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Some notions of I_3-convergence sequences spaces defined by modulus function and strong Cesáro sequence spaces

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Abstract. *In this article, we use the notion of I_3-convergence to introduce and study some triple sequences spaces by using modulus function and strong Cesáro sequences spaces. Those sequences spaces are namely $c^I_{3_0}$, c^I_3, $l^I_{3_\infty}$, m^I_3, $m^I_{3_0}$, C^I_{s3} and $C^I_{s3_0}$.*

Keywords. I_3-convergence · triple sequences spaces · modulus function · strong Cesáro sequences spaces.

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1 Introduction

The notion of I-convergence was itnroduced by Kostyrko, Salat and Wilczynski [8] as a generalization of statical convergence. This notion has been studied by many mathematicians in different fields of the mathematics. Fast [2] in 1951 introduced the concept of statistical convergence, at the same time Steinhaus [15] in 1951 by his own way defined the notion of ordinary and asymptotic convergences. After that, Fridy [3,4] studied the statistical convergence and he created a relation which linked this notion with the notion of summability theory. Some years after, Salat et al. [13] studied some properties of I-convergence, and then this notion started to be studied in double and tripe sequences spaces. Addicionaly, triple sequences on I-convergence was studied by Sahiner and Tripathy [14] in which they showed some interesting results which were useful for Tripathy and Goswami [16,17] who studied this notion by suing Orlicz function and multiple sequences in probabilistic normed spaces, respectively. On the other hand, the idea of modulus was introduced by Nakano [10]. Khan et al. [7] studied I-convergent sequences by using modulus function through Zeweir I-convergent.

Ruckle [12] took the idea of modulus function for constructing the sequence spaces $X(f) = \{x = (x_n) : \sum_{k=1}^{\infty} f(|x_n|) < \infty\}$. Otherwise, the notion of strong Cesáro convergence was initially defined by [5], this notion was defined as: A sequence (x_n) on a normed space $(X, \|\cdot\|)$ is called strongly Cesáro convergence to L if $\lim_{n\to\infty} 1/k \sum_{n=1}^{k} \|x_n - L\| = 0$. In [10,11], the authors extended this notion in several fields. Recently, in 2020, Faisal [1] defined the concept of strongly Cesáso ideal convergent and proved some properties.

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Throughout this paper, a triple sequence x is represented by (x_{nmj}) i.e. a triple infinite array of real numbers where $n, m, j \in \mathbb{N}$, $\mathbb{N}$ denotes the set of natural number. In this paper, we took the notion of triple sequence for studding new results over the sequence spaces $c^I_{3_0}$, c^I_3, $l^I_{3_\infty}$, m^I_3 and $m^I_{3_0}$, they denote the I-null, I-convergent, I- bounded, bounded I-convergence and bounded I-null, respectively. Besides, we introduce the notion of C^I_{s3} which denotes the space of all Cesáro triple I-convergent sequences and the notion of $C^I_{s3_0}$ which denotes the space of Cesáro triple ideal null sequences. Furthermore, ω denotes the class of all sequences.

2 Preliminaries

In this section, we show the definitions and notions which are useful for the developing of this paper.

Definition 2.1 *An ideal I is a collection of subsets of X which satisfies the following conditions:*

1 *If $A \in I$ and $B \subset A$, then $B \in I$.*
2 *If $A, B \in I$, then $A \cup B \in I$.*

Definition 2.2 *A non-empty family of sets $F(I) \subset 2^X$ is said to be filter on X if and only if $\phi \notin F(I)$, for $A, B \in F(I)$, we have that $A \cap B \in F(I)$ and for each $A \in F(I)$ and $A \subset B$, implies that $B \subset F(I)$.*

Definition 2.3 *An ideal $I \subset 2^X$ is called non-trivial if $I \neq 2^X$. A non-trivial ideal $I \subset 2^X$ is called admissible if $\{\{x\} : x \in X\} \subset I$.*

Definition 2.4 *For each ideal I, there is a filter $F(I)$ corresponding to I such that $F(I) = \{H \subseteq \mathbb{N} : H^c \in I\}$, where $H^c = \mathbb{N} - H$.*

Lemma 2.1 *Let $H \in F(I)$ and $J \subseteq \mathbb{N}$. If $J \notin I$, then $J \cap \mathbb{N} \notin I$ (see [6]) .*

Lemma 2.2 *If $I \subset 2^{\mathbb{N}}$ and $J \subseteq \mathbb{N}$. If $J \notin I$, then $J \cap \mathbb{N} \notin I$ (see [6]).*

Definition 2.5 *$I_f = I$ denotes the class of all finite subsets of $\mathbb{N}$. Then, I_f is a non-trivial admissible ideal and I_f convergence coincides with the usual convergence with respect to the metric in X.*

Definition 2.6 *$I = I_\delta$ and $A \subset \mathbb{N}$ with $\delta(A) = 0$. I_δ is a non-trivial admissible ideal.*

Definition 2.7 *A function $f : [0, \infty) \to [0, \infty)$ is said to be modulus if*

1 *$f(t) = 0$ if and only if $t = 0$.*
2 *$f(t + u) \leq f(t) + f(u)$.*
3 *f is non-decreasing.*
4 *f is continuous from the right at zero.*

Definition 2.8 *A modulus function f is said to be Δ_2-condition if for all values of u there exits a constant $K > 0$ such that $f(Lu) \leq KLf(u)$ for all values of $L > 1$.*

Definition 2.9 *A triple sequence (x_{nmj}) is said to be I_3-convergence to a number L if for every $\epsilon > 0$, $\{(m, n, j) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{nmj} - L| \geq \epsilon\} \in I$. In this case, we write $I_3\text{-}\lim x_{nmj} = L$*

Definition 2.10 *A triple sequence (x_{nmj}) is said to be I_3-null if $L = 0$. In this case, we write $I_3\text{-}\lim x_{nmj} = 0$*

Definition 2.11 *A triple sequence (x_{nmj}) is said to be I_3-Cauchy to a number L if for every $\epsilon > 0$ there exits, $h = h_0, l = l_0$ and $b = b_0$ such that $\{(m,n,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : |x_{nmj} - x_{lbh}| \geq \epsilon\} \in I$.*

Definition 2.12 *A triple sequence (x_{nmj}) is said to be I_3-bounded if there exits $M > 0$ such that $\{(m,n,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : |x_{nmj}| > M\} \in I$.*

Definition 2.13 *A triple sequence space Q is said to be solid if $(\gamma_{nmj}x_{nmj}) \in Q$ whenever $(x_{nmj}) \in Q$ and for all sequences (γ_{nmj}) of scalars with $|\gamma_{nmj}| \leq 1$, for all $n,m,j \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$.*

Definition 2.14 *A triple sequence space Q is said to be monotone if it contains the canonical pre-images of all its step spaces.*

Lemma 2.3 *Let M be a sequence space. If M is solid, then M is monotone (see [6]) .*

Definition 2.15 *A triple sequence space Q is said to be convergence free if $(y_{nmj}) \in Q$, whenever $(x_{nmj}) \in Q$ and $x_{nmj} = 0$ implies $y_{nmj} = 0$.*

Definition 2.16 *A triple sequence space Q is said to be sequence algebra if $(x_{nmj}\cdot y_{nmj}) \in Q$, whenever $(x_{nmj}) \in Q$ and $(y_{nmj}) \in Q$.*

Definition 2.17 *A map h defined on a domain $D \subset X$ i.e. $h : D \subset X \to \mathbb{R}$ is said to satisfy Lipschitz condition if $|h(x) - h(y)| \leq K|x - y|$ where K is known as the Lipschitz constant.*

Remark 2.1 A convergence field of I-convergence is a set $F(I) = \{x = (x_n) \in l_\infty :$ there exists $I\text{-}\lim x \in \mathbb{R}\}$. The convergence field $F(I)$ is a closed linear sub-space of l_∞ with respect to the supremum norm $F(I) = l_\infty \cap c^I$ (see [13]).

Otherwise, consider a function $\phi : F(I) \to \mathbb{N}$ such that $\phi(x) = I\text{-}\lim x$, for all $x \in F(I)$, then the function $\phi : F(I) \to \mathbb{R}$ is a Lipschitz function (see [9]).

3 I_3-convergent by modulus function

We define and introduce the following classes of sequence spaces:

$$c_3^I(f) = \{(x_{nmj}) \in \omega : I_3 - \lim f(|x_{nmj}) = L,\ for\ some\ L\} \in I, \tag{3.1}$$

$$c_{3_0}^I(f) = \{(x_{nmj}) \in \omega : I_3 - \lim f(|x_{nmj}) = 0\} \in I, \tag{3.2}$$

$$l_{3_\infty}^I(f) = \{(x_{nmj}) \in \omega : \sup_{nmj} f(|x_{nmj}) < \infty\} \in I. \tag{3.3}$$

Besides, $m_3^I(f)$ and $m_{3_0}^I(f)$ are denoted as:

$$m_3^I(f) = c_3^I(f) \cap l_{3_\infty}^I,$$
$$m_{3_0}^I(f) = c_{3_0}^I(f) \cap l_{3_\infty}^I.$$

Theorem 3.1 *For any modulus function f, the sequences c_3^I, $c_{3_0}^I(f)$, $m_3^I(f)$ and $m_{3_0}^I(f)$ are linear.*

Proof. We just prove the case $c_3^I(f)$, the others are proved similarly.

Let $(x_{nmj}), (y_{nmj}) \in c_3^I(f)$ and let γ, δ be scalars. Then

$$I_3\text{-}\lim f(|x_{nmj} - L_1|) = 0, \text{ for some } L_1 \in c;$$

$$I_3\text{-}\lim f(|y_{nmj} - L_1|) = 0, \text{ for some } L_2 \in c.$$

This is for a given $\epsilon > 0$, thus we have that

$$W_1 = \{(n, m, j) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : f(|x_{nmj} - L_1|) > \frac{\epsilon}{2}\} \in I, \tag{3.4}$$

$$W_2 = \{(n, m, j) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : f(|y_{nmj} - L_2|) > \frac{\epsilon}{2}\} \in I. \tag{3.5}$$

It is well know that f is a modulus function, for that reason we have that

$$\begin{aligned} & f(|\gamma x_{nmj} + \delta y_{nmj}) - (\gamma L_1 + \delta L_2)|) \\ \leq\ & f(|\gamma||x_{nmj} - L_1|) + f(|\delta||y_{nmj} - L_2|) \\ \leq\ & f(|x_{nmj} - L_1|) + f(|y_{nmj} - L_2|). \end{aligned}$$

Now, taking into account (4) and (5), we have that

$$\{(n, m, j) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : f(|(\gamma x_{nmj} + \delta y_{nmj}) - (\gamma L_1 + \delta L_2)|) > \epsilon \subset W_1 \cup W_2.$$

Therefore, this shows that $(\gamma x_{nmj} + \delta y_{nmj}) \in c_3^I(f)$, and hence $c_3^I(f)$ is a linear space.

Theorem 3.2 *Any sequence $x = (x_{nmj}) \in m_3^I(f)$ is I_3-convergent if and only if for every $\epsilon > 0$ there exits $N(\epsilon), M(\epsilon), J(\epsilon) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that*

$$\{(n, m, j) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : f(|x_{nmj} - x_{N(\epsilon)M(\epsilon)J(\epsilon)}|) < \epsilon\} \in m_3^I.$$

Proof. Consider $L = I_3\text{-}\lim x$, then

$$\beta(\epsilon) = \{(n, m, j) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{nmj} - L| < \frac{\epsilon}{2}\} \in m_2^I(f). \text{ For all } \epsilon > 0.$$

Now, fix $N(\epsilon), M(\epsilon), J(\epsilon) \in \beta(\epsilon)$. Then , we have that

$$|x_{N(\epsilon)M(\epsilon)J(\epsilon)} - x_{nmj}| \leq |x_{N(\epsilon)M(\epsilon)J(\epsilon)} - L| + |L - x_{nmj}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Which holds for all $n, m, j \in \beta(\epsilon)$. Hence,

$$\{(n, m, j) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : f(|x_{nmj} - x_{N(\epsilon)M(\epsilon)J(\epsilon)}|) < \epsilon\} \in m_3^I(f).$$

Conversely, consider

$$\{(n, m, j) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : f(|x_{nmj} - x_{N(\epsilon)M(\epsilon)J(\epsilon)}|) < \epsilon\} \in m_3^I(f).$$

This is that

$$\{(n, m, j) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : (|x_{nmj} - x_{N(\epsilon)M(\epsilon)J(\epsilon)}|) < \epsilon\} \in m_3^I(f), \text{ for all } \epsilon > 0.$$

Then, the set

$$W_3(\epsilon) = \{(n, m, j) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : x_{nmj} \in [x_{N(\epsilon)M(\epsilon)J(\epsilon)} - \epsilon, x_{N(\epsilon)M(\epsilon)J(\epsilon)} + \epsilon]\} \in,_3^I (f),$$
$$\text{for all } \epsilon > 0.$$

Now, let $A(\epsilon) = [x_{N(\epsilon)M(\epsilon)J(\epsilon)} - \epsilon, x_{N(\epsilon)M(\epsilon)J(\epsilon)} + \epsilon]$. If we fix $\epsilon > 0$, then we have $W_3(\epsilon) \in m_3^I(f)$, as well as, $W_3(\epsilon/2) \in m_3^I(f)$. Hence, $W_3(\epsilon) \cap W_3(\epsilon/2) \in m_3^I(f)$. This implies that $A(\epsilon) \cap A(\epsilon/2) \neq \emptyset$. This is that

$$\{(n, m, j) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : x_{nmj} \in A\} \in m_3^I(f).$$

Thus, $diam(A) \leq diam(A(\epsilon))$, where the $diam$ of A denotes the length of interval A. In this way, by induction we obtain the sequence of closed intervals

$$A(\epsilon) = I_0 \supseteq I_1 \supseteq ... \supseteq I_{nmj} \supseteq ...$$

With the property that $diam(I_{nmj}) \leq \frac{1}{2} diam(I_{n-1)(m-1)(j-1)})$ for $n, m, j = 2, 3, 4, ...$ and $\{(n, m, j) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : x_{nmj} \in I_{nmj}\} \in m_3^I$ for $n, m, j = 2, 3, 4, ...$ Then, there exits a $\sigma \in \bigcap I_{nmj}$ where $n, m, j \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that $\sigma = I_3\text{-}\lim x$. Therefore, $f(\sigma) = I - 3\text{-}\lim f(x)$. Hence, $L = I_3\text{-}\lim f(x)$.

Theorem 3.3 *Let f and g be modulus functions that satisfy the Δ_2-conditions. If X is any of the spaces c_3^I, $c_{3_0}^I$, m_3^I and $m_{3_0}^I$. Then, the following assertions hold:*

1 $X(g) \subseteq X(f \cdot g)$.
2 $X(f) \cap X(g) \subseteq X(f + g)$.

Proof. 1 Let $(x_{nmj}) \in c_{3_0}^I(g)$. Then,

$$I_3 - \lim_{nmj} g(|x_{nmj}|) = 0. \tag{3.6}$$

Now, let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(r) < \epsilon$ for $0 < r < 1$. Write $y_{nmj} = g(|x_{nmj}|)$ and consider $\lim_{nmj} f(y_{nmj} = \lim f(y_{nmj})_{y_{nmj}^{<\delta}} + \lim f(y_{mnj})_{y_{nmj}^{>\delta}}$. Then, we have that

$$\lim_{nmj} f(y_{nmj}) \leq f(2) \lim_{nmj}(y_{nmj}). \tag{3.7}$$

For $y_{nmj} > \delta$, we have $y_{nmj} < \frac{y_{nmj}}{\delta} < 1 + \frac{y_{nmj}}{\delta}$. It is well known that f is non-decreasing, this implies that

$$f(y_{nmj}) < f(1 + \frac{y_{nmj}}{\delta}) < \frac{1}{2} f(2) + \frac{1}{2} f(\frac{2y_{nmj}}{\delta}).$$

Now, it is well known that f satisfies Δ_2-condition, therefore

$$f(y_{nmj}) < \frac{1}{2} K \frac{y_{nmj}}{\delta} f(2) + \frac{1}{2} K \frac{y_{nmj}}{\delta} f(2) = K \frac{y_{nmj}}{\delta} f(2).$$

In consequence,

$$\lim_{nmj} f(y_{nmj}) \leq max(1, K)\delta^{-1} f(2) \lim_{nmj}(y_{nmj}). \tag{3.8}$$

By (6), (7) and (8), we have that $(x_{nmj}) \in c_{3_0}^I(f \cdot g)$. Therefore, $c_{3_0}^I(g) \subseteq c_{3_0}^I(f \cdot g)$. The others cases are proved similarly.

2 Let $(x_{nmj}) \in c_{3_0}^I(f) \cap c_{3_0}^I(g)$. Then,

$$I_3\text{-}\lim_{nmj} f(|x_{nmj}|) = 0 \text{ and } I_3\text{-}\lim_{nmj} g(|x_{nmj}|) = 0,$$

$$\lim_{nmj}(f+g)(|x_{nmj}|) = \lim_{nmj} f(|x_{nmj}|) + g(|x_{nmj}|) = \lim_{nmj} f(|x_{nmj}|) + \lim_{nmj} g(|x_{nmj}|) = 0.$$

Therefore, $lim_{nmj}(f+g)(|x_{nmj}|) = 0$, which implies that $(x_{nmj}) \in X(f+g)$, this is that $X(f) \cap x(G) \subseteq x(f+g)$.

Theorem 3.4 *The spaces $c^I_{3_0}(f)$ and $m^I_{3_0}(f)$ are solid and monotone.*

Proof. We just prove the case $c^I_3(f)$, the another is proved similarly.

Let $(x_{nmj}) \in c^I_{2_0}(f)$, then I_3-$\lim_{nmj} f(|x_{nmj}|) = 0$. Now, let (γ_{nmj}) be a sequence of scalars with $|\gamma_{nmj}| \leq 1$ for all $n, m, j \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. Then, we have that

$$I_3\text{-}\lim f(|\gamma_{nmj} x_{nmj}|) \leq I_3\text{-}\lim_{nmj} f(|\gamma_{nmj}||x_{nmj}|)$$
$$= |\gamma_{nmj}| I_3\text{-}\lim_{nmj} f(|x_{nmj}|) = 0$$

Thus, I_3-$\lim_{nmj} f(|\gamma_{nmj} x_{nmj}|) = 0$ for all $n, m, j \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ which implies that $\gamma_{nmj} x_{nmj} \in c^I_{3_0}(f)$. Therefore, the space $c^I_{3_0}(f) is solid$ and by the Lemma 2.3, $c^I_{3_0}$ is monotone.

Remark 3.1 The spaces c^I_3 and m^I_3 are neither solid nor monotone in general as can be seen in the following example:

Let $I = I_\delta$ and $f(x) = x^4$ for all $x \in [0, \infty)$. Consider the K-step space $X_K(f)$ of X defined by: Let $(x_{nmj}) \in X$ and let $(y_{nmj}) \in X_K$ be such that

$$(y_{nmj}) = \begin{cases} (x_{nmj}), & \text{if n, m, j is even} \\ 0, & \text{otherwise} \end{cases} \tag{3.9}$$

Suppose that (x_{nmj}) is a sequence defined by $(x_{nmj}) = 1$ for all $n, m, j \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. Then, $(x_{nmj}) \in c^I_3(f)$, but its K-stepspace preimage does not belong to $c^I_3(f)$. Therefore, $c^I_3(f)$ is not monotone and hence c^I_3 i not solid.

Theorem 3.5 *The spaces $c^I_3(f)$ and $c^I_{3_0}(f)$ are sequence algebra.*

Proof. We just prove the case $c^I_{3_0}(f)$, the another is proved similarly.

Let $(x_{nmj}, (y_{nmj}) \in c^I_{3_0}(f)$. Then, I_3-$\lim f(|x_{nmj}|) = 0$ and I_3-$\lim f(|y_{nmj}|) = 0$. Then, we have that I_3-$\lim f(|x_{nmj} \cdot y_{nmj}|) = 0$. Therefore, $(x_{nmj} \cdot y_{nmj}) \in c^I_{3_0}(f)$ is a sequence algebra.

Remark 3.2 The spaces $c^I_3(f)$ and $c^I_{3_0}(f)$ are not convergence free in general as can be seen in the following example:

Let $I = I_f$ and $f(x) = x^5$ for all $x \in [0, \infty)$. Consider the sequences (x_{nmj}) and (y_{nmj}) defined by $x_{nmj} = 1/(n+m+j)$ and $y_{nmj} = n+m+j$ for all $n, m, j \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. Then, $(x_{nmj}) \in c^I_{3_0}$ and C^I_3, but $(y_{nmj}) \notin c^I_{3_0}$ and $(y_{nmj}) \notin c^I_3$. Therefore, $c^I_3(f)$ and $c^I_{3_0}(f)$ are not convergence free.

Theorem 3.6 *Let f be a modulus function. Then, $c_{3_0}(f) \subset c^I_3 \subset l^I_{3_\infty}$ and the inclusions are proper.*

Proof. The inclusion $c^I_{3_0}(f) \subset c^I_3(f)$ is followed by (1) y (2).

Let $x = x_{nmj} \in c^I_3$. Then, there exits $L \in C$ such that I_3-$\lim f(|x_{nmj} - L|) = 0$. Thus, we have that $f(|x_{nmj}|) \leq 1/2 f(|x_{nmj} - L|) + f 1/2(|L|)$. Taking the supremum over n, m and j on both sides, we obtain $x_{nmj} \in l^I_{3_\infty}$.

Now, we will show that the inclusion is proper.

1 $c_{3_0}(f) \subset c_3^I(f)$.
Let $x = (x_{nmj}) \in c_3^I(f)$, then I_3-lim $f(|x_{nmj}|) = L$ for some $L \in C$ and $L \neq 0$, which implies $x \notin c_{3_0}^I(f)$. Therefore, the inclusion is proper.

2 $c_3^I(f) \subset l_{3_\infty}(f)$.
Let $x = (x_{nmj}) \in l_{3_0}^I(f)$, then

$$I_3\text{-lim}\, f(|x_{nmj}|) < \infty,$$
$$I_3\text{-lim}\, f(|x_{nmj} - L + L|) < \infty,$$
$$I_3\text{-lim}\, f(|x_{nmj} - L|) + I_3\text{-lim}\, f(|L|) < \infty,$$
$$I_3\text{-lim}\, f(|x_{nmj}) - L|) < \infty,$$
$$I_3\text{-lim}\, f(|x_{nmj}) - L|) \neq 0.$$

Therefore, $x \notin c_3^I(f)$ and then the inclusion is proper.

Theorem 3.7 *The function $t : m_3^I(f) \to \mathbb{R}$ is the Lipschitz function, wehre $m_3^I(f) = c_3^I(f) \cap l_{3_\infty}^I(f)$, and hence uniformly continuous.*

Proof. Let $x, y \in m_3^I(f)$, where $x \neq y$. Then, the sets

$$W_x = \{(n,m,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : |x_{nmj} - t(x)| \geq \|x - y\|\} \in I,$$
$$W_y = \{(n,m,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : |y_{nmj} - t(y)| \geq \|x - y\|\} \in I.$$

Then, the sets

$$P_x = \{(n,m,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : |x_{nmj} - t(x)| < \|x - y\|\} \in M_3^I(f),$$
$$P_y = \{(n,m,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : |y_{nmj} - t(y)| < \|x - y\|\} \in M_3^I(f),$$

Therefore, we also have that $P_x \cap P_y \in m_3^I(f)$, thus $B \neq \emptyset$. Now, by taking $n, m, j \in B$,

$$|t(x) - t(y)| \leq |t(x) - x_{nmj}| + |x_{nmj} - y_{nmj}| + |y_{nmj} - t(y)| \leq 3\|x - y\|.$$

Consequently, t is a Lipschitz function.

Remark 3.3 The above result is satisfied for $m_{3_0}^I$ and it is proved similarly.

4 I_3-convergent by strong Cesáro sequence spaces

We define and introduce the following classes of sequence spaces:

Definition 4.1 $C_{s3}^I = \{x = (x_{nmj}) \in \omega : \{(n,m,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : I_3\text{-}\lim_{iop\to\infty} \frac{1}{iop} \sum_{nmj=1}^{iop} \|x_{nmj} - L\| = 0\}$ *,for some $L \in C\} \in I$.*

Definition 4.2 $C_{s3_0}^I = \{x = (x_{nmj}) \in \omega : \{(n,m,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : I_3\text{-}\lim_{iop\to\infty} \frac{1}{iop} \sum_{nmj=1}^{iop} \|x_{nmj}\| = 0\}\} \in I$.

Theorem 4.1 *The sequences spaces C_{s3}^I and $C_{s3_0}^I$ are linear.*

Proof. Let $x = (x_{nmj})$ and $y = (y_{nmj})$, where $x, y \in C^I_{s3}$. Then , we have that

$$I_3\text{-}\lim_{iop\to\infty} \frac{1}{iop} \sum_{nmj=1}^{iop} \|x_{nmj} - L\| = 0 \text{ for some } L \in C.$$

$$I_3\text{-}\lim_{iop\to\infty} \frac{1}{iop} \sum_{nmj=1}^{iop} \|y_{nmj} - L_0\| = 0 \text{ for some } L_0 \in C.$$

Now, let

$$\mathcal{V}_1 = \{(n,m,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{iop} \sum_{nmj=1}^{iop} \|x_{nmj} - L\|\}, \tag{4.1}$$

$$\mathcal{V}_2 = \{(n,m,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{iop} \sum_{nmj=1}^{iop} \|y_{nmj} - L_0\|\}. \tag{4.2}$$

Now, let γ and ϕ be two any scalars. Taking into account the properties of norm, we have that

$$\lim_{iop\to\infty} \frac{1}{iop} \sum_{nmj=1}^{iop} \|(\gamma x_{nmj} + \phi y_{nmj}) - (\gamma L + \phi L_0)\|$$
$$\leq \lim_{iop\to\infty} \frac{1}{iop} |\gamma| \|x_{nmj} - L\| + \lim_{iop\to\infty} \frac{1}{iop} |\phi| \|y_{nmj} - L_0\|.$$

Thus, from (10) and (11), we have that for every $\epsilon > 0$

$$\{(n,m,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \lim_{iop\to\infty} \frac{1}{iop} \sum_{nmj=1}^{iop} \|(\gamma x_{nmj} + \phi y_{nmj}) - (\gamma L + \phi L_0)\| \geq \epsilon\} \subset \mathcal{V}_1 \cup \mathcal{V}_2.$$

Therefore, $(\gamma x_{nmj} + \phi y_{nmj}) \in C^I_{s3}$ for all scalars γ, ϕ and $(x_{nmj}), (y_{nmj}) \in C^I_{s3}$. In consequence, this implies that C^I_{s3} is a linear space.

The proof of $C^I_{s3_0}$ is a linear space is proved in the same manner of the C^I_{s3}.

Proposition 4.1 *Let $x = (x_{nmj}) \in \omega$ be any triple sequence, then $C^I_{s3_0} \subset C^I_{s3}$.*

Proof. The proof is followed by the Definitions 4.1 and 4.2.

Theorem 4.2 *The space $C^I_{s3_0}$ is solid.*

Proof. Let $(x_{nmj}) \in C^I_{s3_0}$ be any element. Then, we have that

$$\{(n,m,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : I_3\text{-}\lim_{iop\to\infty} \frac{1}{iop} \sum_{nmj=1}^{iop} \|x_{nmj}\| = 0\}.$$

Now, let (γ_{nmj}) be a triple sequence of scalars such that $|\gamma_{nmj}| \leq 1$, for all $nmj \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$. So, we gets that

$$\frac{1}{iop}\sum_{nmj=1}^{iop}|\gamma_{nmj}| \leq 1.$$

Then , from the above inequality, we have that

$$\frac{1}{iop}\sum_{nmj=1}^{iop}\|\gamma_{nmj}x_{nmj}\| = \frac{1}{iop}\sum_{nmj=1}^{iop}|\gamma_{nmj}|\|x_{nmj}\|$$
$$= \frac{1}{iop}\sum_{nmj=1}^{iop}|\gamma_{nmj}|\frac{1}{iop}\sum_{nmj=1}^{iop}\|x_{nmj}\| \leq \frac{1}{iop}\sum_{nmj=1}^{iop}\|x_{nmj}\|$$

for all $(n, m, j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$. Therefore, the space $C^I_{s3_0}$ is solid.

Theorem 4.3 *A triple sequence $x = (x_{nmj}) \in C^I_{s3}$ is I_3-convergent if and only if for every $\epsilon > 0$, there exits $t = t(\epsilon) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ such that*

$$\{(n, m, j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{iop}\sum_{nmj}^{iop}\|x_{nmj} - x_t\| < \epsilon\} \in F(I).$$

Proof. We begin proof $\Rightarrow$:

Consider $x = (x_{nmj}) \in C^I_{s3}$. Then, I_3-$\lim_{iop\to\infty}\frac{1}{iop}\sum_{nmj}^{iop}\|x_{nmj} - L\| = 0$. Thus, for all $\epsilon > 0$ the set

$$C^{\epsilon}_{s3} = \{(n, m, j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{iop}\sum_{nmj}^{iop}\|x_{nmj} - L\| < \frac{\epsilon}{2}\} \in F(I).$$

Fix a $t(\epsilon) \in C^{\epsilon}_{s3}$, then we obtain

$$\frac{1}{iop}\sum_{nmj}^{iop}\|x_{nmj} - x_t\| \leq \frac{1}{iop}\sum_{nmj}^{ioj}\|x_{nmj} - L\| + \frac{1}{iop}\sum_{nmj}^{iop}\|x_t - L\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Which holds for all $(n, m, j) \in C^{\epsilon}_{s3}$. Therefore,

$$\{(n, m, j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{iop}\sum_{nmj}^{iop}\|x_{nmj} - x_t\| < \epsilon\} \in F(I).$$

Now, we proof $\Leftarrow$:

Consider that for all $\epsilon > 0$, the set

$$\{(n, m, j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{iop}\sum_{nmj}^{iop}\|x_{nmj} - x_t\| < \epsilon\} \in F(I).$$

Then, for every $\epsilon > 0$, we have that

$$M^{\epsilon}_{nmj} = \{(n, m, j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{iop}\sum_{nmj}^{iop}\|x_{nmj}\|$$
$$\in [\frac{1}{iop}\sum_{nmj}^{iop}\|x_t\| - \epsilon, \frac{1}{iop}\sum_{nmj}^{iop}\|x_t\| + \epsilon]\} \in F(I).$$

We will denote $W^{\epsilon}_{nmj} = [\frac{1}{iop}\sum_{nmj}^{iop}\|x_t\| - \epsilon, \frac{1}{iop}\sum_{nmj}^{iop}\|x_t\| + \epsilon]$.

For fixed $\epsilon > 0$, we have that $M^{\epsilon}_{nmj} \in F(I)$, as well as, $M^{\epsilon/2}_{nmj} \in F(I)$. Therefore, $M^{\epsilon}_{nmj} \cap M^{\epsilon/2}_{nmj} \in F(I)$. This implies that $M^{\epsilon}_{nmj} \cap M^{\epsilon/2}_{nmj} \neq \emptyset$. Thus,

$$\{(n,m,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{iop}\sum_{nmj}^{iop}\|x_{nmj}\| \in W_{nmj}\} \in F(I).$$

For this, we have $diam(W_{nmj}) \leq diam(W^{\epsilon}_{nmj})$, where the $diam(W_{nmj})$ denotes the length of the interval of W_{nmj}. In this way, by induction, we have the sequence of closed intervals $W^{\epsilon}_{nmj} = U^0_{nmj} \supseteq U^1_{nmj} \supseteq ... \supseteq U^5_{nmj} \supseteq ....$ With the property that $U^i_{nmj} \leq 1/2diam(U^{i-1}_{nmj})$, for $i = 1,2,3,...$ and

$$\{(n,m,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{iop}\sum_{nmj}^{iop}\|x_{nmj}\| \in U^i_{nmj}\} \in F(I),$$

For $i = 1,2,3,....$ Then, there exists a $L \in \cap U^i_{nmj}$ such that $L = I_3\text{-}\lim_{iop\to\infty} 1/iop\sum_{nmj}^{kj}\|x_{nmj}\|$. This proves that $x = (x_{nmj}) \in C^I_{nmj}$ is I_3-convergent.

Theorem 4.4 *Let $x = (x_{nmj})$ and $y = (y_{nmj})$ be any two double sequences such that $T(x\cdot y) = T(x)T(y)$. Then, the space C^I_{s3} and $C^I_{s3_0}$ are sequence algebra.*

Proof. Let $x = (x_{nmj})$ and $y = (y_{nmj})$ be any two elements of C^I_{s3} with $T(x\cdot y) = T(x)T(y)$. Now, for every $\epsilon > 0$ choose $\lambda > 0$ such that $\epsilon < \lambda$. Then, we have that

$$\{(n,m,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{iop}\sum_{nmj}^{iop}\|T(x_{nmj}) - L_q\| < \frac{\epsilon}{2\lambda}\} \in F(I)$$

and

$$\{(n,m,j) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \frac{1}{iop}\sum_{nmj}^{iop}\|T(y_{nm}) - L_z\| < \frac{\epsilon}{2L_q}\} \in F(I).$$

Taking into account the above and the properties of norm, we have that

$$\frac{1}{iop}\sum_{nmj}^{iop}\|T(x_{nmj}y_{nmj}) - L_qL_z\|$$
$$= \frac{1}{iop}\sum_{nmj}^{iop}\|T(x_{nmj})T(y_{nmj}) - L_qL_z\|$$
$$= \frac{1}{iop}\sum_{nmj}^{iop}\|T(x_{nmj})T(y_{nmj}) - L_qT(y_{nmj}) + L_qT(y_{nmj}) - L_qL_z\|$$
$$\leq \frac{1}{iop}\sum_{nmj}^{iop}\|T(y_{nmj})\|\frac{1}{iop}\sum_{nmj}^{iop}\|T(x_{nmj}) - L_q\| + |L_q|\frac{1}{iop}\sum_{nmj}^{iop}\|T(y_{nmj}) - L_z\|$$
$$< \frac{\epsilon^2}{2\alpha} + |L_q|\frac{\epsilon}{2|L_q|} < \epsilon.$$

Hence, the set

$$\{(n,m,j) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \frac{1}{iop} \sum_{nmj}^{iop} \|T(x_{nmj}y_{nmj}) - L_q L_z\| \geq \epsilon\} \in I.$$

In consequence, $(x_{nmj})(y_{nmj}) \in C_{s3}^I$. Therefore, C_{s3}^I is a sequence algebra.

The proof of $C_{s3_0}^I$ is a sequence algebra is proved in the same manner of the C_{s3}^I.

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