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Inverse boundary problem for a fourth order Boussinesq equation with integral conditions

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Abstract. *The work is devoted to the study of the solvability of the inverse boundary value problem with an unknown time depended coefficient for a fourth order Boussinesq equation . The goal of paper consists of determination of the unknown coefficient and the solution of the considered problem. We introduce the definition of the classical solution, and then the considered problem is reduced to an auxiliary equivalent problem. Further, the existence and uniqueness of the solution of the equivalent problem are proved using a contraction mapping principle. Finally, using equivalency, the unique existence of a classical solution is proved.*

Keywords. Inverse problems, Boussinesq equation, nonlocal integral condition, classical solution, existence, uniqueness.

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1 Introduction

There are many cases where the needs of the practice leads to problems in determining the coefficients or the right-hand side of the differential equations according to some known data of its solutions. Such problems are called inverse value problems of mathematical physics. Inverse value problems arise in various areas of human activity such as seismology, mineral exploration, biology, medicine, quality control of industrial products, etc., that states them in a number of actual problems of modern mathematics.The inverse problems are favorably developing section of up-to-date mathematics. Recently, the inverse problems are widely applied in various fields of science. Different inverse problems for various types of partial differential equations have been studied in many papers. First of all we note the papers of A.N.Tikhonov [9], M.M.Lavrentyev [4,5], A.M.Denisov [2], M.I.Ivanchov [3] and their followers.

Recently, much attention has been paid to the study of various nonlinear evolution equations describing wave processes in media with dispersion. One of them is the Boussinesq

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equation, derived by the author in [1] and describing the propagation of long waves in shallow water. This equation is interesting from both physical and mathematical points of view.

For the Boussinesq equation of the fourth order, inverse problems were considered in [7, 8].

In this paper, we proved the existence and uniqueness of the solution of the inverse boundary value problem for the fourth order Boussinesq equation with integral conditions.

2 Problem statement and its reduction to equivalent problem

Let $T > 0$ be some fixed number and denote by $D_T := \{(x,t):\ 0 < x < 1,\ 0 < t < T\}$. Consider the one-dimensional inverse problem of identifying an unknown pair of functions $\{u(x,t)\,,\ a(t)\}$ for the following fourth order Boussinesq equation [1]

$$u_{tt}(x,t) - 2\alpha u_{txx}(x,t) + \beta u_{xxxx}(x,t) = a(t)u(x,t) + f(x,t) \tag{2.1}$$

with the nonlocal initial conditions

$$u(x,0) = \varphi(x) + \int_0^T p_1(t)u(x,t)dt, \quad u_t(x,0) = \psi(x)$$

$$+ \int_0^T p_2(t)u(x,t)dt \ \ (0 \le x \le 1) \tag{2.2}$$

Neumann boundary conditions

$$u_x(0,t) = u_x(1,t) = u_{xxx}(0,t) = 0 \ \ (0 \le t \le T), \tag{2.3}$$

nonlocal integral condition

$$\int_0^1 u(x,t)dx = 0(0 \le t \le T) \tag{2.4}$$

and overdetermination condition

$$u(0,t) = h(t) \ \ (0 \le t \le T)\,, \tag{2.5}$$

where $\alpha > 0, \beta > \alpha^2$ given numbers ,$f(x,t),\ \varphi(x),\ \psi(x),\ p_i(t)\ (i = 1,2)$, and $h(t)$ are given sufficiently smooth functions of $x \in [0,1]$ and $t \in [0,T]$.

Definition 2.1 *The pair* $\{u(x,t)\,,\ a(t)\}$ *is said to be a classical solution to the problem* (2.1)-(2.5)*, if the functions* $u(x,t) \in \tilde{C}^{4,2}(\bar{D}_T)$ *and* $a(t) \in C[0,T]$ *satisfies an Equation* (2.1) *in the region* D_T*, the condition* (2.2) *on* $[0,1]$*, and the statements* (2.3)-(2.5) *on the interval* $[0,T]$*, where*

$$\tilde{C}^{(4,2)}(D_T) = \left\{u(x,t):\ u(x,t) \in C^2(D_T), u_{txx}(x,t), u_{xxx}(x,t), u_{xxxx}(x,t) \in C(D_T)\right\}.$$

In order to investigate the problem (2.1) - (2.5), first we consider the following auxiliary problem

$$y''(t) = a(t)y(t), \ \ t \in [0,T], \tag{2.6}$$

$$y(0) = \int_0^T p_1(t)y(t)dt, \ \ y'(0) = \int_0^T p_2(t)y(t)dt, \tag{2.7}$$

where $p_1(t),\ p_2(t),\ a(t) \in C[0,T]$ are given functions, and $y = y(t)$ is desired function. Moreover, by the solution of the problem (2.6), (2.7), we mean a function $y(t)$ belonging to $C^2[0,T]$ and satisfying the conditions (2.6), (2.7) in the usual sense.

Lemma 2.1 *[7] Assume that* $p_1(t),\ p_2(t) \in C[0,T],\ \ a(t) \in C[0,T],\ \|a(t)\|_{C[0,T]} \le R = const$, *and the condition*

$$\left(T\,\|p_2(t)\|_{C[0,T]} + \|p_1(t)\|_{C[0,T]} + \frac{T}{2}R\right)T < 1$$

hold. Then the problem (2.6), (2.7) *has a unique trivial solution.*

Now along with the inverse boundary-value problem (2.1) - (2.5), we consider the following auxiliary inverse boundary-value problem: It is required to determine a pair $\{u(x,t), a(t)\}$ of functions $u(x,t) \in \tilde{C}^{4,2}(\bar{D}_T)$ and $a(t) \in C[0,T]$, from relations (2.1)-(2.3), and

$$u_{xxx}(1,t) = 0\,(0 \le t \le T), \tag{2.8}$$

$$\begin{gathered} a\,(t)\,h_1\,(t) + b\,(t)\,g\,(0,t) + f\,(0,t) \\ = h_1''\,(t) - 2\alpha u_{txx}\,(0,t) + \beta u_{xxxx}\,(0,t)\quad (\,0 \le t \le T\,)\,. \end{gathered} \tag{2.9}$$

Using Lemma 1, similarly to [7]. we prove the following

Theorem 2.1 *Suppose that* $f(x,t)\ \in C(\bar{D}_T),\ \varphi(x), \psi(x) \in C[0,1],\ p_i(t) \in C[0,T]\ (i = 1,2),\ h(t) \in C^2[0,T],\ h(t) \ne 0,\ \ \int_0^1 f(x,t)dx = 0\quad (0 \le t \le T)$ *and the compatibility conditions*

$$\int_0^1 \varphi(x)dx = 0, \quad \int_0^1 \psi(x)dx = 0\,, \tag{2.10}$$

$$\varphi(0) + \int_0^T p_1(t)h(t)dt = h(0),\ \ \psi(0) + \int_0^T p_2(t)h(t)dt = h'(0)\ , \tag{2.11}$$

holds. Then the following assertions are valid:

1 *each classical solution* $\{u(x,t), a(t)\}$ *of the problem* (2.1)-(2.5) *is a solution of problem* (2.1)-(2.3), (2.8), (2.9)*, as well;*
2 *each solution* $\{u(x,t), a(t)\}$ *of the problem* (2.1)-(2.3), (2.8), (2.9)*, if*

$$\left(T\,\|p_2(t)\|_{C[0,T]} + \|p_1(t)\|_{C[0,T]} + \frac{T}{2}\,\|a(t)\|_{C[0,T]}\right)T < 1 \tag{2.12}$$

is a classical solution of problem (2.1)-(2.5).

3 Existence and uniqueness of the classical solution

We seek the first component $u(x,t)$ of classical solution $\{u(x,t)\,,\,a(t)\}$ of the problem (2.1)-(2.3), (2.8), (2.9) in the form

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t)\cos\lambda_k x \quad (\lambda_k = k\pi)\,, \tag{3.1}$$

where

$$u_k(t) = l_k \int_0^1 u(x,t)\cos\lambda_k x dx\ (k = 0,1,...),\ \ l_k = \begin{cases} 1, & k = 0, \\ 2, & k = 1,2,\dots\ . \end{cases}$$

Then applying the formal scheme of the Fourier method, from (2.1) and (2.2) we have

$$u_k''(t) + 2\alpha\lambda_k^2 u_k'(t) + \beta\lambda_k^4 u_k(t) = F_k(t;u,a)\ \ (0 \le t \le T;\ k = 0,1,...), \tag{3.2}$$

$$u_k(0)=\varphi_k+\int_0^T p_1(t)u_k(t)dt,\quad u'_k(0)=\psi_k+\int_0^T p_2(t)u_k(t)dt\quad (k=0,1,...),\ (3.3)$$

where

$$F_k(t;u)=f_k(t)+a(t)u_k(t), f_k(t)=l_k\int_0^1 f(x,t)\cos\lambda_k x dx\ (k=0,1,...)\,,$$

$$\varphi_k=l_k\int_0^1\varphi(x)\cos\lambda_k x dx, \psi_k=l_k\int_0^1\psi(x)\cos\ \lambda_k x dx\quad (k=0,1,...),$$

Solving the problem (3.2), (3.3) gives

$$u_0(t)=\varphi_0+\int_0^T p_1(t)u_0(t)dt+t\ \left(\psi_0+\int_0^T p_2(t)u_0(t)dt\right)$$
$$+\int_0^t (t-\tau)F_0(\tau;u,a)d\tau\ \ (0\le t\le T)\,, \tag{3.4}$$

$$u_k(t)=e^{\alpha_k t}\left[\left(\cos\beta_k t-\frac{\alpha_k}{\beta_k}\sin\beta_k t\right)\left(\varphi_k+\int_0^T p_1(t)u_k(t)dt\right)\right.$$
$$\left.+\frac{\sin\beta_k t}{\beta_k}\left(\psi_k+\int_0^T p_2(t)u_k(t)dt\right)\right]$$
$$+\frac{1}{\beta_k}\int_0^t F_k(\tau;u,a)\sin\beta_k\,(t-\tau)\,e^{\alpha_k(t-\tau)}d\tau,(\,k=1,2,...;\ \ 0\le t\le T), \tag{3.5}$$

where

$$\alpha_k=-\alpha\lambda_k^2,\beta_k=\lambda_k^2\sqrt{\beta-\alpha^{2\cdot}}.$$

To determine the first component of the classical solution to the problem (2.1)-(2.3), (2.8), (2.9) we substitute the expressions $u_k(t)\ \ (k=0,\,1,...)$ into (3.1) and obtain

$$u(x,t)=\varphi_0+\int_0^T p_1(t)u_0(t)dt+t\ \left(\psi_0+\int_0^T p_2(t)u_0(t)dt\right)+\int_0^t (t-\tau)F_0(\tau;u)d\tau$$
$$+\sum_{k=1}^{\infty}\left\{e^{\alpha_k t}\left[\left(\cos\beta_k t-\frac{\alpha_k}{\beta_k}\sin\beta_k t\right)\left(\varphi_k+\int_0^T p_1(t)u_k(t)dt\right)\right.\right.$$
$$\left.+\frac{\sin\beta_k t}{\beta_k}\left(\psi_k+\int_0^T p_2(t)u_k(t)dt\right)\right]$$
$$\left.+\frac{1}{\beta_k}\int_0^t F_k(\tau;u,a)\sin\beta_k\,(t-\tau)\,e^{\alpha_k(t-\tau)}d\tau\right\}\cos\lambda_k x\,. \tag{3.6}$$

It follows from (2.11) and (3.1) that

$$a(t)=[h(t)]^{-1}\left\{h''(t)-f(0,t)\,+\,\sum_{k=1}^{\infty}\lambda_k^2\left(2\alpha u'_k\,(t)+\beta\lambda_k^2 u_k\,(t)\right)\right\}. \tag{3.7}$$

Differentiating (3.5) we get:

$$u'_k(t)=e^{\alpha_k t}\left[-\frac{1}{\beta_k}\left(\alpha_k^2+\beta_k^2\right)\left(\varphi_k+\int_0^T p_1(t)u_k(t)dt\right)\sin\beta_k t\right.$$

$$+\left(\frac{\alpha_k}{\beta_k}\sin\beta_k t+\cos\beta_k t\right)\left(\psi_k+\int_0^T p_2(t)u_k(t)dt\right)\Bigg]$$
$$+\frac{1}{\beta_k}\int_0^t F_k(\tau;u)\left(\alpha_k\sin\beta_k\left(t-\tau\right)\right.$$
$$\left.+\beta_k\cos\beta_k\left(t-\tau\right)\right)e^{\alpha_k(t-\tau)}d\tau(\,0\le t\le T). \tag{3.8}$$

Further, from (3.5) and (3.8), we obtain:

$$2\alpha u_k'\left(t\right)+\beta\lambda_k^2 u_k\left(t\right)$$
$$=e^{\alpha_k t}\left[\left(\beta\lambda_k^2\cos\beta_k t-\frac{1}{\beta_k}\left(\beta\lambda_k^2\alpha_k+2\alpha\left(\alpha_k^2+\beta_k^2\right)\right)\sin\beta_k t\right)\right.$$
$$\times\left(\varphi_k+\int_0^T p_1(t)u_k(t)dt\right)$$
$$\left.+\left(\frac{1}{\beta_k}\left(\beta\lambda_k^2+2\alpha\alpha_k\right)\sin\beta_k t+2\alpha\cos\beta_k t\right)\left(\psi_k+\int_0^T p_2(t)u_k(t)dt\right)\right]$$
$$+\frac{1}{\beta_k}\int_0^t F_k(\tau;u)\left(\left(2\alpha\alpha_k+\beta\lambda_k^2\right)\sin\beta_k\left(t-\tau\right)\right.$$
$$\left.\left.+2\alpha\beta_k\cos\beta_k\left(t-\tau\right)\right)e^{\alpha_k(t-\tau)}d\tau\right\}. \tag{3.9}$$

Then from (3.9), taking into account (3.7), respectively, we find:

$$a(t)=[h(t)]^{-1}\left\{h''(t)-f(0,t)\right.$$
$$+\sum_{k=1}^{\infty}\lambda_k^2\left\{e^{\alpha_k t}\left[\left(\beta\lambda_k^2\cos\beta_k t-\frac{1}{\beta_k}\left(\beta\lambda_k^2\alpha_k+2\alpha\left(\alpha_k^2+\beta_k^2\right)\right)\sin\beta_k t\right)\right.\right.$$
$$\times\left(\varphi_k+\int_0^T p_1(t)u_k(t)dt\right)$$
$$\left.+\left(\frac{1}{\beta_k}\left(\beta\lambda_k^2+2\alpha\alpha_k\right)\sin\beta_k t+2\alpha\cos\beta_k t\right)\left(\psi_k+\int_0^T p_2(t)u_k(t)dt\right)\right]$$
$$+\frac{1}{\beta_k}\int_0^t F_k(\tau;u,a,b)\left(\left(2\alpha\alpha_k+\beta\lambda_k^2\right)\sin\beta_k\left(t-\tau\right)\right.$$
$$\left.\left.+2\alpha\beta_k\cos\beta_k\left(t-\tau\right)\right)e^{\alpha_k(t-\tau)}d\tau\right\}. \tag{3.10}$$

Thus, the solution of problem (2.1) - (2.3), (2.8), (2.9) was reduced to the solution of system (3.6), (3.10) with respect to unknown functions $u(x,t)$ and $a(t)$.

Lemma 3.1 *If* $\{u(x,t),a(t)\}$ *is any solution to problem* (2.1) - (2.3)*,* (2.8)*,* (2.9)*, then the functions*

$$l_k(t)=m_k\int_0^1 u(x,t)\cos\lambda_k x dx,\ \ (\,k=0,1,2\ldots)$$

satisfies the system (3.4)*,* (3.5) *in* $C[0,T]$.

It follows from Lemma 3.1 that

Corollary 3.1 *Let system* (3.6)*,* (3.10) *have a unique solution. Then problem* (2.1) - (2.3)*,* (2.8)*,* (2.9) *cannot have more than one solution, i.e. if the problem* (2.1) - (2.3)*,* (2.8)*,* (2.9) *has a solution, then it is unique.*

With the purpose to study the problem (2.1) - (2.3), (2.8), (2.9), we consider the following functional spaces.

Denote by $B_{2,T}^5$ [8] a set of all functions of the form

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t) \cos \lambda_k x, \ \ \lambda_k = k\pi,$$

considered in the region D_T, where each of the function $u_k(t) \ \ (k = 0, \ 1, \ 2, ...)$ is continuous over an interval $[0\,, \, T]$ and satisfies the following condition:

$$J(u) \equiv \| \, u_0(t) \, \|_{C[0\,,\,T]} + \left\{ \sum_{k=1}^{\infty} \left(\lambda_k^5 \, \| \, u_k(t) \, \|_{C[0\,,\,T]} \right)^2 \right\}^{\frac{1}{2}} < +\infty \, .$$

The norm in this set is defined by

$$\| \, u(x,t) \, \|_{B_{2,T}^5} = J(u) \, .$$

It is known that $B_{2,T}^5$ is Banach space . Obviously, $E_T^5 = B_{2,T}^5 \times C[0,T]$ with the norm $\|z(x,t)\|_{E_T^5} = \|u(x,t)\|_{B_{2,T}^5} + \|a(t)\|_{C[0,T]}$ is also Banach space.

Now consider the operator

$$\varPhi(u,a) = \{\varPhi_1(u,a) \, , \, \varPhi_2(u,a) \, \} \, ,$$

in the space E_T^3, where

$$\varPhi_1(u,a) = \tilde{u}(x,t) \equiv \sum_{k=0}^{\infty} \tilde{u}_k(t) \cos \lambda_k x \, , \varPhi_2(u,a) = \tilde{a}(t)$$

and the functions $\tilde{u}_0(t)$,$\tilde{u}_k(t), \ \ k = \ 1, \, 2, ...$, and $\tilde{a}(t)$ are equal to the right-hand sides of (3.4), (3.5), and (3.10), respectively.

It is easy to see that

$$\left| \cos \beta_k t - \frac{\alpha_k}{\beta_k} \sin \beta_k t \right| \leq 1 + \frac{\alpha}{\sqrt{\beta - \alpha^2}} \equiv \varepsilon_1, \left| \frac{1}{\beta_k} \sin \beta_k t \right| \leq \frac{1}{\sqrt{\beta - \alpha^2}} \frac{1}{\lambda_k^2} \equiv \varepsilon_2 \frac{1}{\lambda_k^2},$$

$$\left| \beta \lambda_k^2 \cos \beta_k t - \frac{1}{\beta_k} \left(\beta \lambda_k^2 \alpha_k + 2\alpha \left(\alpha_k^2 + \beta_k^2 \right) \right) \sin \beta_k t \right| \leq \left(\frac{3\alpha}{\sqrt{\beta - \alpha^2}} + 1 \right) \beta \lambda_k^2 \equiv \varepsilon_3 \lambda_k^2,$$

$$\left| \frac{1}{\beta_k} \left(\beta \lambda_k^2 + 2\alpha \alpha_k \right) \sin \beta_k t + 2\alpha \cos \beta_k t \right| \leq \frac{\beta + 2\alpha^2}{\sqrt{\beta - \alpha^2}} + 2\alpha \equiv \varepsilon_4.$$

$$\frac{1}{\beta_k} \left| \left(2\alpha \alpha_k + \beta \lambda_k^2 \right) \sin \beta_k \left(t - \tau \right) + 2\alpha \beta_k \cos \beta_k \left(t - \tau \right) \right| \leq \varepsilon_4.$$

Then, with the help of simple transformations, we find:

$$\|\tilde{u}_0(t)\|_{C[0,T]} \leq |\varphi_0| + T\,|\psi_0| + T(\|p_1(t)\|_{C[0,T]} + T\,\|p_2(t)\|_{C[0,T]})\,\|u_0(t)\|_{C[0,T]}$$

$$+T\sqrt{T}\left(\int_0^T |f_0(\tau)|^2\, d\tau\right)^{\frac{1}{2}} + T^2\,\|a(t)\|_{C[0,T]}\,\|u_0(t)\|_{C[0,T]}\ , \tag{3.11}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|\tilde{u}_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le \sqrt{7}\varepsilon_1\left(\sum_{k=1}^{\infty}(\lambda_k^5\,|\varphi_k|)^2\right)^{\frac{1}{2}} + \sqrt{7}\varepsilon_1\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\psi_k|)^2\right)^{\frac{1}{2}}$$

$$+\sqrt{7}\left(\|p_1(t)\|_{C[0,T]} + \|p_2(t)\|_{C[0,T]}\right) T\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+\varepsilon_1\sqrt{7T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^3\,|f_k(\tau)|)^2 d\tau\right)^{\frac{1}{2}}$$

$$+\sqrt{7}\varepsilon_1 T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}, \tag{3.12}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \le \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h''(t) - f(0,t)\right\|_{C[0,T]}\right.$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\frac{\sqrt{6}}{12}\varepsilon_3\left(\sum_{k=1}^{\infty}(\lambda_k^5\,|\varphi_k|)^2\right)^{\frac{1}{2}} + \frac{\sqrt{6}}{12}\varepsilon_4\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\psi_k|)^2\right)^{\frac{1}{2}}\right.$$

$$+\frac{\sqrt{6}}{12}(\varepsilon_3+\varepsilon_4)T\left(\|p_1(t)\|_{C[0,T]} + \|p_2(t)\|_{C[0,T]}\right)\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+\frac{\sqrt{6T}}{12}\varepsilon_4\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^3\,|f_k(\tau)|)^2 d\tau\right)^{\frac{1}{2}}$$

$$\left.\left.+\ \frac{\sqrt{6}}{12}\varepsilon_4 T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]\right\}. \tag{3.13}$$

Suppose that the data for problem (2.1)-(2.3), (2.8), (2.9) satisfy the assumptions:
1. $\varphi(x) \in C^4[0,1],\ \varphi^5(x) \in L_2(0,1)$ and $\varphi'(0) = \varphi'(1) = \varphi'''(0) = \varphi'''(1) = 0.$
2. $\psi(x) \in C^2[0,1],\ \ \psi^{(3)}(x) \in L_2(0,1)$ and $\psi'(0) = \psi'(1) = 0.$
3. $f(x,t),\ f_x(x,t),\ f_{xx}(x,t) \in C(D_T),\ f_{xxx}(x,t) \in L_2(D_T),$
$f_x(0,t) = f_x(1,t) = 0\ (0 \le t \le T).$
4. $\alpha > 0, \beta > \alpha^2, p_i(t) \in C[0,T]\ \ (i = 1,2), h(t) \in C^2[0,T],$
$h(t) \neq 0,\ (0 \le t \le T).$
Then from (3.11)-(3.13) we get:

$$\|\tilde{u}(x,t)\|_{B^5_{2,T}} \le A_1(T)$$

$$+B_1(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^5_{2,T}} + C_1(T)\,\|u(x,t)\|_{B^5_{2,T}}\ , \tag{3.14}$$

$$\|\tilde{a}(t)\|_{C[0,T]} \le A_2(T) + B_2(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^5_{2,T}} + C_2(T)\,\|u(x,t)\|_{B^5_{2,T}}\ , \tag{3.15}$$

where

$$A_1(T) = \|\varphi(x)\|_{L_2(0,1)} + T\,\|\psi(x)\|_{L_2(0,1)}$$

$$+T\sqrt{T}\,\|f(x,t)\|_{L_2(D_T)}+2\sqrt{7}\varepsilon_1\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}$$

$$+2\sqrt{7}\varepsilon_2\left\|\psi^{(3)}(x)\right\|_{L_2(0,1)}+2\varepsilon_2\sqrt{7T}\,\|f_{xxx}(x,t)\|_{L_2(D_T)}\,,$$

$$B_1(T)=(T+2\sqrt{7}\varepsilon_2)T, C_1(T)=T(1+2\sqrt{7})\,\|p_1(t)\|_{C[0,T]}+T(T+2\sqrt{7})\,\|p_2(t)\|_{C[0,T]}\ ,$$

$$A_2(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h''(t)-f(0,t)\right\|_{C[0,T]}\right.$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\frac{\sqrt{6}}{6}\varepsilon_3\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+\frac{\sqrt{6}}{6}\varepsilon_3\left\|\psi'''(x)\right\|_{L_2(0,1)}\right.$$

$$\left.\left.+\ \frac{\sqrt{6T}}{6}\varepsilon_4\,\|f_{xxx}(x,t)\|_{L_2(D_T)}\right]\right\},$$

$$B_2(T)=\ \frac{\sqrt{6}}{12}\varepsilon_4\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}T,$$

$$C_2(T)=\frac{\sqrt{6}}{6}(\varepsilon_4+\varepsilon_3)\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}T\left(\|p_1(t)\|_{C[0,T]}+\|p_2(t)\|_{C[0,T]}\right).$$

From the inequalities (3.14),(3.15) we conclude:

$$\|\tilde{u}(x,t)\|_{B^5_{2,T}}+\|\tilde{a}(t)\|_{C[0,T]}$$

$$\leq A(T)+B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^5_{2,T}}\ +C(T)\,\|u(x,t)\|_{B^5_{2,T}}\ , \tag{3.16}$$

where

$$A(T)=A_1(T)+A_2(T)\,,\ B(T)=B_1(T)+B_2(T)\,,\ C(T)=C_1(T)+C_2(T)\,.$$

Thus, we can prove the following theorem

Theorem 3.1 *Assume that statements 1-4 and the condition*

$$(B(T)(A(T)+2)+C(T))(A(T)+2)<1 \tag{3.17}$$

holds, then problem (2.1)-(2.3), (2.8), (2.9) *has a unique solution in the ball* $K=K_R(\|z\|_{E^5_T}\leq R\leq A(T)+2)$ *of the space* E^5_T*.*

Proof. In the space E^5_T, consider the operator equation

$$z=\Phi z\,, \tag{3.18}$$

where $z=\{u,\,a\}$, and the components $\Phi_i(u,a)\ \ (i=1,2)$, of operator $\Phi(u,a)$ defined by the right sides of (3.6) and (3.10).
Consider the operator $\Phi(u,a)$in the ball $K=K_R$ out of E^5_T. Similarly to (3.16), we obtain that for any the estimates are valid: respectively and the following inequalities hold:

$$\|\Phi z\|_{E^5_T}\leq A(T)+B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^5_{2,T}}+C(T)\,\|u(x,t)\|_{B^5_{2,T}}$$

$$\leq A(T)+B(T)R^2+C(T)R=A(T)+(B(T)(A(T)+2)+C(T))(A(T)+2)\,, \tag{3.19}$$

$$\|\Phi z_1 - \Phi z_2\|_{E_T^5} \le B(T)R(\|u_1(x,t) - u_2(x,t)\|_{B_{2,T}^5} + \|a_1(t) - a_2(t)\|_{C[0,T]})$$
$$+C(T)\,\|u_1(x,t) - u_2(x,t)\|_{B_{2,T}^5}\,, \tag{3.20}$$

Then it follows from (3.17), (3.19), and (3.20) that the operator Φ acts in the ball $K = K_R$, and satisfy the conditions of the contraction mapping principle. Therefore the operator Φ has a unique fixed point $\{z\} = \{u\,,\,a\}$ in the ball $K = K_R$, which is a solution of equation (3.18); i.e. the pair $\{u, a\}$ is the unique solution of the systems (3.6) and (3.10) in $K = K_R$.

hen the function $u(x,t)$ as an element of space $B_{2,T}^5$ is continuous and has continuous derivatives $u_x(x,t), u_{xx}(x,t), u_{xxx}(x,t)$ and $u_{xxxx}(x,t)$ in D_T.

Similarly [7], one can prove that $u_t(x,t)$, $u_{tx}(x,t), u_{txx}(x,t)$, $u_{tt}(x,t)$ are continuous in D_T .

It is easy to verify that Eq. (2.1) and conditions (2.2), (2.3), (2.8), (2.9) satisfy in the usual sense. So, $\{u(x,t), a(t)\}$ is a solution of (2.1)-(2.3), (2.8), (2.9), and by Lemma 3.1 it is unique in the ball $K = K_R$. The proof is complete.

In summary, from Theorem 2.1 and Theorem 3.1, straightforward implies the unique solvability of the original problem (2.1) - (2.5).

Theorem 3.2 *Suppose that all assumptions of Theorem 3.1, and the conditions*

$$\int_0^1 f(x,t)dx = 0\,, \quad (0 \le t \le T), \quad \int_0^1 \varphi(x)dx = 0, \quad \int_0^1 \psi(x)dx = 0\,,$$

$$\varphi(0) + \int_0^T p_1(t)h(t)dt = h(0), \quad \psi(0) + \int_0^T p_2(t)h(t)dt = h'(0)\,.$$

holds. Then problem (2.1) - (2.5) *has a unique classical solution in the ball* $K = K_R(\|z\|_{E_T^5} \le A(T) + 2)$ *of the space* E_T^5.

4 Conclusion

The existence and uniqueness of the solution of one inverse boundary value problem for one Bussinsk equation of the fourth order with integral conditions is proved in the work. First, the original problem is reduced to an equivalent problem, for which the theorem of existence and uniqueness of the solution is proved. Using these facts, the existence and uniqueness of the classical solution of one inverse boundary value problem for one Bussinsk equation of the fourth order with integral conditions is proved.

References

1. Boussinesq J.: Theorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide content dans ce canal des vitesses sensiblement pareilles de surface au fond *J. Math. Pures Appl.* **17**, 55-108 (1872).
2. Denisov A.M.: *Introduction to Theory of Inverse Problems*, (in Russian) M: MSU (1994).
3. Ivanchov M.I.: *Inverse problems for equation of parabolic type*, Ukraine (2003).
4. Lavrent'ev M.M.: On an inverse problem for a wave equation, (in Russian) *Dokl. AN SSSR*, **157** (3), 520-521 (1964)
5. Lavrent'ev M.M., Romanov V.G., Shishatsky S.T.: *Ill-Posed Problems of Mathematical Physics and Analysis*, (in Russian) M. Nauka (1980).

6. Megraliev Y.T., Alizade F.Kh.: An inverse boundary value problem for a single fourth order Boussinesq equation with an integral condition, (in Russian) *Chebyshevskii Sb.*, **14** (4) , 167-179 (2013).
7. Ya.T. Mehraliev and F.Kh. Alizade, Inverse boundary value problem for a Boussinesq type equation of fourth order with nonlocal time integral conditions of the second kind, (in Russian) *Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki*, **26** (4), 503-514 (2016).
8. Mehraliev Ya.T.: On an inverse boundary value problem for a second order elliptic equation with integral condition, *Visnyk of the Lviv Univ. Series Mech. Math.* Issue 77, 145–156 (2012).
9. Tikhonov A.I.: On stability of inverse problems, (in Russian) *Dokl. AN SSSR*, **39** (5), 195-198 (1943).