Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **44** (1), 1-11 (2024).

Global attractors in a two-species chemotaxis system with two chemicals and variable logistic sources

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Received: 02.08.2023 / Revised: 28.12.2023 / Accepted: 07.02.2024

Abstract. *This paper deals with the higher dimension quasilinear parabolic-parabolic chemotaxis model involving a source term of logistic type* $u_t = \Delta u - \chi_1 \nabla \cdot (u \nabla \upsilon) + \eta_1 u - \mu_1 u^{m(x)}$, $\upsilon_t = \Delta \upsilon - \upsilon + \omega$, $\omega_t = \Delta \omega - \chi_2 \nabla \cdot (\omega \nabla z) + \eta_2 \omega - \mu_2 \omega^{m(x)}$, $0 = \Delta z - z + u$, *subject to the homogeneous Neumann boundary conditions in a* $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ *with smooth boundary. It is shown that for the attraction-repulsion case with* $\chi_2 \leq 0$*, the global boundedness of solutions can be ensured by* μ_1, $\mu_2 > 0$ *without any other assumptions, due to the contribution of the logistic sources included in addition to the repulsion mechanism. While for the attraction-attraction case with* $\chi_2 > 0$*, the global boundedness of solutions has to require logistic coefficients* μ_1, $\mu_2 > 0$ *such that* μ_2 *properly large.*

Keywords. Two-species chemotaxis system, variable logistic source, global boundedness.

Mathematics Subject Classification (2010): 35B35, 35B40, 35K55, 92C17.

1 Introduction and preliminaries

In this paper, we consider a quasilinear parabolic-parabolic chemotaxis model with a source term of variable logistic type,

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla \cdot (u \nabla \upsilon) + \eta_1 u - \mu_1 u^{m(x)}, & (x,t) \in \Omega \times (0,T), \\ \upsilon_t = \Delta \upsilon - \upsilon + \omega, & (x,t) \in \Omega \times (0,T), \\ \omega_t = \Delta \omega - \chi_2 \nabla \cdot (\omega \nabla z) + \eta_2 \omega - \mu_2 \omega^{m(x)}, & (x,t) \in \Omega \times (0,T), \\ 0 = \Delta z - z + u, & (x,t) \in \Omega \times (0,T), \\ \frac{\partial u}{\partial \nu} = \frac{\partial \upsilon}{\partial \nu} = \frac{\partial \omega}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \upsilon(x,0) = \upsilon_0(x), \omega(x,0) = \omega_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ and $\frac{\partial}{\partial \nu}$ denotes the derivative with respect to the outer normal of $\partial\Omega$, parameters χ_1, μ_1, $\mu_2 > 0$, η_1, η_2, $\chi_2 \in \mathbb{R}$ and variable

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exponent $m : \Omega \to (2, \infty)$ is a measurable function. We introduce m^- and m^+ such that

$$2 < m^- := \operatorname*{ess\,inf}_{x\in\Omega} m(x) \le m(x) \le m^+ := \operatorname*{ess\,sup}_{x\in\Omega} m(x) < +\infty. \tag{1.2}$$

The nonnegative initial data

$$u_0, \omega_0 \in C\left(\overline{\Omega}\right) \text{ with } u_0, \omega_0 \neq 0, \text{ and } \upsilon_0 \in W^{1,\infty}(\Omega). \tag{1.3}$$

Next, we denote $L^p(\Omega)$ Lebesgue and $W^{1,p}(\Omega)$, $W^{2,p}(\Omega)$ Sobolev spaces and the norms of these spaces by $\|u\|_{L^p(\Omega)} = \|u\|_p$, $\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p$, $\|u\|_{2,p} = \|u\|_p + \|\Delta u\|_p$ $(1 \le p \le \infty)$ respectively.

In the model (1.1), u and ω represent the densities of two species, υ and z denote the concentrations of chemical substances secreted by ω and u respectively. The system (1.1) means that the population u is attracted by the signals υ produced by the population ω, whereas the population ω is attracted (with $\chi_2 > 0$) or repelled (with $\chi_2 < 0$) by the signals z produced by the population u. The logistic source $\eta_1 u - \mu_1 u^{m(x)}$ and $\eta_2\omega - \mu_2\omega^{m(x)}$ included in (1.1) prevent the unlimited growth of cell densities. When $m(x) \equiv 2,\ \forall x \in \Omega$, in the single-species Keller-Segel chemotaxis system with logistic source

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla\upsilon) + f(u), & (x,t) \in \Omega\times(0,T),\\ \tau\upsilon_t = \Delta\upsilon - \upsilon + \omega, & (x,t)\in\Omega\times(0,T),\\ \frac{\partial u}{\partial\nu} = \frac{\partial\upsilon}{\partial\nu} = 0, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0) = u_0(x), \tau\upsilon(x,0) = \tau\upsilon_0(x), & x\in\Omega, \end{cases} \tag{1.4}$$

where $f(u) = \eta u - \mu u^2$ and χ, $\mu > 0$, $\eta \in \mathbb{R}$, $\tau \in \{0,1\}$, the population u is attracted by the signals υ secreted by itself. Now rich dynamical properties of solutions to (1.4) have been established [9,14,15,19,26,29,30]. For the parabolic-elliptic case with $\tau = 0$, it was proved by Tello and Winkler [21] that when $\mu = \eta > 0$, if $N \le 2$ or $\mu > \frac{N-2}{N}\chi$, then the solutions of the parabolic-elliptic chemotaxis system are globally bounded, and the equilibrium $(1,1)$ is a global attractor if in addition $\mu > 2\chi$. For the parabolic-parabolic case with $\tau = 1$, Winkler [27] showed that if μ is sufficiently large, then there exist globally bounded solutions for the system (1.4).

There are also many works about system (1.4) when $f(u) = \eta u - \mu u^\alpha$ $(\eta \ge 0, \mu > 0$, $\alpha > 1)$ being a damping source term in order to improve the system's consistency with biological reality or adapt it to complex biological situations. The term f describes cell proliferation and cell death in biological systems. Here, the parameter α characterizes the dying growth of cells as a constant number. As can be seen, the constancy of α makes the cell death kinetics independent of the variable x, and the growth of dying cells remains "isotropic" in different directions. This situation gives incomplete results in practice. However, in the kinetics of cell growth, if the death of cells depends on the variable x, that is, in models such as $f(x,u) = \eta u - \mu u^{\alpha(x)}$ with $\eta \ge 0, \mu > 0$ and $\alpha(x)$, $\forall x \in \Omega$ is a measurable function, cell death growth in each x direction will have different values ("anisotropic"). Such models are clearer and more perfect in practice than the previous one. The mathematical study of such models has some difficulties with respect to "classical" models. We refer the interested reader to the recently published articles [3–5] with logistic source involving the exponents depending on the spatial variables.

Recently, multi-species and multi-stimulus problems of Keller-Segel systems have been more and more studied as well [12,17,18,22,23,25,31–33]. Among them, Tao and Winkler [20] considered the two-species chemotaxis model with two chemicals

$$\begin{cases} u_t = \Delta u - \chi_1\nabla\cdot(u\nabla\upsilon), & (x,t)\in\Omega\times(0,T),\\ 0 = \Delta\upsilon - \upsilon + \omega, & (x,t)\in\Omega\times(0,T),\\ \omega_t = \Delta\omega - \chi_2\nabla\cdot(\omega\nabla z), & (x,t)\in\Omega\times(0,T),\\ 0 = \Delta z - z + u, & (x,t)\in\Omega\times(0,T). \end{cases} \tag{1.5}$$

For the attraction-repulsion case (i.e. $\chi_1 = 1, \chi_2 = -1$), it was proved that if $N \leq 3$, Eq. (1.5) possesses global bounded solutions for any nonnegative $u_0(x), \omega_0(x) \in C(\overline{\Omega})$. For the attraction-attraction case (i.e. $\chi_1 = \chi_2 = 1$), if either $N = 2$ and $\int_\Omega u_0 + \int_\Omega \omega_0$ lies below some threshold, or $N \geq 3$ and $\|u_0\|_\infty$, $\|\omega_0\|_\infty$ small enough, then solutions are globally bounded.

Concerning the two-species with two chemicals chemotaxis system with two chemicals and variable logistic sources (1.1), the main result of this paper is the following theorem.

Theorem 1.1 . *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded domain with smooth boundary, parameters χ_1, μ_1, $\mu_2 > 0$, η_1, η_2, $\chi_2 \in \mathbb{R}$, variable exponent m satisfies (1.2) and nonnegative initial data u_0, υ_0, ω_0 satisfy (1.3). If $\chi_2 \leq 0$ or $\chi_2 > 0$ and μ_2 is large enough then the solution of (1.1) is globally bounded.*

Remark 1. When $m(x) \equiv 2$ for all $x \in \Omega$ given in (1.1), Tian et al. [24] showed that for the attraction-repulsion case with $\chi_2 \leq 0$, the global boundedness of solutions can be ensured by $\mu_1, \mu_2 > 0$ without any other assumptions, due to the contribution of the logistic sources included in addition to the repulsion mechanism. While for the attraction-attraction case with $\chi_2 > 0$, the global boundedness of solutions has to require small sensitivity coefficients $\chi_1 > 0$, $\chi_2 > 0$ or large logistic coefficients μ_1, μ_2 such that $\frac{\mu_1\mu_2}{\chi_1\chi_2}$ properly large, rather than the simple condition μ_1, $\mu_2 > 0$ for the attraction-repulsion case. Our results are generalized and extended of those obtained by Tian et al..

Now, we introduce a well-known regularity property on the parabolic or elliptic equations under the Neumann boundary condition, and state the local existence result to the system (1.1).

Lemma 1.1 . *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded domain with smooth boundary. Then for any nonnegative $(u_0(x), \upsilon_0(x), \omega_0(x)) \in C(\overline{\Omega}) \times W^{1,\delta}(\Omega) \times C(\overline{\Omega})$ $(\delta > N)$, there exist nonnegative functions $(u, \upsilon, \omega, z) \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max}))$ with $T_{\max} \in (0, \infty]$ classically solving (1.1) in $\Omega \times (0, T_{\max})$. Moreover, if $T_{\max} < \infty$, then*

$$\lim_{t \to T_{\max}} \left(\|u(\cdot, t)\|_\infty + \|\omega(\cdot, t)\|_\infty \right) = \infty.$$

The local existence of solutions to (1.1) can be proved by the standard parabolic theory in a suitable framework of fixed point theory, refer to [13,20], and the reference therein.

Lemma 1.2 *(Lemma 2.4 in [28] or Lemma 2.3 in [16]). Let $\rho_0 \in W^{1,\infty}(\Omega)$ and $f \in C^0(\overline{\Omega} \times [0, T_*))$ with $T_* \in (0, \infty]$. Suppose that $\rho \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max}))$ solves*

$$\begin{cases} \tau\rho_t = \Delta\rho - \rho + f, \ (x,t) \in \Omega \times (0,T), \\ \frac{\partial\rho}{\partial\nu} = 0, \qquad\quad (x,t) \in \partial\Omega \times (0,T), \\ \tau\rho(x,0) = \tau\rho_0(x), \qquad\quad x \in \Omega \end{cases}$$

with $\tau = \{0, 1\}$. Assume that $\frac{1}{2} + \frac{N}{2}(\frac{1}{p} - \frac{1}{q}) < 1$ and $1 \leq p, q \leq \infty$, then

$$\|\nabla\rho(\cdot, t)\|_q \leq C(1 + \sup_{s \in (0,t)} \|f(\cdot, s)\|_p) \tag{1.6}$$

for each $t \in (0, T_*)$ with $C > 0$.

When $\tau = 1$, the estimate (2.1) was proved by smoothing estimates for the Neumann heat semigroup in [28]. When $\tau = 0$, (2.1) can be obtained by the Sobolev imbedding theorem (see Chapter 4 in [1]) and elliptic equations regularity estimates (see Theorem 6.30 in [8]).

The following lemma as a variation of Maximal Sobolev Regularity [10] is crucial for our proof.

Lemma 1.3 *([6] or [7]). Let $r \in (1, \infty)$. Consider the following evolution equation*

$$\begin{cases} \upsilon_t = \Delta \upsilon - \upsilon + \omega, \ (x,t) \in \Omega \times (0,T), \\ \frac{\partial \upsilon}{\partial \nu} = 0, \qquad\qquad (x,t) \in \partial\Omega \times (0,T), \\ \upsilon(x,0) = \upsilon_0(x), \qquad\qquad x \in \Omega. \end{cases}$$

For each

$$\upsilon_0 \in W^{2,r}(\Omega) \ (r > N), \omega \in L^r((0,T); L^r(\Omega)),$$

there exists a unique solution

$$\upsilon \in W^{1,r}((0,T); L^r(\Omega)) \cap L^r((0,T); W^{2,r}(\Omega)), r > N.$$

Moreover, there exists a $C_r > 0$, such that if $s_0 \in [0, T))$, $\upsilon(\cdot, s_0) \in W^{2,r}(\Omega)(r > N)$ with $\frac{\partial \upsilon(\cdot, s_0)}{\partial \nu} = 0$, then

$$\int_{s_0}^T \int_\Omega e^{rs} |\Delta \upsilon|^r \le C_r \int_{s_0}^T \int_\Omega e^{rs} \omega^r + C_r e^{rs_0} \left(\|\upsilon(\cdot, s_0)\|_r^r + \|\Delta \upsilon(\cdot, s_0)\|_r^r \right).$$

Given $s_0 \in (0, T_{\max})$ such that $s_0 \le 1$, from the regularity principle asserted by Lemma 1.2*, we know that $u(\cdot, s_0)$, $\upsilon(\cdot, s_0)$, $\omega(\cdot, s_0) \in C^2(\overline{\Omega})$ with $\frac{\partial \upsilon(\cdot, s_0)}{\partial \nu} = 0$ on $\partial\Omega$. So, we can pick $M > 0$ such that*

$$\sup_{0 \le s \le s_0} \|u(\cdot, s)\|_\infty, \ \sup_{0 \le s \le s_0} \|\upsilon(\cdot, s)\|_\infty, \ \sup_{0 \le s \le s_0} \|\omega(\cdot, s)\|_\infty, \ \|\Delta \upsilon(\cdot, s_0)\|_\infty \le M. \quad (1.7)$$

Lemma 1.4 *(Lemma 2.2 in [11]). Let (u, υ, ω, z) be a solution to* (1.1) *ensured by* 1.1*. Then for any $l, h > 0$, $\theta > 1$, there is $c_0 = c_0(h, \theta, l) > 0$ such that*

$$\int_\Omega z^\theta \le h \int_\Omega u^{l\theta} + c_0 \quad (1.8)$$

for all $t \in (0, T_{\max})$.

2 Proof of Main Results

Proof. (Proof of Theorem 1.1) The key step of the proof is to show that for any $\gamma > 1$, there exists $C = C(\gamma, m^-, \mu_1, \mu_2, \eta_1, \eta_2, \chi_1, |\Omega|) > 0$ such that

$$\|u(\cdot, t)\|_\gamma \le C, \quad (2.1)$$

and

$$\|\omega(\cdot, t)\|_\gamma \le C \quad (2.2)$$

for all $t \in (0, T_{\max})$.

Multiply the first equation given in (1.1) by $u^{\gamma-1}$ for arbitrary $\gamma > 1$, and then integrate over Ω by parts we have

$$\begin{aligned} \frac{1}{\gamma} \frac{d}{dt} \int_\Omega u^\gamma \le & -\frac{4(\gamma-1)}{\gamma^2} \int_\Omega \left| \nabla u^{\frac{\gamma}{2}} \right|^2 - \frac{(\gamma-1)\chi_1}{\gamma} \int_\Omega u^\gamma \Delta \upsilon \\ & + \eta_1 \int_\Omega u^\gamma - \mu_1 \int_\Omega u^{m(\cdot)+\gamma-1} \\ \le & -\frac{m^- + \gamma - 1}{\gamma} \int_\Omega u^\gamma - \frac{(\gamma-1)\chi_1}{\gamma} \int_\Omega u^\gamma \Delta \upsilon \\ & + \left(\eta_1 + \frac{m^- + \gamma - 1}{\gamma} \right) \int_\Omega u^\gamma - \mu_1 \int_\Omega u^{m(\cdot)+\gamma-1} \end{aligned} \quad (2.3)$$

for $t \in (s_0, T_{\max})$ with s_0 defined in Lemma 1.3.

By the conditions (1.2), we derive

$$\int_{\Omega} u^{m(\cdot)+\gamma-1} \geq \int_{\Omega\cap\{x:u\geq 1\}} u^{m^-+\gamma-1} = \int_{\Omega} u^{m^-+\gamma-1} - \int_{\Omega\cap\{x:u<1\}} u^{m^-+\gamma-1}$$
$$\geq \int_{\Omega} u^{m^-+\gamma-1} - \int_{\Omega} 1 = \int_{\Omega} u^{m^-+\gamma-1} - |\Omega| . \tag{2.4}$$

By using Young's inequality we can see that

$$-\frac{(\gamma-1)\chi_1}{\gamma}\int_{\Omega} u^{\gamma}\Delta v \leq \chi_1 \int_{\Omega} u^{\gamma}|\Delta v|$$
$$\leq \frac{\chi_1}{m^-+\gamma-1}\int_{\Omega}|\Delta v|^{m^-+\gamma-1} + \frac{\chi_1(m^-+\gamma-2)}{m^-+\gamma-1}\int_{\Omega} u^{\frac{\gamma\left(m^-+\gamma-1\right)}{m^-+\gamma-2}} \tag{2.5}$$

for all $t \in (s_0, T_{\max})$ with $\chi_1 > 0$.

From (2.4) , (2.5) and (2.3), we have

$$\frac{1}{\gamma}\frac{d}{dt}\int_{\Omega} u^{\gamma} + \frac{m^-+\gamma-1}{\gamma}\int_{\Omega} u^{\gamma}$$
$$\leq \frac{\chi_1}{m^-+\gamma-1}\int_{\Omega}|\Delta v|^{m^-+\gamma-1} + \frac{\chi_1(m^-+\gamma-2)}{m^-+\gamma-1}\int_{\Omega} u^{\frac{\gamma\left(m^-+\gamma-1\right)}{m^-+\gamma-2}}$$
$$+\left(\eta_1 + \frac{m^-+\gamma-1}{\gamma}\right)\int_{\Omega} u^{\gamma} - \mu_1\int_{\Omega} u^{m^-+\gamma-1} + \mu_1|\Omega| . \tag{2.6}$$

On the other hand, by the elementary inequality we have

$$a_0\xi^i - b_0\xi^j \leq a_0\left(\frac{a_0}{b_0}\right)^{\frac{i}{j-i}}, \forall \xi > 0, \tag{2.7}$$

where $a_0 \geq 0$, $b_0 > 0$ and $0 \leq i < j$. By using inequality (2.7) for the third and fourth terms in (2.6), we obtain

$$\left(\eta_1 + \frac{m^-+\gamma-1}{\gamma}\right)\int_{\Omega} u^{\gamma} - \frac{\mu_1}{3}\int_{\Omega} u^{m^-+\gamma-1} \leq C_1, \tag{2.8}$$

where $C_1 = \left(\eta_1 + \frac{m^-+\gamma-1}{\gamma}\right)\left(\frac{3\left(\eta_1+\frac{m^-+\gamma-1}{\gamma}\right)}{\mu_1}\right)^{\frac{\gamma}{m^--1}}|\Omega| > 0$. Since $m^- > 2$, we have $\frac{\gamma\left(m^-+\gamma-1\right)}{m^-+\gamma-2} < m^-+\gamma-1$. Then by (2.7) for the two and fourth terms in (2.6), we see that

$$\frac{\chi_1(m^-+\gamma-2)}{m^-+\gamma-1}\int_{\Omega} u^{\frac{\gamma\left(m^-+\gamma-1\right)}{m^-+\gamma-2}} - \frac{\mu_1}{3}\int_{\Omega} u^{m^-+\gamma-1} \leq C_2, \tag{2.9}$$

where $C_2 = \frac{\chi_1\left(m^-+\gamma-2\right)}{m^-+\gamma-1}\left(\frac{3\chi_1\left(m^-+\gamma-2\right)}{(m^-+\gamma-1)\mu_1}\right)^{\frac{\gamma}{m^--2}}|\Omega| > 0$.

Combine (2.6) with (2.8) and (2.9) to get

$$\begin{aligned}&\frac{1}{\gamma}\frac{d}{dt}\int_{\Omega}u^{\gamma}+\frac{m^{-}+\gamma-1}{\gamma}\int_{\Omega}u^{\gamma}\\&\leq\frac{\chi_1}{m^{-}+\gamma-1}\int_{\Omega}|\Delta v|^{m^{-}+\gamma-1}-\frac{\mu_1}{3}\int_{\Omega}u^{m^{-}+\gamma-1}+C_3\end{aligned}\tag{2.10}$$

with $C_3=C_1+C_2+\mu_1|\Omega|$. By applying the variation-of-constants formula to (2.10), we have

$$\begin{aligned}\frac{1}{\gamma}\int_{\Omega}u^{\gamma}\leq&\frac{1}{\gamma}e^{-\left(m^{-}+\gamma-1\right)(t-s_0)}\int_{\Omega}u^{\gamma}(\cdot,s_0)-\frac{\mu_1}{3}\int_{s_0}^{t}e^{-\left(m^{-}+\gamma-1\right)(t-s)}\int_{\Omega}u^{m^{-}+\gamma-1}\\&+\frac{\chi_1}{m^{-}+\gamma-1}\int_{s_0}^{t}e^{-\left(m^{-}+\gamma-1\right)(t-s)}\int_{\Omega}|\Delta v|^{m^{-}+\gamma-1}+C_3\int_{s_0}^{t}e^{-\left(m^{-}+\gamma-1\right)(t-s)}\end{aligned}\tag{2.11}$$

for all $t\in(s_0,T_{\max})$. Then from (2.11), we get

$$\begin{aligned}\frac{1}{\gamma}\int_{\Omega}u^{\gamma}\leq&\frac{\chi_1}{m^{-}+\gamma-1}\int_{s_0}^{t}e^{-\left(m^{-}+\gamma-1\right)(t-s)}\int_{\Omega}|\Delta v|^{m^{-}+\gamma-1}\\&-\frac{\mu_1}{3}\int_{s_0}^{t}e^{-\left(m^{-}+\gamma-1\right)(t-s)}\int_{\Omega}u^{m^{-}+\gamma-1}+C_4,\end{aligned}$$

where $C_4=\frac{1}{\gamma}\int_{\Omega}u^{\gamma}(\cdot,s_0)+\frac{C_3}{m^{-}+\gamma-1}$.

By Lemma 1.3, we know that there exists a $C_{m^{-},\gamma}>0$ such that

$$\begin{aligned}&\int_{s_0}^{t}e^{-\left(m^{-}+\gamma-1\right)(t-s)}\int_{\Omega}|\Delta v|^{m^{-}+\gamma-1}\\&=e^{-\left(m^{-}+\gamma-1\right)t}\int_{s_0}^{t}\int_{\Omega}e^{\left(m^{-}+\gamma-1\right)s}|\Delta v|^{m^{-}+\gamma-1}\\&\leq C_{m^{-},\gamma}e^{-\left(m^{-}+\gamma-1\right)t}\Big(\int_{s_0}^{t}\int_{\Omega}e^{\left(m^{-}+\gamma-1\right)s}\omega^{m^{-}+\gamma-1}\\&\quad+e^{\left(m^{-}+\gamma-1\right)s_0}\|v(\cdot,s_0)\|_{2,m^{-}+\gamma-1}^{m^{-}+\gamma-1}\Big),\end{aligned}$$

and thus,

$$\begin{aligned}\frac{1}{\gamma}\int_{\Omega}u^{\gamma}\leq&\frac{\chi_1C_{m^{-},\gamma}}{m^{-}+\gamma-1}\int_{s_0}^{t}\int_{\Omega}e^{-\left(m^{-}+\gamma-1\right)(t-s)}\omega^{m^{-}+\gamma-1}\\&-\frac{\mu_1}{3}\int_{s_0}^{t}\int_{\Omega}e^{-\left(m^{-}+\gamma-1\right)(t-s)}u^{m^{-}+\gamma-1}+C_5\end{aligned}\tag{2.12}$$

for all $t\in(s_0,T_{\max})$ with $C_5=C_4+C_{m^{-},\gamma}\|v(\cdot,s_0)\|_{2,m^{-}+\gamma-1}^{m^{-}+\gamma-1}$.

Similarly, multiply $\omega^{\gamma-1}$ for arbitrary $\gamma>1$ to the third equation of (1.1), and integrate by parts over Ω to get

$$\begin{aligned}\frac{1}{\gamma}\frac{d}{dt}\int_{\Omega}\omega^{\gamma}\leq&-\frac{4(\gamma-1)}{\gamma^2}\int_{\Omega}\left|\nabla\omega^{\frac{\gamma}{2}}\right|^2-\frac{(\gamma-1)\chi_2}{\gamma}\int_{\Omega}\omega^{\gamma}\Delta z\\&+\eta_2\int_{\Omega}\omega^{\gamma}-\mu_2\int_{\Omega}\omega^{m^{-}+\gamma-1}+\mu_2|\Omega|\end{aligned}\tag{2.13}$$

for all $t \in (s_0, T_{\max})$. By substituting the fourth equation given in (1.1) into the inequality (2.13), we have

$$\frac{1}{\gamma}\frac{d}{dt}\int_\Omega \omega^\gamma \leq -\frac{4(\gamma-1)}{\gamma^2}\int_\Omega \left|\nabla \omega^{\frac{\gamma}{2}}\right|^2 - \frac{(\gamma-1)\chi_2}{\gamma}\int_\Omega \omega^\gamma (z-u) + \eta_2\int_\Omega \omega^\gamma - \mu_2\int_\Omega \omega^{m^-+\gamma-1} + \mu_2|\Omega|. \tag{2.14}$$

We deal with the case of $\chi_2 \leq 0$ at first. By (2.14), we know that

$$\frac{1}{\gamma}\frac{d}{dt}\int_\Omega \omega^\gamma \leq -\frac{4(\gamma-1)}{\gamma^2}\int_\Omega \left|\nabla \omega^{\frac{\gamma}{2}}\right|^2 - \frac{(\gamma-1)\chi_2}{\gamma}\int_\Omega \omega^\gamma z + \eta_2\int_\Omega \omega^\gamma - \mu_2\int_\Omega \omega^{m^-+\gamma-1} + \mu_2|\Omega| \tag{2.15}$$

for all $t \in (s_0, T_{\max})$. By Young's inequality and (1.8) with $l = \frac{m^-+\gamma-1}{\gamma+1} > 0$, $h = \frac{\mu_1}{3C_7}$, there exist $C_6, C_7, C_8 > 0$ such that

$$\begin{aligned} -\frac{(\gamma-1)\chi_2}{\gamma}\int_\Omega \omega^\gamma z &\leq C_6\int_\Omega \omega^{\gamma+1} + C_7\int_\Omega z^{\gamma+1} \\ &\leq C_6\int_\Omega \omega^{\gamma+1} + \frac{\mu_1}{3}\int_\Omega u^{m^-+\gamma-1} + C_8. \end{aligned} \tag{2.16}$$

Substitute (2.16) into (2.15), we obtain

$$\begin{aligned} &\frac{1}{\gamma}\frac{d}{dt}\int_\Omega \omega^\gamma + \frac{m^-+\gamma-1}{\gamma}\int_\Omega \omega^\gamma \\ &\leq C_6\int_\Omega \omega^{\gamma+1} + \frac{\mu_1}{3}\int_\Omega u^{m^-+\gamma-1} + \left(\eta_2 + \frac{m^-+\gamma-1}{\gamma}\right)\int_\Omega \omega^\gamma \\ &- \left(\mu_2 - \frac{2\chi_1 C_{m^-,\gamma}}{m^-+\gamma-1}\right)\int_\Omega \omega^{m^-+\gamma-1} \\ &- \frac{2\chi_1 C_{m^-,\gamma}}{m^-+\gamma-1}\int_\Omega \omega^{m^-+\gamma-1} + C_9 \end{aligned} \tag{2.17}$$

with $C_9 = C_8 + \mu_2|\Omega| > 0$. Next by using (2.7) for μ_2 large, we obtain

$$C_6\int_\Omega \omega^{\gamma+1} - \left(\mu_2 - \frac{2\chi_1 C_{m^-,\gamma}}{m^-+\gamma-1}\right)\int_\Omega \omega^{m^-+\gamma-1} \leq C_{10} \tag{2.18}$$

with $C_{10} = C_6\left(\frac{C_6}{\left(\mu_2 - \frac{2\chi_1 C_{m^-,\gamma}}{m^-+\gamma-1}\right)}\right)^{\frac{\gamma+1}{m^--2}}|\Omega| > 0$ and

$$\left(\eta_2 + \frac{m^-+\gamma-1}{\gamma}\right)\int_\Omega \omega^\gamma - \frac{\chi_1 C_{m^-,\gamma}}{m^-+\gamma-1}\int_\Omega \omega^{m^-+\gamma-1} \leq C_{11} \tag{2.19}$$

with $C_{11} = \left(\eta_2 + \frac{m^-+\gamma-1}{\gamma}\right)\left[\frac{\left(\eta_2+\frac{m^-+\gamma-1}{\gamma}\right)(m^-+\gamma-1)}{C_{m^-,\gamma}\chi_1}\right]^{\frac{\gamma}{m^--1}}|\Omega| > 0.$

Combine (2.17) with (2.18) and (2.19), we have

$$\begin{aligned}
&\frac{1}{\gamma}\frac{d}{dt}\int_{\Omega}\omega^{\gamma}+\frac{m^{-}+\gamma-1}{\gamma}\int_{\Omega}\omega^{\gamma}\\
&\leq\frac{\mu_1}{3}\int_{\Omega}u^{m^{-}+\gamma-1}-\frac{\chi_1 C_{m^{-},\gamma}}{m^{-}+\gamma-1}\int_{\Omega}\omega^{m^{-}+\gamma-1}+C_{12}
\end{aligned}\tag{2.20}$$

for all $t\in(s_0,T_{\max})$ with s_0 defined in Lemma 1.3 and $C_{12}=C_9+C_{10}+C_{11}$. Applying the variation-of-constants formula to (2.20) , we obtain

$$\begin{aligned}
\frac{1}{\gamma}\int_{\Omega}\omega^{\gamma}\leq&\frac{\mu_1}{3}\int_{s_0}^{t}\int_{\Omega}e^{-\left(m^{-}+\gamma-1\right)(t-s)}u^{m^{-}+\gamma-1}\\
&-\frac{\chi_1 C_{m^{-},\gamma}}{m^{-}+\gamma-1}\int_{s_0}^{t}\int_{\Omega}e^{-\left(m^{-}+\gamma-1\right)(t-s)}\omega^{m^{-}+\gamma-1}+C_{13}.
\end{aligned}\tag{2.21}$$

Combine (2.21) with (2.12) to get

$$\int_{\Omega}u^{\gamma}+\int_{\Omega}\omega^{\gamma}\leq C_{14}$$

for all $t\in(s_0,T_{\max})$ with $C_{14}=(C_5+C_{13})\,\gamma>0$. So $\int_{\Omega}u^{\gamma}$,$\int_{\Omega}\omega^{\gamma}\leq C_{14}$.

Next consider the case of $\chi_2>0$. We know from (2.14) that

$$\begin{aligned}
\frac{1}{\gamma}\frac{d}{dt}\int_{\Omega}\omega^{\gamma}\leq&-\frac{4(\gamma-1)}{\gamma^2}\int_{\Omega}\left|\nabla\omega^{\frac{\gamma}{2}}\right|^2+\frac{(\gamma-1)\,\chi_2}{\gamma}\int_{\Omega}\omega^{\gamma}u\\
&+\eta_2\int_{\Omega}\omega^{\gamma}-\mu_2\int_{\Omega}\omega^{m^{-}+\gamma-1}+\mu_2\,|\Omega|\,.
\end{aligned}\tag{2.22}$$

Then, it is from (2.22) with Young's inequality that

$$\frac{(\gamma-1)\,\chi_2}{\gamma}\int_{\Omega}\omega^{\gamma}u\leq\frac{\mu_1}{3}\int_{\Omega}u^{m^{-}+\gamma-1}+C_{15}\int_{\Omega}\omega^{\frac{\gamma\left(m^{-}+\gamma-1\right)}{m^{-}+\gamma-2}}\tag{2.23}$$

with $C_{15}>0$. From (2.22) and (2.23), we get

$$\begin{aligned}
&\frac{1}{\gamma}\frac{d}{dt}\int_{\Omega}\omega^{\gamma}+\frac{m^{-}+\gamma-1}{\gamma}\int_{\Omega}\omega^{\gamma}\\
\leq&\frac{\mu_1}{3}\int_{\Omega}u^{m^{-}+\gamma-1}+C_{15}\int_{\Omega}\omega^{\frac{\gamma\left(m^{-}+\gamma-1\right)}{m^{-}+\gamma-2}}+\left(\eta_2+\frac{m^{-}+\gamma-1}{\gamma}\right)\int_{\Omega}\omega^{\gamma}\\
&-\left(\mu_2-\frac{2\chi_1 C_{m^{-},\gamma}}{m^{-}+\gamma-1}\right)\int_{\Omega}\omega^{m^{-}+\gamma-1}\\
&-\frac{2\chi_1 C_{m^{-},\gamma}}{m^{-}+\gamma-1}\int_{\Omega}\omega^{m^{-}+\gamma-1}+\mu_2\,|\Omega|\,.
\end{aligned}$$

By using (2.7) for μ_2 large and since $m^{-}>2$, we obtain

$$C_{15}\int_{\Omega}\omega^{\frac{\gamma\left(m^{-}+\gamma-1\right)}{m^{-}+\gamma-2}}-\frac{2\chi_1 C_{m^{-},\gamma}}{m^{-}+\gamma-1}\int_{\Omega}\omega^{m^{-}+\gamma-1}\leq C_{16},$$

where $C_{16} = C_{15} \left(\frac{\left(m^- + \gamma - 1\right) C_{15}}{2\chi_1 C_{m^-,\gamma}} \right)^{\frac{\gamma\left(m^- + \gamma - 1\right)}{(m^- + \gamma - 1)(m^- + \gamma - 2) - \gamma(m^- + \gamma - 1)}} |\Omega| > 0$ and

$$\left(\eta_2 + \frac{m^- + \gamma - 1}{\gamma} \right) \int_\Omega \omega^\gamma - \frac{\chi_1 C_{m^-,\gamma}}{m^- + \gamma - 1} \int_\Omega \omega^{m^- + \gamma - 1} \leq C_{17},$$

where $C_{17} = \left(\eta_2 + \frac{m^- + \gamma - 1}{\gamma} \right) \left(\frac{\left(\eta_2 + \frac{m^- + \gamma - 1}{\gamma} \right) \left(m^- + \gamma - 1 \right)}{\chi_1 C_{m^-,\gamma}} \right)^{\frac{\gamma}{m^- - 1}} |\Omega| > 0$. Next applying the variation-of-constants formula we obtain

$$\begin{aligned} \frac{1}{\gamma} \int_\Omega \omega^\gamma \leq & \frac{\mu_1}{3} \int_{s_0}^t \int_\Omega e^{-\left(m^- + \gamma - 1\right)(t-s)} u^{m^- + \gamma - 1} \\ & - \frac{\chi_1 C_{m^-,\gamma}}{m^- + \gamma - 1} \int_{s_0}^t \int_\Omega e^{-\left(m^- + \gamma - 1\right)(t-s)} \omega^{m^- + \gamma - 1} + C_{18} \end{aligned} \tag{2.24}$$

for all $t \in (s_0, T_{\max})$. Combine (2.24) with (2.12) to get

$$\int_\Omega u^\gamma, \int_\Omega \omega^\gamma \leq C_{19}$$

for all $t \in (s_0, T_{\max})$ with $C_{19} = (C_5 + C_{18}) \gamma > 0$. Together with (1.7), this verifies (2.1) and (2.2).

Finally, we can use the standard Alikakos-Moser iteration [2] to derive our main result. Take $\gamma = \gamma_0 > N$ in (2.1) and (2.2). Applying Lemma 1.2 to the second and fourth equations of (1.1), we obtain the L^∞ boundedness of ∇v and ∇z. Thus, all assumptions of [19], Lemma A.1 are satisfied. This concludes

$$\|u\|_\infty, \|\omega\|_\infty \leq C$$

with some $C > 0$ for all $t \in (0, T_{\max})$. By Lemma 1.1, we know that $T_{\max} = +\infty$. The proof of Theorem 1.1 is completed.

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