-2}\right)\|u_x(t)\|^2 \\ &\geq \eta^*\,\|u_x(t)\|^2 + (h*u)\,(t) \geq 0,\ \forall t \in [0,\tilde{T}], \end{aligned} \tag{3.14}$$

where $\eta^* > 0$ as in (3.11).

Next, we prove that $I\,(t) > 0, \forall t \geq 0$. We put $T_* = \sup\{T > 0 : I(t) > 0,\ \forall t \in [0,T]\}$. If $T_* < +\infty$ then, by the continuity of $I(t)$, we have $I(T_*) \geq 0$.

In case of $I(T_*) > 0$, by the same arguments as above, we can deduce that there exists $\tilde{T}_* > T_*$ such that $I(t) > 0, \forall t \in [0,\tilde{T}_*]$.We obtain a contradiction to the definition of T_*.

In case of $I(T_*) = 0$, it implies from (3.14) that

$$0 = I(T_*) \geq \eta^*\,\|u_x(T_*)\|^2 + (h*u)\,(T_*)\,.$$

Therefore

$$u\,(T_*) = (h*u)\,(T_*) = 0.$$

By the fact that the function $s \mapsto h\,(T_* - s)\,\|u_x\,(s) - u_x\,(T_*)\|^2$ is continuous on $[0,T_*]$, $h\,(T_* - s) > 0, \forall s \in [0,T_*]$, and

$$(h*u)\,(T_*) = \int_0^{T_*} h\,(T_* - s)\,\|u_x\,(s) - u_x\,(T_*)\|^2\,ds = 0,$$

it follows that $\|u_x\,(s) - u_x\,(T_*)\| = 0, \forall s \in [0,T_*]$, it means that $u\,(s) = u\,(T_*) = 0, \forall s \in [0,T_*]$. Then, $u\,(s) = 0$. It leads to $I\,(0) = 0$. We get a contradiction with the fact that $I\,(0) > 0$.

Hence, we conclude that $T_* = +\infty$, i.e. $I(t) > 0,\ \forall t \geq 0$. Lemma 3.2 is proved completely. □

Next, we put

$$E_1(t) = \left\|u'(t)\right\|^2 + \|u_x(t)\|^2 + (h*u)\,(t) + I(t). \tag{3.15}$$

In order to show our stability result, we need the following lemma.

Lemma 3.3. *Under the assumptions of Lemma 3.2, there exist the positive constants* $\beta_1, \bar{\beta}_1, \beta_2, \bar{\beta}_2$ *such that*

$$\begin{aligned} &\text{(i) } \beta_1 E_1(t) \le \mathcal{L}(t) \le \beta_2 E_1(t),\ \forall t \ge 0, \\ &\text{(ii) } \bar{\beta}_1 E_1(t) \le E(t) \le \bar{\beta}_2 E_1(t),\ \forall t \ge 0, \end{aligned} \tag{3.16}$$

for δ *is small enough.*

Proof. The functional $\mathcal{L}(t)$ is rewritten as follows

$$\begin{aligned} \mathcal{L}(t) = {} & \frac{1}{2}\left\|u'(t)\right\|^2 + \left(\frac{1}{2} - \frac{1}{p}\right)(h * u)(t) \\ & + \left(\frac{1}{2} - \frac{1}{p}\right)\left[\left(\mu_1(t) - \overline{h}(t)\right)\|u_x(t)\|^2 + \int_0^{\|u_x(t)\|^2} \mu_2(z)dz\right] \\ & + \frac{1}{p} I(t) + \delta\langle u'(t), u(t)\rangle + \frac{\delta\lambda}{2}\|u_x(t)\|^2 + \frac{\delta\lambda_1}{2}\|u(t)\|^2 . \end{aligned}$$

From the following inequalities

$$\begin{aligned} &\left|\langle u'(t), u(t)\rangle\right| \le \frac{1}{2}\left\|u'(t)\right\|^2 + \frac{1}{2}\|u_x(t)\|^2, \\ &-\mu_2^* \|u_x(t)\|^2 \ge -\int_0^{\|u_x(t)\|^2} \mu_2(z)dz, \end{aligned}$$

we deduce that

$$\begin{aligned} \mathcal{L}(t) \ge {} & \frac{1-\delta}{2}\left\|u'(t)\right\|^2 + \left(\frac{1}{2} - \frac{1}{p}\right)(h * u)(t) \\ & + \frac{1}{2}\left[\frac{(p-2)\,l}{p} - \delta\right]\|u_x(t)\|^2 + \frac{1}{p} I(t) \\ \ge {} & \beta_1 E_1(t), \end{aligned}$$

where we choose $\beta_1 = \min\left\{\frac{1-\delta}{2},\ \frac{(p-2)l}{2p} - \frac{\delta}{2}, \left(\frac{1}{2} - \frac{1}{p}\right),\ \frac{1}{p}\right\}$, with δ is small enough, $0 < \delta < \min\left\{1;\ \frac{(p-2)l}{p}\right\}$.

Similarly, we can prove that

$$\begin{aligned} \mathcal{L}(t) \le {} & \frac{1}{2}\left\|u'(t)\right\|^2 + \left(\frac{1}{2} - \frac{1}{p}\right)(h * u)(t) \\ & + \left(\frac{1}{2} - \frac{1}{p}\right)\left[\mu_1(t)\|u_x(t)\|^2 + \int_0^{\|u_x(t)\|^2} \mu_2(z)dz\right] + \frac{1}{p} I(t) \\ & + \frac{\delta}{2}\left(\left\|u'(t)\right\|^2 + \|u_x(t)\|^2\right) + \frac{\delta\lambda}{2}\|u_x(t)\|^2 + \frac{\delta\lambda_1}{2}\|u(t)\|^2 . \end{aligned}$$

Put $\mu_{2\max} = \max_{0 \le z \le R_*^2} \mu_2(z)$, we have $\int_0^{\|u_x(t)\|^2} \mu_2(z)dz \le \mu_{2\max}\|u_x(t)\|^2$, hence

$$\begin{aligned} \mathcal{L}(t) \le {} & \frac{1+\delta}{2}\left\|u'(t)\right\|^2 + \frac{p-2}{2p}(h * u)(t) \\ & + \frac{1}{2}\left[\frac{p-2}{p}\left(\mu_1(0) + \mu_{2\max}\right) + \delta\left(1 + \lambda + \lambda_1\right)\right]\|u_x(t)\|^2 + \frac{1}{p} I(t) \\ \le {} & \beta_2 E_1(t), \end{aligned} \tag{3.17}$$

where $\beta_2 = \max\left\{\dfrac{1+\delta}{2},\ \frac{p-2}{2p}\left(\mu_1(0)+\mu_{2\max}\right)+\frac{\delta}{2}\left(1+\lambda+\lambda_1\right)\right\}$.

The proof of (ii) is similar. Hence, Lemma 3.3 is proved completely. □

Lemma 3.4. *Under the assumptions of Lemma 3.2, the functional $\psi(t)$ defined by* (3.6) *satisfies*

$$\begin{aligned}\psi'(t) \leq \left\|u'(t)\right\|^2 + \left(\frac{d}{p}+\frac{1}{2\varepsilon_3}\right)(h*u)(t) - \frac{d\delta_1}{p}I(t) + \frac{1}{2\varepsilon_2}\left\|F(t)\right\|^2, \qquad (3.18)\\ -\left[\frac{d(1-\delta_1)\eta^*}{p}+\left(1-\frac{d}{p}\right)\mu_1^* + \left(1-\frac{d}{p\chi_*}\right)\mu_2^* - \mu_3^* - \frac{\varepsilon_2}{2} - \left(1-\frac{d}{p}+\frac{\varepsilon_3}{2}\right)\overline{h}(\infty)\right]\left\|u_x(t)\right\|^2\end{aligned}$$

for all $\varepsilon_2>0, \varepsilon_3>0$, $\delta_1\in(0,1)$.

Proof. Multiplying $(3.1)_1$ by $u(x,t)$ and integrating over $[0,1]$, we obtain

$$\begin{aligned}\psi'(t) = \left\|u'(t)\right\|^2 - \left[\mu_1(t)+\mu_2\left(\left\|u_x(t)\right\|^2\right)+\mu_3\left(\langle u_x(t),u_x'(t)\rangle\right)\right]\left\|u_x(t)\right\|^2\\ + \int_0^t h(t-s)\langle u_x(s),u_x(t)\rangle\, ds + \langle g(u(t)),u(t)\rangle + \langle F(t),u(t)\rangle.\end{aligned}$$

It is easy to obtain the following estimates

$$\begin{aligned}-\mu_1(t)\left\|u_x(t)\right\|^2 &\leq -\mu_1^*\left\|u_x(t)\right\|^2,\\ -\left\|u_x(t)\right\|^2\mu_2\left(\left\|u_x(t)\right\|^2\right) &\leq -\mu_2^*\left\|u_x(t)\right\|^2,\\ -\mu_3\left(\langle u_x(t),u_x'(t)\rangle\right)\left\|u_x(t)\right\|^2 &\leq \mu_3^*\left\|u_x(t)\right\|^2, \qquad (3.19)\\ \int_0^{\left\|u_x(t)\right\|^2}\mu_2(z)dz &\leq \frac{1}{\chi_*}\left\|u_x(t)\right\|^2\mu_2\left(\left\|u_x(t)\right\|^2\right),\\ -I(t) &\leq -\eta^*\left\|u_x(t)\right\|^2,\\ \langle F(t),u(t)\rangle &\leq \frac{\varepsilon_2}{2}\left\|u_x(t)\right\|^2 + \frac{1}{2\varepsilon_2}\left\|F(t)\right\|^2, \forall\varepsilon_2>0,\end{aligned}$$

$$\begin{aligned}\langle g(u(t)),u(t)\rangle &\leq d\int_0^1 dx\int_0^{u(x,t)} g(z)\,dz = \frac{d}{p}(h*u)(t)\\ &\quad + \frac{d}{p}\left[\left(\mu_1(t)-\overline{h}(t)\right)\left\|u_x(t)\right\|^2 + \int_0^{\left\|u_x(t)\right\|^2}\mu_2(z)dz - I(t)\right] \qquad (3.20)\\ &\leq \frac{d}{p}\left[\left(\mu_1(t)-\overline{h}(t)\right)\left\|u_x(t)\right\|^2 + \int_0^{\left\|u_x(t)\right\|^2}\mu_2(z)dz\right]\\ &\quad + \frac{d}{p}(h*u)(t) - \frac{d(1-\delta_1)}{p}I(t) - \frac{d\delta_1}{p}I(t),\end{aligned}$$

$$\begin{aligned}\int_0^t h(t-s)\langle u_x(s),u_x(t)\rangle\, ds &= \int_0^t h(t-s)\langle u_x(s)-u_x(t),u_x(t)\rangle\, ds + \overline{h}(t)\left\|u_x(t)\right\|^2 \qquad (3.21)\\ &\leq \frac{1}{2\varepsilon_3}(h*u)(t) + \left(1+\frac{\varepsilon_3}{2}\right)\overline{h}(t)\left\|u_x(t)\right\|^2\end{aligned}$$

for all $\varepsilon_3 > 0$.

Then, it follows from (3.19)-(3.21) that the inequality (3.18) is valid. □

Using Lemmas 3.1- 3.4, we can state and prove our main result in this section as follows.

Theorem 3.5. *Assume that $(A_1)-(A_6)$ hold. Let $\tilde{u}_0 \in H_0^1 \cap H^2$ such that $I(0) > 0$, the initial energy $E(0)$ satisfy (3.11) and*

$$\overline{h}(\infty) + \bar{d}d\left(R_*^{\alpha-2} + R_*^{\beta-2}\right) < \mu_1^* + \left(1 + \frac{d}{p} - \frac{d}{p\chi_*}\right)\mu_2^* - \mu_3^*. \tag{3.22}$$

Then, there exist positive constants C, Λ such that

$$\left\|u'(t)\right\|^2 + \|u_x(t)\|^2 \leq C\exp\left(-\Lambda\int_0^t \xi(s)\,ds\right), \text{ for all } t \geq 0. \tag{3.23}$$

Proof. From the definition of $\mathcal{L}(t)$, Lemma 3.1 (ii) and (3.21), we deduce that

$$\begin{aligned}\mathcal{L}'(t) \leq& -\left(\lambda_1 - \frac{\varepsilon_1}{2} - \delta\right)\left\|u'(t)\right\|^2 - \frac{1}{2}\xi(t)(h*u)(t) + \delta\left(\frac{d}{p} + \frac{1}{2\varepsilon_3}\right)(h*u)(t) \\ & - \delta\theta_1\|u_x(t)\|^2 - \frac{d\delta\delta_1}{p}I(t) + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\|F(t)\|^2,\end{aligned} \tag{3.24}$$

where

$$\begin{aligned}\theta_1 &= \theta_1(\delta_1, \varepsilon_2, \varepsilon_3) \\ &= \frac{d(1-\delta_1)\eta^*}{p} + \left(1 - \frac{d}{p}\right)\mu_1^* + \left(1 - \frac{d}{p\chi_*}\right)\mu_2^* - \mu_3^* - \frac{\varepsilon_2}{2} - \left(1 - \frac{d}{p} + \frac{\varepsilon_3}{2}\right)\overline{h}(\infty),\end{aligned}$$

satisfying

$$\begin{aligned}&\lim_{\delta_1\to 0_+,\,\varepsilon_2\to 0_+,\varepsilon_3\to 0_+}\theta_1(\delta_1, \varepsilon_2, \varepsilon_3) \\ &= \frac{d\eta^*}{p} + \left(1 - \frac{d}{p}\right)\mu_1^* + \left(1 - \frac{d}{p\chi_*}\right)\mu_2^* - \mu_3^* - \left(1 - \frac{d}{p}\right)\overline{h}(\infty) \\ &\equiv \hat{\theta}_1.\end{aligned}$$

Note that, the conditions (3.22) leads to $\hat{\theta}_1 > 0$.Therefore, we can choose $\delta_1 \in (0,1)$ and $\varepsilon_2 > 0,\ \varepsilon_3 > 0$ small enough such that

$$\theta_1 = \theta_1(\delta_1, \varepsilon_2, \varepsilon_3) > 0.$$

Moreover, by choosing $\varepsilon_1 > 0$, $\delta > 0$ small enough, we get that

$$\bar{\theta}_1 = \lambda_1 - \frac{\varepsilon_1}{2} - \delta > 0,\ 0 < \delta < \min\left\{1;\ \frac{(p-2)\,l}{p}\right\}. \tag{3.25}$$

Put

$$\begin{aligned}&\bar{\theta}_2 = \delta\theta_1,\ \bar{\theta}_3 = \frac{d\delta\delta_1}{p}, \\ &\bar{\theta}_4 = \delta\left(\frac{d}{p} + \frac{1}{2\varepsilon_3}\right),\ \bar{\theta}_* = \min\left\{\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3\right\}.\end{aligned} \tag{3.26}$$

It follows from (3.24)-(3.26) that

$$\begin{aligned}\mathcal{L}'(t) &\leq -\bar{\theta}_1 \left\|u'(t)\right\|^2 - \bar{\theta}_2 \left\|u_x(t)\right\|^2 - \bar{\theta}_3 I(t) + \bar{\theta}_4 (h*u)(t) + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\|F(t)\|^2 \\ &\leq -\bar{\theta}_* \left[\left\|u'(t)\right\|^2 + \|u_x(t)\|^2 + I(t) + (h*u)(t)\right] \\ &\quad + \left(\bar{\theta}_* + \bar{\theta}_4\right)(h*u)(t) + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\|F(t)\|^2 \qquad (3.27) \\ &\leq -\bar{\theta}_* E_1(t) + \left(\bar{\theta}_* + \bar{\theta}_4\right)(h*u)(t) + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\|F(t)\|^2 \\ &\leq -\frac{\bar{\theta}_*}{\bar{\beta}_2} E(t) + \left(\bar{\theta}_* + \bar{\theta}_4\right)(h*u)(t) + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\|F(t)\|^2.\end{aligned}$$

Combiming Lemma 3.1 (ii) and (3.27), we obtain

$$\begin{aligned}&\xi(t)\mathcal{L}'(t) \leq -\frac{\bar{\theta}_*}{\bar{\beta}_2}\xi(t)E(t) + \left(\bar{\theta}_* + \bar{\theta}_4\right)\xi(t)(h*u)(t) + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\xi(0)\|F(t)\|^2 \\ &\leq -\frac{\bar{\theta}_*}{\bar{\beta}_2}\xi(t)E(t) - 2\left(\bar{\theta}_* + \bar{\theta}_4\right)E'(t) + \frac{\bar{\theta}_* + \bar{\theta}_4}{\varepsilon_1}\|F(t)\|^2 + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\xi(0)\|F(t)\|^2 \\ &\leq -\frac{\bar{\theta}_*}{\bar{\beta}_2}\xi(t)E(t) - 2\left(\bar{\theta}_* + \bar{\theta}_4\right)E'(t) + \tilde{C}_0 e^{-\gamma_0 t}, \qquad (3.28)\end{aligned}$$

where $\tilde{C}_0 = \frac{1}{2}\left[\frac{2\left(\bar{\theta}_* + \bar{\theta}_4\right)}{\varepsilon_1} + \left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\xi(0)\right]C_0.$

We consider the functional

$$L(t) = \xi(t)\mathcal{L}(t) + 2\left(\bar{\theta}_* + \bar{\theta}_4\right)E(t),$$

then

$$\begin{aligned}L(t) &\leq \xi(0)\mathcal{L}(t) + 2\left(\bar{\theta}_* + \bar{\theta}_4\right)E(t) \\ &\leq \xi(0)\beta_2 E_1(t) + 2\left(\bar{\theta}_* + \bar{\theta}_4\right)E(t) \\ &\leq \left[\frac{\beta_2}{\bar{\beta}_1}\xi(0) + 2\left(\bar{\theta}_* + \bar{\theta}_4\right)\right]E(t) \equiv \hat{\beta}_2 E(t),\end{aligned}$$

and

$$\begin{aligned}L'(t) &= \xi'(t)\mathcal{L}(t) + \xi(t)\mathcal{L}'(t) + 2\left(\bar{\theta}_* + \bar{\theta}_4\right)E'(t) \\ &\leq -\frac{\bar{\theta}_*}{\bar{\beta}_2}\xi(t)E(t) + \tilde{C}_0 e^{-\gamma_0 t} \qquad (3.29) \\ &\leq -\frac{\bar{\theta}_*}{\bar{\beta}_2\hat{\beta}_2}\xi(t)L(t) + \tilde{C}_0 e^{-\gamma_0 t}.\end{aligned}$$

Choose $0 < \Lambda < \min\left\{\frac{\bar{\theta}_*}{\bar{\beta}_2\hat{\beta}_2}, \frac{\gamma_0}{\xi(0)}\right\}$, we get from (3.29) that

$$L'(t) + \Lambda\xi(t)L(t) \leq \tilde{C}_0 e^{-\gamma_0 t}. \qquad (3.30)$$

Integrating (3.30) with respect to time variable, we obtain

$$L(t) \leq \left(L(0) + \frac{\tilde{C}_0}{\gamma_0 - \Lambda\xi\left(0\right)} \right) \exp\left(-\Lambda \int_0^t \xi\left(s\right) ds \right). \tag{3.31}$$

On the other hand, we have

$$L(t) = \xi\left(t\right) \mathcal{L}(t) + 2\left(\bar{\theta}_* + \bar{\theta}_4\right) E(t) \geq 2\left(\bar{\theta}_* + \bar{\theta}_4\right) E(t) \geq 2\left(\bar{\theta}_* + \bar{\theta}_4\right) \bar{\beta}_1 E_1(t), \tag{3.32}$$

$$E_1\left(t\right) \geq \left\|u'(t)\right\|^2 + \left\|u_x(t)\right\|^2.$$

Combining (3.31) and (3.31) we obtain (3.23). Theorem 3.5 is proved completely. □

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