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Basicity of a perturbed system of exponents in rearrangement invariant spaces

Vafa G. Alili

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Abstract. *The paper considers a perturbed system of exponents* $E_\lambda \equiv 1 \cup \left\{e^{\pm i\lambda_n t}\right\}_{n\in N}$*, where the sequence* $\{\lambda_n\}$ *is defined by the expression* $\lambda_n = \sqrt[m]{|P_m(n)|}$*, and* $P_m(x) = x^m + a_{m-1}x^{m-1} + ... + a_0$ *is a polynomial of the* m*-th degree with real coefficients. The basicity problem of this system in rearrangement invariant space* $X(-\pi,\pi)$ *over the interval* $(-\pi,\pi)$ *is studied. A sufficient condition for the system* E_λ *to be a basis in* $X(-\pi,\pi)$ *is found depending on* m*, the coefficient* a_{m-1}*, and on the Boyd indices* α_X *and* β_X *of the space* $X(-\pi,\pi)$*. Some special cases of the space* $X(-\pi,\pi)$ *are considered.*

Keywords. system of exponents, basicity, rearrangement invariant space, Boyd indices.

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1 Introduction

Consider the following perturbed system of exponents

$$E_\lambda \equiv 1 \cup \left\{e^{\pm i\lambda_n t}\right\}_{n\in N},$$

where $\lambda_n = \sqrt[m]{|P_m(n)|}$, $P_m(x) = x^m + a_{m-1}x^{m-1} + ... + a_0-$ is a polynomial of the m-th degree with real coefficients . In connection with both theoretical and practical points of view, interest in studying the basis properties (completeness, minimality, basicity and etc.) of a system of the form E_λ in various functional spaces has always been very high and is not weakening at the present time. This direction is associated with the names of very famous mathematicians such as Paley-Wiener [28], N. Levinson [19], M.I. Kadets [21] and others. In the case when $\lambda_n = n + \alpha\, sign\, n$, $n \in Z$, the criterion for the basicity of the system E_λ in $L_p(-\pi,\pi)$, $1 < p < +\infty$, was found by A.M. Sedletsky [31] with respect to the real parameter $\alpha \in R$. For the complex case of the parameter α this result was carried over by G.G. Devdariani [15, 16]. Another method for establishing the basicity of the system E_λ, in $L_p(-\pi,\pi)$, when $\lambda_n = n + \alpha\, sign\, n$, $n \in Z$, was proposed by E.I. Moiseev [23, 24] and he also found a criterion for the basicity of a system of sines and cosines. Further development of this method belongs to B.T. Bilalov [4–9]. In the work of S.R. Sadigova and A.E. Guliyeva [29] established the basicity of the same system of exponents in the weighted Lebesgue space $L_{p;\nu}(-\pi,\pi)$, $1 < p < +\infty$, with a weight function $\nu(\cdot)$ from

V.G. Alili
Baku Slavic University, Baku, Azerbaijan
E-mail: alilivefa@mail.ru

the Mackenhoupt class $A_p(-\pi,\pi)$. The criterion for this system to be a basis in a Morrey-type space was found by B.T. Bilalov [10,11]. A similar result for the Lebesgue space with a variable summability exponent was obtained in [12], the weighted case of the space was considered in [25,26]. In [27], a sufficient condition for the basicity of the system E_λ in Lebesgue spaces with a variable summability exponent is found. Similar problems have been also studied in [2,17,18,30].

The present paper considers a perturbed system of exponents $E_\lambda \equiv 1 \cup \left\{e^{\pm i\lambda_n t}\right\}_{n\in N}$, where the sequence $\{\lambda_n\}$ is defined by the expression $\lambda_n = \sqrt[m]{|P_m(n)|}$, and $P_m(x) = x^m + a_{m-1}x^{m-1} + ... + a_0$−is a polynomial of the mth degree with real coefficients. The basicity problem of this system in rearrangement invariant space $X(-\pi,\pi)$ over the interval $(-\pi,\pi)$ is studied. A sufficient condition for the system E_λ to be a basis in $X(-\pi,\pi)$ is found depending on m, the coefficient a_{m-1}, and on the Boyd indices α_X and β_X of the space $X(-\pi,\pi)$. Some special cases of the space $X(-\pi,\pi)$ are considered.

2 Auxiliary facts

First, we give the following standard notation used in the article.

$N-$ will be a set of all positive integers; $Z_+ = \{0\} \cup N$;$Z-$ will be a set of all integers; $\mathbb{C}$ will stand for the field of complex numbers; $L[\cdot]$ −will be a linear span; $\bar{M}$−will be a closure of the set M; $KerT-$ will be a kernel of the operator T; R_T−will be a range of the operator T; $[X]-$ is an algebra of bounded operators in X; $\dim M-$ dimension of M; $\dot{+}$−is a direct sum; X^*−is a dual space to X; T^* is conjugate to T operator; $X/_M$−is a quotient space of a space X in the subspace M; B-space-is a Banach space ; $\exists!-$ there exists a unique; $p' : \frac{1}{p} + \frac{1}{p'} = 1$−is the conjugate number to p; γ is a unit circle in $\mathbb{C}$.

We will need some concepts and facts from the basis theory.

Definition 2.1 *The system $\{x_n^+; x_n^-\}_{n\in N} \subset X$ is called a double basis (or simply a basis) in the B-space X, if $\forall x \in X$; $\exists!\,\{\lambda_n^+; \lambda_n^-\}_{n\in N} \subset C$:*

$$\left\| \sum_{k=1}^{n_1} \lambda_k^+ x_k^+ + \sum_{k=1}^{n_2} \lambda_k^- x_k^- - x \right\|_X \to 0\,,\; n_1; n_2 \to \infty.$$

We also need some concepts and facts from the theory of close bases.

Definition 2.2 *The systems $\{\varphi_n\}_{n\in N}$ and $\{\psi_n\}_{n\in N} \subset X$ in B-space X are said to be p-close if*

$$\sum_n \|\varphi_n - \psi_n\|_X^p < +\infty.$$

Let us define the concept of a p-Bessel system.

Definition 2.3 *A minimal system $\{x_n\}_{n\in N} \subset X$ in a B-space X with conjugate system $\{x_n^*\}_{n\in N} \subset X^*$ is called p-Besselian if*

$$\left(\sum_n |x_n^*(f)|^p\right)^{\frac{1}{p}} \le M\,\|f\|_X\;,\;\forall f \in X.$$

The following theorem is true.

Theorem 2.1 ([14]) *Let p-Besselian system $\{x_n\}_{n\in N} \subset X$ form a basis for B-space X and the system $\{y_n\}_{n\in N} \subset X$ be a p'-close to $\{x_n\}_{n\in N}$. Then the following properties of the system $\{y_n\}_{n\in N} \subset X$ in X are equivalent: i) $\{y_n\}_{n\in N}$ is complete; ii) $\{y_n\}_{n\in N}$ is minimal; iii) $\{y_n\}_{n\in N}$ ω-linearly independent; iv) $\{y_n\}_{n\in N}$ forms a basis isomorphic to $\{x_n\}_{n\in N}$.*

Let us recall the definition of ω-linear independence.

Definition 2.4 *The system $\{x_n\}_{n\in N} \subset X$ is called ω-linearly independent in B-space X if it follows from $\sum_{n=1}^{\infty} \lambda_n x_n = 0$ that $\lambda_n = 0,\ \forall n \in N$.*

More details of these and other facts can be found, for example, from the monograph [14].

We also accept the following

Definition 2.5 *A system $\{x_n\}_{n\in N} \subset X$ in B-spaces X is called defective if, after adding to it and eliminating a finite number of elements from it, it becomes complete and minimal in X.*

We will need the following theorem from the monograph [27, p. 129].

Theorem 2.2 ([27]) *The system of exponents $\left\{e^{i\lambda_n t}\right\}$ is complete in $C\,[a,b]$ if and only if its closed linear span contains on other exponential function $e^{i\lambda t}$.*

We will need some concepts and facts from the theory of Banach function spaces (see e.g. [3,20,22]).

Let $(R;\mu)$ be a measure space. Let $\mathscr{M}^+$ be the cone of μ-measurable functions on R whose values lie in $[0,+\infty]$. The characteristic function of a μ-measurable subset E of R denote by χ_E .

Definition 2.6 *A mapping $\rho : \mathscr{M}^+ \to [0,+\infty]$ is called a Banach function norm (or simply a function norm) if, for all $f, g, f_n, n \in N$, in $\mathscr{M}^+$, for all constants $a \geq 0$ and for all μ-measurable subsets $E \subset R$, the following properties hold:*

(P1) $\rho\,(f) = 0 \Leftrightarrow f = 0$ μ-a.e.; $\rho\,(af) = a\rho\,(f)$; $\rho\,(f+g) \leq \rho\,(f) + \rho\,(g)$;

(P2) $0 \leq g \leq f$ μ-a.e. $\Rightarrow \rho\,(g) \leq \rho\,(f)$;

(P3) $0 \leq f_n \uparrow f$ μ-a.e. $\Rightarrow \rho\,(f_n) \uparrow \rho\,(f)$;

(P4) $\mu\,(E) < +\infty \Rightarrow \rho\,(\chi_E) < +\infty$;

(P5) $\mu\,(E) < +\infty \Rightarrow \int_E f d\mu \leq C_E \rho\,(f)$, for some constant $C_E : 0 < C_E < +\infty$, depending on E and ρ, but independent of f .

Let $\mathscr{M}$ denote the collection of all extended scalar-valued (real or complex) μ-measurable functions and $\mathscr{M}_0 \subset \mathscr{M}$ the subclass of functions that are finite μ-a.e. .

Definition 2.7 *Let ρ be a function norm. The collection $X = X\,(\rho)$ of all functions f in $\mathscr{M}$ for which $\rho\,(|f|) < +\infty$, is called a Banach function space. For each $f \in X$, define $\|f\|_X = \rho\,(|f|)$.*

It is valid the following

Theorem 2.3 *Let ρ be a function norm and let $X = X\,(\rho)$ and $\|\cdot\|_X$ be as above. Then under the natural vector space operations, $(X; \|\cdot\|_X)$ is a normed linear space for which the inclusions*

$$\mathscr{M}_s \subset X \subset \mathscr{M}_0,$$

hold, where $\mathscr{M}_s$ is the set of μ-simple functions. In particular, if $f_n \to f$ in X, then $f_n \to f$ in measure on sets of finite measure, and hence some subsequence converges point wise μ-a.e. to f.

A space X equipped with the norm $\|f\|_X = \rho(|f|)$ is called a Banach function space. Let

$$\rho'(g) = \sup\left\{\int_\gamma f(\tau)\, g(\tau)\, |dt| : f \in \mathscr{M}^+; \rho(f) \le 1\right\}, \forall g \in \mathscr{M}^+.$$

A space

$$X' = \left\{g \in \mathscr{M} : \rho'(|g|) < +\infty\right\},$$

is called an associate space (Kothe dual) of X.

The functions $f; g \in \mathscr{M}_0$ are called equimeasurable if

$$|\{\tau \in R : |f(\tau)| > \lambda\}| = |\{\tau \in R : |g(\tau)| > \lambda\}|, \forall \lambda \ge 0.$$

Banach function norm $\rho : \mathscr{M}^+ \to [0, \infty]$ is called rearrangement invariant if for arbitrary equimeasurable functions $f; g \in \mathscr{M}_0^+$ the relation $\rho(f) = \rho(g)$ holds. In this case, Banach function space X with the norm $\|\cdot\|_X = \rho(|\cdot|)$ is said to be rearrangement invariant function space (r.i.s. for short). Classical Lebesgue, Orlicz, Lorentz, Lorentcz–Orlicz spaces are r.i.s.

Let α_X and β_X be upper and lower Boyd indices for the space X (regarding the Boyd indices see e.g. [3,20,22]). To obtain our main results, we will significantly use the following result of [3] (see also [13]).

Theorem 2.4 *For every p and q such that*

$$1 \le q < \frac{1}{\beta_X} \le \frac{1}{\alpha_X} < p \le \infty,$$

we have

$$L_p \subset X \subset L_q,$$

with the inclusion maps being continuous.

We will use some results related to Fourier series in r.i.s. Let's state some relevant concepts and notations.

Definition 2.8 *Let X be a Banach function space. The closure in $X$of the set of simple functions $\mathscr{M}_s$ is denoted by X_b.*

Recall the definition of resonant space

Definition 2.9 *Suppose $f(\cdot)$ belongs to $\mathscr{M}_0$. The decreasing rearrangement of $f(\cdot)$ is the function f^* defined on $[0, \infty)$ by*

$$f^*(t) = \inf\{\lambda : \mu_f(\lambda) \le t\}, \ t \ge 0,$$

where $\mu_f(\lambda) = \mu\{t : |f(t)| > \lambda\}$, $\lambda \ge 0$ is a distribution function of $f(\cdot)$.

It is valid the following well known

Theorem 2.5 *(Hardy, Littlewood). If $f(\cdot)$ and $g(\cdot)$ belong to $\mathscr{M}_0$, then*

$$\int_R |f\, g|\, d\mu \le \int_0^\infty f^*(s)\, g^*(s)\, ds. \tag{2.1}$$

An immediate consequence of the Hardy-Littlewood inequality (2.1) *is that*

$$\int_{\mathscr{M}} |f\, \tilde{g}|\, d\mu \le \int_0^\infty f^*(t)\, g^*(t)\, dt, \tag{2.2}$$

for every function $\tilde{g}$ on $\mathscr{M}$ equimeasurable with g.

Definition 2.10 *If the supremum on $\tilde{g}$ of the integrals on the left of* (2.2) *coincide with the value on the right, such measure spaces is called resonant. If the supremum is in fact attained, then the measure space will be called strongly resonant.*

In the sequel we will consider the case $R = \gamma$ and μ will be Lebesgue measure (linear) on γ. Moreover, we will identify the circle γ and the segment $[-\pi, \pi)$ by $e^{it} : [-\pi, \pi) \leftrightarrow \gamma$, and we will identify the function $f : \gamma \to \mathbb{C}$ with $f : [-\pi, \pi) \to \mathbb{C}$ by $f(t) = f\left(e^{it}\right)$.

We denote by T_s the translation operator $(T_s f)(t) = f\left(e^{i(s+t)}\right)$, $-\pi < s; t \leq \pi$ and by $\omega_X(f, \cdot)$the X -modulus of continuity of f:

$$\omega_X(f; \delta) = \sup_{|s| \leq \delta} \|T_s f - f\|_X , \ 0 \leq \delta \leq \pi.$$

Definition 2.11 *Let X be a rearrangement-invariant Banach space (r.i.s.) over a resonant space $(R; \mu)$. For each finite value of t belonging to the range of μ, let E be a subset of R with $\mu(E) = t$ and let*

$$\varphi_X(t) = \|\chi_E\|_X \cdot$$

The function φ_X is called the fundamental function of X.

If f belongs to $L_1(\gamma)$, then for each integer n the n-th Fourier coefficient of f is defined by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(e^{i\theta}\right) e^{-in\theta} d\theta , \ n \in Z.$$

So called the “multiplier” operator m is defined initially on trigonometric polynomials $P\left(e^{i\theta}\right) = \sum_{n=-r}^{r} a_n e^{in\theta}$ by $mP\left(e^{i\theta}\right) = \sum_{n=-r}^{r} -isign\, n\, a_n e^{in\theta}$.

So, it is evidently that

$$\left(\overset{\wedge}{mP}\right)(n) = \begin{cases} -isign\, n\, a_n, \ \forall n = \overline{-r, r}, \\ 0, \quad n \neq \overline{-r, r} \ , \end{cases}$$

for arbitrary trigonometric polynomial $P\left(e^{i\theta}\right) = \sum_{n=-r}^{r} a_n e^{in\theta}$.

Let $S_n' s$ be partial sums of the Fourier series of the function f :

$$S_n(f) = \sum_{|k| \leq n} \hat{f}(k) e^{ikt}.$$

In the sequel we also need the following

In the sequel we also need the following

Theorem 2.6 *Suppose X is a r.i.s. on γ whose fundamental function satisfies $\varphi_X(+0) = 0$. Then the following conditions are equivalent:*

i) Fourier series converge in norm in X_b;
ii) the partial-sum operators S_n are uniformly bounded on X_b;
iii) the multiplier operator m is bounded on X_b;
iv) the conjugate-function operator is bounded on X_b;
v) the Calderon operator

$$S\, f^*(t) = \int_0^1 f^*(s) \min\left(1, \frac{s}{t}\right) \frac{ds}{s},$$

is bounded on $(X_b)^-$ $-$the Luxemburg representation of X_b on the interval $[0, 1]$.

The conjugate-function operator $\tilde{f}$ is defined by

$$\tilde{f}\left(e^{i\theta}\right)=\frac{1}{2\pi}\lim_{\varepsilon\to+0}\int_{\varepsilon<|s|\le\pi} f\left(e^{i(\theta-s)}\right)\cot\frac{s}{2}ds, \forall\theta: -\pi<\theta\le\pi.$$

If any one of these conditions holds, then $mf=\tilde{f}$ a.e. for $\forall f\in X_b$.

Corollary 2.1 *Let X be a separable r.i.s. on $[-\pi,\pi]$. Fourier series converge in norm in X if and only if the Boyd indices of X satisfy $0<\alpha_X;\beta_X<1$.*

We will need also the following lemma from the work [3].

Lemma 2.1 ([3]) *Let $X=X(\rho)$ be a Banach function space and suppose $f_n\in X$, $n\in N$.*

i) If $0\le f_n\uparrow f$ μ-a.e., then either $f\notin X$ and $\|f_n\|_X\uparrow+\infty$, or $f\in X$ and $\|f_n\|_X\uparrow\|f\|_X$.

ii) (Fatous lemma) If $f_n\to f$ μ-a.e., and if $\lim_{n\to\infty}\inf\|f_n\|_X<+\infty$, then $f\in X$ and $\|f\|_X\le\lim_{n\to\infty}\inf\|f_n\|_X$.

More details on these results can be found, e.g. in the monographs [3,20,22].

To obtain the main results, we need some results from the work [13].

Lemma 2.2 *Let $X(-\pi,\pi)$ be a r.i.s. with Boyd indices $\alpha_X;\beta_X\in(0,1)$. Then $X_b(-\pi,\pi)=X_s(-\pi,\pi)=\overline{C^\infty[-\pi,\pi]}$ (the closure is taken in the norm $X(-\pi,\pi)$).*

So, consider the following system of exponents

$$E^\alpha\equiv\left\{e^{i(n-\alpha\, sign\, n)t}\right\}_{n\in Z},$$

where $\alpha\in C$ is some complex parameter. In [13] the following theorem was proved.

Theorem 2.7 ([13]) *Let $X(-\pi,\pi)$ be a r.i.s. with Boyd indices $\alpha_X;\beta_X\in(0,1)$. If the system E^α forms a basis for $X_b(-\pi,\pi)$, then it is isomorphic in $X_b(-\pi,\pi)$ to the system $E_\lambda\equiv 1\cup\left\{e^{\pm i\lambda_n t}\right\}_{n\in N}$, and the isomorphism is given by the operator*

$$T\left[e^{int}\right]=e^{-i\alpha t}e^{int},\ \forall n\in Z_+;$$
$$T\left[e^{-int}\right]=e^{i\alpha t}e^{-int},\ \forall n\in N.$$

To present further results, we also need the following characterization of the space $X(-\pi,\pi)$:

$$\gamma_X=\inf\left\{\alpha\in R:\ |t|^\alpha\in X(-\pi,\pi)\right\}.$$

The set of all weight functions on $(-\pi,\pi)$ will be denoted by $W(-\pi,\pi)$, i.e. $w\in W(-\pi,\pi)$, means that $w(\cdot):\ [-\pi,\pi]\to R_+=[0,+\infty]-$ is a measurable (according to Lebesgue) function and $\left|w^{-1}\{0,+\infty\}\right|=0$. Denote by $\mathscr{K}$ the following singular Cauchy operator

$$\mathscr{K}f=\frac{1}{2\pi i}\int_\gamma\frac{f(\xi)\,d\xi}{\xi-\tau},\ \tau\in\gamma.$$

Assume

$$A_X\equiv\left\{w\in W(-\pi,\pi):\ \mathscr{K}\in\left[X_w(-\pi,\pi)\right]\right\},$$

where $X_w(-\pi,\pi)$ is a weighted space

$$X_w(-\pi,\pi)\equiv\left\{f:\ fw\in X(-\pi,\pi)\right\},$$

with the norm

$$\|f\|_{X_w}=\|f\,w\|_X.$$

Thus, in [13] the following theorem was proved.

Theorem 2.8 **([13])** *Let* $X(-\pi,\pi)$ *be a r.i.s. with Boyd indices* $\alpha_X;\beta_X \in (0,1)$ *and let the following conditions be satisfied*

$$\gamma_{X'} < 2Re\alpha < \alpha_X; \; w_0 \in A_X,$$

where $w_0(t) = \left|t^2 - \pi^2\right|^{2Re\alpha}$. *Then the system* E^α *forms a basis for* $X_b(-\pi,\pi)$.

In the case where the Boyd indices coincide, i.e. $\alpha_X = \beta_X$, then, as established in [13] $\gamma_X = \alpha_X \Rightarrow \gamma_{X'} = \alpha_{X'}$, holds. In this case, the result of this theorem is strengthened as follows.

Theorem 2.9 **([13])** *Let* $X(-\pi,\pi)$ *be a r.i.s. with Boyd indices* $\alpha_X = \beta_X \in (0,1)$ *and* $2Re\alpha+\alpha_X \notin Z$. *Then the system* E^α *forms a basis for* $X_b(-\pi,\pi)$ *if and only if* $[2Re\alpha + \alpha_X] = 0$. *For* $[2Re\alpha + \alpha_X] < 0$ *this system is not complete, but is minimal in* $X_b(-\pi,\pi)$*; for* $[2Re\alpha + \alpha_X] > 0$ *it is complete, but is not minimal in* X_b*, moreover, its defect is equal to* $d(E^\alpha) = |\,[2Re\alpha + \alpha_X]\,|$.

Before turning to the main results, we prove the following

Theorem 2.10 *Let* $X(-\pi,\pi)$ *be a r.i.s. with Boyd indices* $\alpha_X;\beta_X \in (0,1)$. *The system* E_λ *is complete in* $X_b(-\pi,\pi)$ *if and only if* $\overline{L[E_\lambda]}$ *contains an exponent* $e^{i\lambda t}$ *different from* E_λ.

Proof. The necessity is obvious. Let us prove the sufficiency. Let

$$\left(e^{i\lambda t}\right) \notin E_\lambda \,\&\, \left(e^{i\lambda t}\right) \in \overline{L[E_\lambda]}.$$

From the axioms of the norm for a Banach function space it immediately follows

$$\|f\,g\|_X \le \|f\|_{L_\infty(-\pi,\pi)} \|g\|_X .$$

From this inequality and from $\left\{e^{i\lambda t}\right\} \in \overline{L[E_\lambda]}$ it follows that $1 \in \overline{L\left[\left\{e^{i(\lambda_n-\lambda)t}\right\}_{n\in Z}\right]}$. Further, taking into account that $|f(t)| \le |g(t)|$, a.e. $t \in (-\pi,\pi) \Rightarrow \|f\|_X \le \|g\|$, we have

$$\left\|\int_0^x \left(1 - \sum_n a_n e^{i(\lambda_n-\lambda)t}\right) dt\right\|_X = \left\|x - \sum_n b_n e^{i(\lambda_n-\lambda)x} + \sum_n b_n\right\|_X ,$$

where $b_n = \frac{a_n}{i(\lambda_n-\lambda)}$. It is clear that $\sum_n b_n \in \overline{L\left[\left\{e^{i(\lambda_n-\lambda)t}\right\}_{n\in Z}\right]}$. We also have

$$\left\|\int_0^x \left(1 - \sum_n a_n e^{i(\lambda_n-\lambda)t}\right) dt\right\|_X \le c\left\|\int_{-\pi}^{\pi} \left|e^{i\lambda t} - \sum_n a_n e^{i\,\lambda_n t}\right| dt\right\|_X \le$$

$$\le c\left\|e^{i\lambda t} - \sum_n a_n e^{i\,\lambda_n t}\right\|_{L_1(-\pi,\pi)} \le c\left\|e^{i\lambda t} - \sum_n a_n e^{i\,\lambda_n t}\right\|_X .$$

It follows directly from these relations that $x \in \overline{L\left[\left\{e^{i(\lambda_n-\lambda)t}\right\}_{n\in Z}\right]}$. Continuing this process, as a result we get that

$$L\left[\{x^n\}_{n\in Z_+}\right] \subset \overline{L\left[\left\{e^{i(\lambda_n-\lambda)t}\right\}_{n\in Z}\right]}.$$

Since the polynomials are dense in $C[-\pi,\pi]$ it follows that

$$C[-\pi,\pi] \subset \overline{L\left[\left\{e^{i(\lambda_n-\lambda)t}\right\}_{n\in Z}\right]}.$$

Then, paying attention to Lemma 2.2, hence we obtain that

$$\overline{L\left[\left\{e^{i(\lambda_n-\lambda)t}\right\}_{n\in Z}\right]} \equiv X_b[-\pi,\pi].$$

Theorem is proved.

3 Main results

So, consider the system of exponents

$$E_\lambda \equiv 1 \cup \left\{e^{\pm i\lambda_n t}\right\}_{n\in N},$$

where $\lambda_n = \sqrt[m]{|P_m(n)|}$, $P_m(x) = x^m + a_{m-1}x^{m-1} + ... + a_0$ is a m-th degree polynomial. We will consider the case when the coefficients a_k, $k = \overline{0, m-1}$, are real. As shown in [27], the following asymptotic formula holds

$$\lambda_n = n + \frac{a_{m-1}}{m} + \underline{\underline{O}}\left(\frac{1}{n}\right), \; n \to \infty.$$

Assume $\mu_n = n + \frac{a_{m-1}}{m}$, $n \in N$. We have

$$\left|e^{i\lambda_n t} - e^{i\mu_n t}\right| \leq 2\pi|\lambda_n - \mu_n| = \underline{\underline{O}}\left(\frac{1}{n}\right), \; n \to \infty.$$

Hence it immediately follows

Lemma 3.1 *The system E_λ is r-close in $X(-\pi,\pi)$ to the system of exponent*

$$E_\mu = 1 \cup \left\{e^{\pm i\mu_n t}\right\}_{n\in N},$$

for $\forall r > 1$.

The validity of the lemma follows from the following obvious relation

$$\sum_n \left\|e^{i\lambda_n t} - e^{i\mu_n t}\right\|_X^r \leq c\sum_n \left\|\underline{\underline{O}}\left(\frac{1}{n}\right)\right\|_X^r \leq c\sum_n \frac{1}{n^r} < +\infty.$$

Denote $\alpha = -\frac{a_{m-1}}{m}$. Consequently, $\mu_n = n - \alpha$. Let us assume that all the conditions of Theorem 2.8 are satisfied, i.e.

$$\alpha_X; \beta_X \in (0,1); \gamma_{X'} < 2Re\alpha < \alpha_X; \; w_0 \in A_X.$$

Then by the results of this theorem the system E^α forms a basis for $X_b(-\pi,\pi)$. By Theorem 2.7, it is isomorphic to the classical system $E \equiv \left\{e^{int}\right\}_{n\in Z}$ in $X_b(-\pi,\pi)$. Consider the following functionals

$$e_n^*(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-int} dt, \; n \in Z.$$

We have

$$|e_n^*(f)| \le c\|f\|_{L_1(-\pi,\pi)} \le c\|f\|_X \ , \ \forall n \in Z,$$

where c is a constant, independent of f and n. Hence it follows that $\{e_n^*\} \subset X^*(-\pi,\pi)$, and moreover $\gamma = \sup_n \|e_n^*\| < +\infty$. Since the system E forms a basis for $X_b(-\pi,\pi)$, then it is clear that

$$1 \le \left\|e^{int}\right\|_X \|e_n^*\| \le const < +\infty \, , \ \forall n \in Z,$$

holds. It is quite obvious that $\left\|e^{int}\right\|_X \equiv \|1\|_X \equiv const > 0 \, , \ \forall n \in Z$. Then from the previous relation we get

$$0 \le \delta \le \|e_n^*\| \le \gamma < +\infty \, , \ \forall n \in Z.$$

Let us show that for some $r \in (1,2]$, the system E_μ is r'-Besselian in $X(-\pi,\pi)$ ($\frac{1}{r} + \frac{1}{r'} = 1$). Let us first establish this fact for the system E. As $\alpha_X; \beta_X \in (0,1)$, then it is clear that $\exists r; p \in (1,+\infty)$, for which

$$1 < r < \frac{1}{\beta_X} \le \frac{1}{\alpha_X} < p < +\infty.$$

Then by Theorem 2.4 we have continuous embeddings $L_p(-\pi,\pi) \subset X(-\pi,\pi) \subset L_r(-\pi,\pi)$. Therefore, the inequality

$$\|f\|_{L_r(-\pi,\pi)} \le c\|f\|_X \, , \forall f \in X(-\pi,\pi) \, ,$$

holds. It is quite obvious that we can assume that r belongs to the interval $(1,2]$, i.e.$1 < r \le 2$. Then, according to the classical Hausdorff-Young theorem, we have

$$\left(\sum_n |e_n^*(f)|^{r'}\right)^{\frac{1}{r'}} \le c_r \|f\|_{L_r(-\pi,\pi)} \le c\|f\|_X \ , \ \forall f \in X(-\pi,\pi) \, .$$

This implies that the system of exponent E is r'-Besselian in $X(-\pi,\pi)$.

Now let us show that the system E_μ is also r'-Besselian in $X(-\pi,\pi)$. So, the system E_μ forms a basis for $X(-\pi,\pi)$ and therefore it is isomorphic to the system E in $X(-\pi,\pi)$. Denote by $\left\{e_{\alpha;n}^*\right\}_{n\in Z}$ the system biorthogonal to E_μ. Let $T \in [X(-\pi,\pi)]$ be an automorphism such that $T[E_\mu] = E$, i.e. T transforms the system E_μ to the system E. We have

$$\delta_{nk} = e_n^*(e_k) = e_n^*(T[e_{\alpha;k}]) = (T^* e_n^*)(e_{\alpha;k}) \ , \ \forall n,k \in Z,$$

where

$$e_{\alpha;0} \equiv 1; \ e_{\alpha;n} = e^{i(n-\alpha\, sign\, n)t}, \ \forall n \ne 0.$$

Hence, from the uniqueness of the biorthogonal system to the basis, we obtain

$$T^* e_n^* = e_{\alpha;n}^* \, , \ \forall n \in Z.$$

Consequently

$$\left(\sum_n \left|e_{\alpha;n}^*(f)\right|^{r'}\right)^{\frac{1}{r'}} = \left(\sum_n |T^* e_n^*(f)|^{r'}\right)^{\frac{1}{r'}} = \left(\sum_n |e_n^*(Tf)|^{r'}\right)^{\frac{1}{r'}} \le c\|Tf\|_X \le$$

$$\le c\|T\|_{[X]}\|f\|_X \ , \ \forall f \in X(-\pi,\pi) \, .$$

This implies that the system E_μ is also r'-Besselian in $X(-\pi,\pi)$. As a result, we obtain that all the conditions of Theorem 2.1 [14] are satisfied with respect to the systems E_λ and E_μ, and hence the following theorem is true

Theorem 3.1 *Let $X(-\pi,\pi)$ be a r.i.s. with Boyd indices $\alpha_X;\beta_X \in (0,1)$. Let the conditions*

$$\gamma_{X'} < -2\alpha < \alpha_X;\ w_0 \in A_X,$$

be fulfilled, where $w_0(t) = \left|t^2-\pi^2\right|^{2\alpha}$, $\alpha = -\frac{a_{m-1}}{m}$.

Then, with respect to the system E_λ the following properties are equivalent in $X(-\pi,\pi)$:
i) E_λ is complete;
ii) E_λ is minimal;
iii) E_λ is ω-linearly independent;
iv) E_λ forms a basis isomorphic to E_μ.

In what follows, we will assume that the sequence $\{\lambda_n\}$ satisfies the condition

$$\lambda_n \neq 0\,,\ \forall n \neq 0\,\&\,\lambda_i \neq \lambda_j,\ i \neq j. \tag{3.1}$$

Let us assume that all the conditions of Theorem 3.1 are satisfied. Consequently, with respect to the systems E_λ and E_μ all the conditions of Theorem 2.1 [14] are true, and as a result, the system E_λ forms a defect basis for $X(-\pi,\pi)$, and there exists a Fredholm operator $F \in [X(-\pi,\pi)]$, which transfers the system E_λ to the system E_μ, i.e. $F[E_\lambda] = E_\mu$. It is easy to see that for sufficiently large $n_0 \in N$, the system $E_{\lambda;n_0} \equiv \left\{e^{\pm i\lambda_n t}\right\}_{n>n_0}$ is minimal in $X(-\pi,\pi)$, and hence its defect is $2n_0+1$. Then it follows directly from Theorem 2.10 that $e^{i\lambda_k t} \notin \overline{L[E_{\lambda;n_0}]}\,,\ \forall k:\ |k| \leq n_0$. Therefore, the system $\left\{e^{i\lambda_k t}\right\} \cup E_{\lambda;n_0}$ is minimal in $X(-\pi,\pi)$. Continuing this process, we finally obtain that if condition (3.1) is satisfied, then the system E_λ is minimal in $X(-\pi,\pi)$. Applying Theorem 3.1 to E_λ we see that it forms a basis in $X(-\pi,\pi)$ isomorphic to E_μ. So, the following main theorem is true.

Theorem 3.2 *Let $X(-\pi,\pi)$ be a r.i.s. with Boyd indices $\alpha_X;\beta_X \in (0,1)$. Let the following conditions be satisfied for the sequence $\{\lambda_n\}$*

$$\gamma_{X'} < -2\alpha < \alpha_X;\ w_0 \in A_X;$$

$$\lambda_n \neq 0\,,\ \forall n \neq 0\,\&\,\lambda_i \neq \lambda_j,\ i \neq j.$$

Then the system E_λ forms a basis for $X(-\pi,\pi)$, isomorphic to the classical system of exponents E.

Let us consider the case when the Boyd indices coincide, i.e. . $\alpha_X = \beta_X$. Whereas, as follows from the results of [1], $\gamma_X = -\alpha_X \Rightarrow \gamma_{X'} = -\alpha_{X'}$ holds. Taking into account that $\alpha_X + \alpha_{X'} = 1$ (since $\alpha_X;\beta_X \in (0,1)$), then with respect to the parameter α we obtain the condition

$$-\alpha_{X'} < -2\alpha < 1-\alpha_{X'};\ w_0 \in A_X.$$

And now as the space $X(-\pi,\pi)$ we take $X(-\pi,\pi) \equiv L_p(-\pi,\pi)$, $1 < p < +\infty$. In this case, as is known, the Boyd indices are equal.

$$\alpha_X = \beta_X = \frac{1}{p} \Rightarrow \alpha_{X'} = \frac{1}{p'} \Rightarrow -\frac{1}{p'} < -2\alpha < 1-\frac{1}{p'} \Leftrightarrow -\frac{1}{p} < 2\alpha < 1-\frac{1}{p}.$$

It is well known that this inequality implies that the weight w_0 satisfies the Mackenhoupt condition $A_p(\gamma)$ and, as a result, $S \in [L_{p;w_0}(\gamma)]$. As a result, from Theorem 3.2 we obtain the following

Corollary 3.1 *Let the sequence* $\{\lambda_n\}$ *satisfy the condition*

$$-\frac{1}{p} < 2\alpha < \frac{1}{p'},$$

$$\lambda_n \neq 0\,,\ \forall n \neq 0\,\&\,\lambda_i \neq \lambda_j,\, i \neq j.$$

Then the system E_λ *forms a basis for* $L_p(-\pi,\pi)$, $1 < p < +\infty$, *isomorphic to the classical system of exponents* E.

It should be noted that a result similar to this corollary for the Lebesgue space $L_{p(\cdot)}(-\pi,\pi)$ with variable summability exponent was obtained in [27].

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