

Fractional maximal operator associated with Schrödinger operator and its commutators on vanishing generalized Morrey spaces

Ali Akbulut*, Suleyman Celik, Mehriban N.Omarova

Received: 21.06.2023 / Revised: 19.01.2024 / Accepted: 02.03.2024

Abstract. Let $\mathcal{L} = -\Delta + V$ be a Schrödinger operator, where the non-negative potential V belongs to the reverse Hölder class $RH_{n/2}$, let b belong to a new $BMO_\theta(\rho)$ space which is larger than the classical BMO space, and let $M_{\beta,V}^\theta$ be the fractional maximal operator associated with \mathcal{L} . In this paper, we study the boundedness of the operator $M_{\beta,V}^\theta$ and its commutators $[b, M_{\beta,V}^\theta]$ with $b \in BMO_\theta(\rho)$ on generalized Morrey spaces $M_{p,\varphi}^{\alpha,V}$ associated with Schrödinger operator and vanishing generalized Morrey spaces $VM_{p,\varphi}^{\alpha,V}$ associated with Schrödinger operator. We find the sufficient conditions on the pair (φ_1, φ_2) which ensures the boundedness of the operators $M_{\beta,V}^\theta$ from one vanishing generalized Morrey space $VM_{p,\varphi_1}^{\alpha,V}$ to another $VM_{q,\varphi_2}^{\alpha,V}$, $1/p - 1/q = \beta/n$.

Keywords Schrödinger operator; fractional maximal operator; commutator; BMO; generalized Morrey space.

MR(2010) Subject Classification 42B35, 35J10

1 Introduction and results

In this paper, we consider the Schrödinger differential operator

$$\mathcal{L} = -\Delta + V(x) \text{ on } \mathbb{R}^n, n \geq 3,$$

where $V(x)$ is a nonnegative potential belonging to the reverse Hölder class RH_q for $q \geq n/2$.

* Corresponding author

A. Akbulut
Department of Mathematics, Ahi Evran University, Kirsehir, Turkey
E-mail: akbulut72@gmail.com

S. Celik
Department of Mathematics, Ahi Evran University, Kirsehir, Turkey
E-mail: aydnsm125@gmail.com

M.N. Omarova
Baku State University, Baku, Azerbaijan
Azerbaijan University of Architecture and Construction, Baku, Azerbaijan
Institute of Mathematics and Mechanics, Baku, Azerbaijan
E-mail: mehriban_omarova@yahoo.com

A nonnegative locally L_q integrable function $V(x)$ on \mathbb{R}^n is said to belong to RH_q , $1 < q \leq \infty$ if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B(x,r)|} \int_{B(x,r)} V^q(y) dy \right)^{1/q} \leq \left(\frac{C}{|B(x,r)|} \int_{B(x,r)} V(y) dy \right) \quad (1.1)$$

holds for every $x \in \mathbb{R}^n$ and $0 < r < \infty$, where $B(x,r)$ denotes the ball centered at x with radius r . In particular, if V is a nonnegative polynomial, then $V \in RH_\infty$. Obviously, $RH_{q_2} \subset RH_{q_1}$, if $q_1 < q_2$. It is worth pointing out that the RH_q class is such that, if $V \in RH_q$ for some $q > 1$, then there exists an $\epsilon > 0$, which depends only n and the constant C in (1.1), such that $V \in RH_{q+\epsilon}$. Throughout this paper, we always assume that $0 \neq V \in RH_{n/2}$.

For $x \in \mathbb{R}^n$, the function $\rho(x)$ is defined by

$$\rho(x) := \frac{1}{m_V(x)} = \sup_{r>0} \left\{ r : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y) dy \leq 1 \right\}.$$

Obviously, $0 < m_V(x) < \infty$ if $V \neq 0$. In particular, $m_V(x) = 1$ with $V = 1$ and $m_V(x) \sim 1 + |x|$ with $V(x) = |x|^2$.

According to [3], the new BMO space $BMO_\theta(\rho)$ with $\theta \geq 0$ is defined as a set of all locally integrable functions b such that

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_B| dy \leq C \left(1 + \frac{r}{\rho(x)} \right)^\theta$$

for all $x \in \mathbb{R}^n$ and $r > 0$, where $b_B = \frac{1}{|B|} \int_B b(y) dy$. A norm for $b \in BMO_\theta(\rho)$, denoted by $[b]_\theta$, is given by the infimum of the constants in the inequalities above. Clearly, $BMO \subset BMO_\theta(\rho)$.

The classical Morrey spaces were originally introduced by Morrey in [16] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [7, 8, 11, 16]. The classical version of Morrey spaces is equipped with the norm

$$\|f\|_{M_{p,\lambda}} := \sup_{x \in \mathbb{R}^n} \sup_{r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))},$$

where $0 \leq \lambda < n$ and $1 \leq p < \infty$. The generalized Morrey spaces are defined with r^λ replaced by a general non-negative function $\varphi(x,r)$ satisfying some assumptions (see, for example, [2, 9–11, 15, 17, 18] and etc).

The vanishing Morrey space $VM_{p,\lambda}$ in its classical version was introduced in [24], where applications to PDE were considered. We also refer to [5] and [19] for some properties of such spaces. This is a subspace of functions in $M_{p,\lambda}(\mathbb{R}^n)$, which satisfy the condition

$$\lim_{r \rightarrow 0} \sup_{x \in \mathbb{R}^n, 0 < t < r} t^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,t))} = 0.$$

Moreover, various Morrey spaces are defined in the process of study. Guliyev, Mizuhara and Nakai [9, 17, 18] introduced generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$ (see, also [2, 11, 20]).

We now present the definition of generalized Morrey spaces (including weak version) associated with Schrödinger operator, which introduced by Guliyev in [12].

Definition 1.1 Let $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$, $1 \leq p < \infty$, $\alpha \geq 0$, and $V \in RH_q$, $q \geq 1$. We denote by $M_{p,\varphi}^{\alpha,V} = M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ the generalized Morrey space associated with Schrödinger operator, the space of all functions $f \in L_{loc}^p(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{p,\varphi}^{\alpha,V}} = \sup_{x \in \mathbb{R}^n, r > 0} \left(1 + \frac{r}{\rho(x)}\right)^\alpha \varphi(x, r)^{-1} r^{-n/p} \|f\|_{L_p(B(x,r))}.$$

Also $WM_{p,\varphi}^{\alpha,V} = WM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ we denote the weak generalized Morrey space associated with Schrödinger operator, the space of all functions $f \in WL_{loc}^p(\mathbb{R}^n)$ with

$$\|f\|_{WM_{p,\varphi}^{\alpha,V}} = \sup_{x \in \mathbb{R}^n, r > 0} \left(1 + \frac{r}{\rho(x)}\right)^\alpha \varphi(x, r)^{-1} r^{-n/p} \|f\|_{WL_p(B(x,r))} < \infty.$$

Remark 1.1 (i) When $\alpha = 0$, and $\varphi(x, r) = r^{(\lambda-n)/p}$, $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ is the classical Morrey space $L_{p,\lambda}(\mathbb{R}^n)$ introduced by Morrey in [16];

(ii) When $\varphi(x, r) = r^{(\lambda-n)/p}$, $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ is the Morrey space associated with Schrödinger operator $L_{p,\lambda}^{\alpha,V}(\mathbb{R}^n)$ studied by Tang and Dong in [22];

(iii) When $\alpha = 0$, $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ is the generalized Morrey space $M_{p,\varphi}(\mathbb{R}^n)$ introduced by Guliyev, Mizuhara and Nakai in [9, 17, 18].

(iv) The generalized Morrey space associated with Schrödinger operator $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ was introduced by Guliyev in [12].

For brevity, in the sequel we use the notations

$$\mathfrak{A}_{p,\varphi}^{\alpha,V}(f; x, r) := \left(1 + \frac{r}{\rho(x)}\right)^\alpha r^{-n/p} \varphi(x, r)^{-1} \|f\|_{L_p(B(x,r))}$$

and

$$\mathfrak{A}_{\Phi,\varphi}^{W,\alpha,V}(f; x, r) := \left(1 + \frac{r}{\rho(x)}\right)^\alpha r^{-n/p} \varphi(x, r)^{-1} \|f\|_{WL_p(B(x,r))}.$$

Definition 1.2 The vanishing generalized Morrey space associated with Schrödinger operator $VM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ such that

$$\limsup_{r \rightarrow 0} \sup_{x \in \mathbb{R}^n} \mathfrak{A}_{p,\varphi}^{\alpha,V}(f; x, r) = 0. \quad (1.2)$$

The vanishing weak generalized Morrey space associated with Schrödinger operator $VWM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ is defined as the spaces of functions $f \in WM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ such that

$$\limsup_{r \rightarrow 0} \sup_{x \in \mathbb{R}^n} \mathfrak{A}_{\Phi,\varphi}^{W,\alpha,V}(f; x, r) = 0.$$

The vanishing spaces $VM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ and $VWM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ are Banach spaces with respect to the norm

$$\begin{aligned} \|f\|_{VM_{p,\varphi}^{\alpha,V}} &\equiv \|f\|_{M_{p,\varphi}^{\alpha,V}} = \sup_{x \in \mathbb{R}^n, r > 0} \mathfrak{A}_{p,\varphi}^{\alpha,V}(f; x, r), \\ \|f\|_{VWM_{p,\varphi}^{\alpha,V}} &\equiv \|f\|_{WM_{p,\varphi}^{\alpha,V}} = \sup_{x \in \mathbb{R}^n, r > 0} \mathfrak{A}_{W,p,\varphi}^{\alpha,V}(f; x, r), \end{aligned}$$

respectively.

Given a function $f \in L^1_{loc}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator M is defined by

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy,$$

and the fractional operator function M_β is defined by

$$M_\beta f(x) := \sup_{r>0} \frac{1}{|B(x,r)|^{1-\frac{\beta}{n}}} \int_{B(x,r)} |f(y)| dy, \quad 0 < \beta < n.$$

Definition 1.3 Let $\mathcal{L} = -\Delta + V$ with $V \in RH_{n/2}$. A variant of Hardy-Littlewood maximal operator M_V^θ (see [3]) is defined by

$$M_V^\theta f(x) := \sup_{r>0} \frac{1}{\Psi_\theta(B(x,r))|B(x,r)|} \int_{B(x,r)} |f(y)| dy,$$

and a variant of fractional maximal operator $M_{\beta,V}^\theta$ (see [23]) is defined by

$$M_{\beta,V}^\theta f(x) := \sup_{r>0} \frac{1}{(\Psi_\theta(B(x,r))|B(x,r)|)^{1-\frac{\beta}{n}}} \int_{B(x,r)} |f(y)| dy, \quad 0 < \beta < n.$$

The fractional integral associated with \mathcal{L} is defined by

$$\mathcal{I}_\beta f(x) = \mathcal{L}^{-\beta/2} f(x) = \int_0^\infty e^{-t\mathcal{L}} f(x) \frac{dt}{t^{-\beta/2+1}}$$

for $0 < \beta < n$. Let $b \in BMO_\theta(\rho)$. The commutator of \mathcal{I}_β is defined by

$$[b, \mathcal{I}_\beta]f(x) = b(x)\mathcal{I}_\beta f(x) - \mathcal{I}_\beta(bf)(x).$$

We now present the definition of generalized Morrey spaces related to certain nonnegative potentials.

In this paper, we consider the boundedness of the fractional integral operator $M_{\beta,V}^\theta$ on the generalized Morrey spaces $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ and the vanishing generalized Morrey spaces $VM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$. When b belongs to the new BMO space $BMO_\theta(\rho)$, we also show that $[b, M_{\beta,V}^\theta]$ is bounded on $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ to $M_{q,\varphi}^{\alpha,V}(\mathbb{R}^n)$. Our main results are as follows.

Theorem 1.1 Let $V \in RH_{n/2}$, $\alpha \geq 0$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ and $\varphi_1 \in \Omega_p^{\alpha,V}$, $\varphi_2 \in \Omega_q^{\alpha,V}$ satisfies the condition

$$\sup_{r<t<\infty} \frac{\text{ess inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{q}}} \leq c_0 \varphi_2(x,r), \quad (1.3)$$

where c_0 does not depend on x and r . Then the operator $M_{\beta,V}^\theta$ is bounded on $M_{p,\varphi_1}^{\alpha,V}$ to $M_{q,\varphi_2}^{\alpha,V}$ for $p > 1$ and from $M_{1,\varphi_1}^{\alpha,V}$ to $WM_{\frac{n}{n-\beta},\varphi_2}^{\alpha,V}$. Moreover, for $p > 1$

$$\|M_{\beta,V}^\theta f\|_{M_{q,\varphi_2}^{\alpha,V}} \leq C \|f\|_{M_{p,\varphi_1}^{\alpha,V}},$$

and for $p = 1$

$$\|M_{\beta,V}^\theta f\|_{WM_{\frac{n}{n-\beta},\varphi_2}^{\alpha,V}} \leq C \|f\|_{M_{1,\varphi_1}^{\alpha,V}},$$

where C does not depend on f .

Theorem 1.2 Let $V \in RH_{n/2}$, $\alpha \geq 0$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ and $\varphi_1 \in \Omega_p^{\alpha,V}$, $\varphi_2 \in \Omega_q^{\alpha,V}$ satisfies the condition

$$\sup_{r < t < \infty} \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname{ess\,inf}_{t < s < \infty} \varphi_1(x, s) s^{\frac{n}{p}}}{t^{\frac{n}{q}}} \leq c_0 \varphi_2(x, r), \quad (1.4)$$

where c_0 does not depend on x and r . If $b \in BMO_\theta(\rho)$, then the operator $[b, M_{\beta,V}^\theta]$ is bounded from $M_{p,\varphi_1}^{\alpha,V}$ to $M_{q,\varphi_2}^{\alpha,V}$ and

$$\|[b, M_{\beta,V}^\theta]f\|_{M_{q,\varphi_2}^{\alpha,V}} \leq C[b]_\theta \|f\|_{M_{p,\varphi_1}^{\alpha,V}},$$

where C does not depend on f .

Theorem 1.3 Let $V \in RH_{n/2}$, $\alpha \geq 0$, $1 \leq p < \infty$ and $\varphi_1 \in \Omega_{p,1}^{\alpha,V}$, $\varphi_2 \in \Omega_{q,1}^{\alpha,V}$ satisfies the conditions

$$c_\delta := \int_\delta^\infty \sup_{x \in \mathbb{R}^n} \varphi_1(x, t) \frac{dt}{t} < \infty$$

for every $\delta > 0$, and

$$\int_r^\infty \varphi_1(x, t) \frac{dt}{t^{1-\beta}} \leq C_0 \varphi_2(x, r), \quad (1.5)$$

where C_0 does not depend on $x \in \mathbb{R}^n$ and $r > 0$. Then the operator $M_{\beta,V}^\theta$ is bounded from $VM_{p,\varphi_1}^{\alpha,V}$ to $VM_{q,\varphi_2}^{\alpha,V}$ for $p > 1$ and from $VM_{1,\varphi_1}^{\alpha,V}$ to $VWM_{\frac{n}{n-\beta},\varphi_2}^{\alpha,V}$.

Theorem 1.4 Let $V \in RH_{n/2}$, $b \in BMO_\theta(\rho)$, $1 < p < \infty$, and $\varphi_1 \in \Omega_{p,1}^{\alpha,V}$, $\varphi_2 \in \Omega_{q,1}^{\alpha,V}$ satisfies the conditions

$$\sup_{r < t < \infty} \left(1 + \ln \frac{t}{r}\right) \varphi_1(x, t) t^\beta \leq c_0 \varphi_2(x, r), \quad (1.6)$$

where c_0 does not depend on x and r ,

$$\lim_{r \rightarrow 0} \frac{\ln \frac{1}{r}}{\inf_{x \in \mathbb{R}^n} \varphi_2(x, r)} = 0 \quad (1.7)$$

and

$$c_\delta := \int_\delta^\infty \left(1 + |\ln t|\right) \sup_{x \in \mathbb{R}^n} \varphi_1(x, t) \frac{dt}{t^{1-\beta}} < \infty \quad (1.8)$$

for every $\delta > 0$. Then the operator $[b, M_{\beta,V}^\theta]$ is bounded from $VM_{p,\varphi_1}^{\alpha,V}$ to $VM_{q,\varphi_2}^{\alpha,V}$.

In this paper, we shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant C , independent of all important parameters, such that $A \leq CB$. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2 Some Preliminaries

We would like to recall the important properties concerning the critical function.

Lemma 2.1 [21] *Let $V \in RH_{n/2}$. For the associated function ρ there exist C and $k_0 \geq 1$ such that*

$$C^{-1}\rho(x)\left(1 + \frac{|x-y|}{\rho(x)}\right)^{-k_0} \leq \rho(y) \leq C\rho(x)\left(1 + \frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{1+k_0}} \quad (2.1)$$

for all $x, y \in \mathbb{R}^n$.

Lemma 2.2 *Suppose $x \in B(x_0, r)$. Then for $k \in \mathbb{N}$ we have*

$$\frac{1}{\left(1 + \frac{2^k r}{\rho(x)}\right)^N} \lesssim \frac{1}{\left(1 + \frac{2^k r}{\rho(x_0)}\right)^{N/(k_0+1)}}.$$

Proof. By (2.1) we have

$$\begin{aligned} \frac{1}{\left(1 + \frac{2^k r}{\rho(x)}\right)^N} &\lesssim \frac{1}{\left(1 + \frac{2^k r}{\rho(x_0)\left(1 + \frac{|x-x_0|}{\rho(x_0)}\right)^{\frac{k_0}{k_0+1}}}\right)^N} \\ &\lesssim \frac{\left(1 + \frac{|x-x_0|}{\rho(x_0)}\right)^{\frac{k_0 N}{k_0+1}}}{\left(1 + \frac{2^k r}{\rho(x_0)}\right)^N} \lesssim \frac{1}{\left(1 + \frac{2^k r}{\rho(x_0)}\right)^{N/(k_0+1)}}. \end{aligned}$$

We give some inequalities about the new BMO space $BMO_\theta(\rho)$.

Lemma 2.3 [3] *Let $1 \leq s < \infty$. If $b \in BMO_\theta(\rho)$, then*

$$\left(\frac{1}{|B|} \int_B |b(y) - b_B|^s dy\right)^{1/s} \leq [b]_\theta \left(1 + \frac{r}{\rho(x)}\right)^{\theta'}$$

for all $B = B(x, r)$, with $x \in \mathbb{R}^n$ and $r > 0$, where $\theta' = (k_0 + 1)\theta$ and k_0 is the constant appearing in (2.1).

Lemma 2.4 [3] *Let $1 \leq s < \infty$, $b \in BMO_\theta(\rho)$, and $B = B(x, r)$. Then*

$$\left(\frac{1}{|2^k B|} \int_{2^k B} |b(y) - b_B|^s dy\right)^{1/s} \leq [b]_\theta k \left(1 + \frac{2^k r}{\rho(x)}\right)^{\theta'}$$

for all $k \in \mathbb{N}$, with θ' as in Lemma 2.3.

Let K_β be the kernel of \mathcal{I}_β . The following result give the estimate on the kernel $K_\beta(x, y)$.

Lemma 2.5 [4] *If $V \in RH_{n/2}$, then for every N , there exists a constant C such that*

$$|K_\beta(x, y)| \leq \frac{C}{\left(1 + \frac{|x-y|}{\rho(x)}\right)^N} \frac{1}{|x-y|^{n-\beta}}. \quad (2.2)$$

Finally, we recall a relationship between essential supremum and essential infimum.

Lemma 2.6 [25] *Let f be a real-valued nonnegative function and measurable on E . Then*

$$\left(\operatorname{ess\,inf}_{x \in E} f(x) \right)^{-1} = \operatorname{ess\,sup}_{x \in E} \frac{1}{f(x)}.$$

Lemma 2.7 *Let $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$, $1 \leq p < \infty$, $\alpha \geq 0$, and $V \in RH_q$, $q \geq 1$.*

(i) *If*

$$\sup_{t < r < \infty} \left(1 + \frac{r}{\rho(x)} \right)^\alpha \frac{r^{-\frac{n}{p}}}{\varphi(x, r)} = \infty \quad \text{for some } t > 0 \text{ and for all } x \in \mathbb{R}^n, \quad (2.3)$$

then $M_{p, \varphi}^{\alpha, V}(\mathbb{R}^n) = \Theta$.

(ii) *If*

$$\sup_{0 < r < \tau} \left(1 + \frac{r}{\rho(x)} \right)^\alpha \varphi(x, r)^{-1} = \infty \quad \text{for some } \tau > 0 \text{ and for all } x \in \mathbb{R}^n, \quad (2.4)$$

then $M_{p, \varphi}^{\alpha, V}(\mathbb{R}^n) = \Theta$.

Proof. (i) Let (2.3) be satisfied and f be not equivalent to zero. Then $\sup_{x \in \mathbb{R}^n} \|f\|_{L_p(B(x, t))} > 0$, hence

$$\begin{aligned} \|f\|_{M_{p, \varphi}^{\alpha, V}} &\geq \sup_{x \in \mathbb{R}^n} \sup_{t < r < \infty} \left(1 + \frac{r}{\rho(x)} \right)^\alpha \varphi(x, r)^{-1} r^{-\frac{n}{p}} \|f\|_{L_p(B(x, r))} \\ &\geq \sup_{x \in \mathbb{R}^n} \|f\|_{L_p(B(x, t))} \sup_{t < r < \infty} \left(1 + \frac{r}{\rho(x)} \right)^\alpha \varphi(x, r)^{-1} r^{-\frac{n}{p}}. \end{aligned}$$

Therefore $\|f\|_{M_{p, \varphi}^{\alpha, V}} = \infty$.

(ii) Let $f \in M_{p, \varphi}^{\alpha, V}(\mathbb{R}^n)$ and (2.4) be satisfied. Then there are two possibilities:

Case 1: $\sup_{0 < r < t} \left(1 + \frac{r}{\rho(x)} \right)^\alpha \varphi(x, r)^{-1} = \infty$ for all $t > 0$.

Case 2: $\sup_{0 < r < t} \left(1 + \frac{r}{\rho(x)} \right)^\alpha \varphi(x, r)^{-1} < \infty$ for some $t \in (0, \tau)$.

For Case 1, by Lebesgue differentiation theorem, for almost all $x \in \mathbb{R}^n$,

$$\lim_{r \rightarrow 0^+} \frac{\|f \chi_{B(x, r)}\|_{L_p}}{\|\chi_{B(x, r)}\|_{L_p}} = |f(x)|. \quad (2.5)$$

We claim that $f(x) = 0$ for all those x . Indeed, fix x and assume $|f(x)| > 0$. Then by (2.5) there exists $t_0 > 0$ such that

$$r^{-\frac{n}{p}} \|f\|_{L_p(B(x, r))} \geq 2^{-1} v_n^{\frac{1}{p}} |f(x)|$$

for all $0 < r \leq t_0$. Consequently,

$$\begin{aligned} \|f\|_{M_{p, \varphi}^{\alpha, V}} &\geq \sup_{0 < r < t_0} \left(1 + \frac{r}{\rho(x)} \right)^\alpha \varphi(x, r)^{-1} r^{-\frac{n}{p}} \|f\|_{L_p(B(x, r))} \\ &\geq 2^{-1} v_n^{\frac{1}{p}} |f(x)| \sup_{0 < r < t_0} \left(1 + \frac{r}{\rho(x)} \right)^\alpha \varphi(x, r)^{-1}. \end{aligned}$$

Hence $\|f\|_{M_{p, \varphi}^{\alpha, V}} = \infty$, so $f \notin M_{p, \varphi}(\mathbb{R}^n)$ and we have arrived at a contradiction.

Note that Case 2 implies that $\sup_{s < r < \tau} \left(1 + \frac{r}{\rho(x)}\right)^\alpha \varphi(x, r)^{-1} = \infty$, hence

$$\begin{aligned} \sup_{s < r < \infty} \left(1 + \frac{r}{\rho(x)}\right)^\alpha \varphi(x, r)^{-1} r^{-\frac{n}{p}} &\geq \sup_{s < r < \tau} \left(1 + \frac{r}{\rho(x)}\right)^\alpha \varphi(x, r)^{-1} r^{-\frac{n}{p}} \\ &\geq \tau^{-\frac{n}{p}} \sup_{s < r < \tau} \left(1 + \frac{r}{\rho(x)}\right)^\alpha \varphi(x, r)^{-1} = \infty, \end{aligned}$$

which is the case in (i).

Remark 2.1 We denote by $\Omega_p^{\alpha, V}$ the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0, \infty)$ such that for all $t > 0$,

$$\sup_{x \in \mathbb{R}^n} \left\| \left(1 + \frac{r}{\rho(x)}\right)^\alpha \frac{r^{-\frac{n}{p}}}{\varphi(x, r)} \right\|_{L_\infty(t, \infty)} < \infty, \quad \text{and} \quad \sup_{x \in \mathbb{R}^n} \left\| \left(1 + \frac{r}{\rho(x)}\right)^\alpha \varphi(x, r)^{-1} \right\|_{L_\infty(0, t)} < \infty,$$

respectively. In what follows, keeping in mind Lemma 2.7, we always assume that $\varphi \in \Omega_p^{\alpha, V}$.

Remark 2.2 We denote by $\Omega_{p,1}^{\alpha, V}$ the sets of all positive measurable functions φ on $\mathbb{R}^n \times (0, \infty)$ such that

$$\inf_{x \in \mathbb{R}^n} \inf_{r > \delta} \left(1 + \frac{r}{\rho(x)}\right)^{-\alpha} \varphi(x, r) > 0, \quad \text{for some } \delta > 0, \quad (2.6)$$

and

$$\lim_{r \rightarrow 0} \left(1 + \frac{r}{\rho(x)}\right)^\alpha \frac{r^{n/p}}{\varphi(x, r)} = 0,$$

For the non-triviality of the space $VM_{p,\varphi}^{\alpha, V}(\mathbb{R}^n)$ we always assume that $\varphi \in \Omega_{p,1}^{\alpha, V}$.

3 Proof of Theorem 1.1

We first prove the following conclusions

Theorem 3.1 *Let $V \in RH_{n/2}$. If $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ then the inequality*

$$\|M_{\beta, V}^\theta f\|_{L_q(B(x_0, r))} \lesssim r^{\frac{n}{q}} \sup_{2r < t < \infty} \frac{\|f\|_{L_p(B(x_0, t))}}{t^{\frac{n}{q}}}$$

holds for any $f \in L_{loc}^p(\mathbb{R}^n)$. Moreover, for $p = 1$ the inequality

$$\|M_{\beta, V}^\theta f\|_{WL_{\frac{n}{n-\beta}}(B(x_0, r))} \lesssim r^{n-\beta} \sup_{2r < t < \infty} \frac{\|f\|_{L_1(B(x_0, t))}}{t^{n-\beta}}$$

holds for any $f \in L_{loc}^1(\mathbb{R}^n)$.

Proof. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ and $\lambda B = B(x_0, \lambda r)$ for any $\lambda > 0$. We write f as $f = f_1 + f_2$, where $f_1(y) = f(y)\chi_{B(x_0, 2r)}(y)$, and $\chi_{B(x_0, 2r)}$ denotes the characteristic function of $B(x_0, 2r)$. Then

$$\|M_{\beta, V}^\theta f\|_{L_q(B(x_0, r))} \leq \|M_{\beta, V}^\theta(f_1)\|_{L_q(B(x_0, r))} + \|M_{\beta, V}^\theta(f_2)\|_{L_q(B(x_0, r))}.$$

Since $f_1 \in L_p(\mathbb{R}^n)$ and from the boundedness of \mathcal{I}_β from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ it follows that

$$\begin{aligned} \|M_{\beta, V}^\theta(f_1)\|_{L_q(B(x_0, r))} &\lesssim \|f\|_{L_p(B(x_0, 2r))} \\ &\lesssim r^{\frac{n}{q}} \|f\|_{L_p(B(x_0, 2r))} \int_{2r}^\infty \frac{dt}{t^{\frac{n}{q}+1}} \\ &\lesssim r^{\frac{n}{q}} \int_{2r}^\infty \frac{\|f\|_{L_p(B(x_0, t))}}{t^{\frac{n}{q}}} \frac{dt}{t}. \end{aligned} \quad (3.1)$$

To estimate $\|M_{\beta, V}^\theta(f_2)\|_{L_p(B(x_0, r))}$, observe that $x \in B$, $y \in (2B)^c$ implies $|x - y| \approx |x_0 - y|$. Then by (2.2) we have

$$\begin{aligned} \sup_{x \in B} |M_{\beta, V}^\theta(f_2)(x)| &\leq \int_{(2B)^c} |K_\beta(x, y) f(y)| dy \\ &\lesssim \int_{(2B)^c} \frac{|f(y)|}{|x_0 - y|^{n-\beta}} dy \\ &\lesssim \sum_{k=1}^\infty (2^{k+1}r)^{-n+\beta} \int_{2^{k+1}B} |f(y)| dy. \end{aligned}$$

By Hölder's inequality we get

$$\begin{aligned} \sup_{x \in B} |M_{\beta, V}^\theta(f_2)(x)| &\lesssim \sum_{k=1}^\infty \|f\|_{L_p(2^{k+1}B)} (2^{k+1}r)^{-1-\frac{n}{p}+\beta} \int_{2^{k+1}B} dt \\ &\lesssim \sum_{k=1}^\infty \int_{2^{k+1}B} \frac{\|f\|_{L_p(B(x_0, t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \\ &\lesssim \int_{2r}^\infty \frac{\|f\|_{L_p(B(x_0, t))}}{t^{\frac{n}{q}}} \frac{dt}{t}. \end{aligned} \quad (3.2)$$

Then

$$\|M_{\beta, V}^\theta(f_2)\|_{L_q(B(x_0, r))} \lesssim r^{\frac{n}{q}} \int_{2r}^\infty \frac{\|f\|_{L_p(B(x_0, t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \quad (3.3)$$

holds for $1 \leq p < n/\beta$. Therefore, by (3.1) and (3.3) we get

$$\|M_{\beta, V}^\theta f\|_{L_q(B(x_0, r))} \lesssim r^{\frac{n}{q}} \int_{2r}^\infty \frac{\|f\|_{L_p(B(x_0, t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \quad (3.4)$$

holds for $1 \leq p < n/\beta$.

When $p = 1$, by the boundedness of $M_{\beta, V}^\theta$ from $L_1(\mathbb{R}^n)$ to $WL_{\frac{n}{n-\beta}}(\mathbb{R}^n)$, we get

$$\|M_{\beta, V}^\theta(f_1)\|_{WL_{\frac{n}{n-\beta}}(B(x_0, r))} \lesssim \|f\|_{L_1(B(x_0, 2r))} \lesssim r^{n-\beta} \int_{2r}^\infty \frac{\|f\|_{L_1(B(x_0, t))}}{t^{n-\beta}} \frac{dt}{t}.$$

By (3.3) we have

$$\|M_{\beta,V}^\theta(f_2)\|_{WL_{\frac{n}{n-\beta}}(B(x_0,r))} \leq \|M_{\beta,V}^\theta(f_2)\|_{L_{\frac{n}{n-\beta}}(B(x_0,2r))} \lesssim r^{n-\beta} \int_{2r}^\infty \frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta}} \frac{dt}{t}.$$

Then

$$\|M_{\beta,V}^\theta f\|_{WL_{\frac{n}{n-\beta}}(B(x_0,r))} \lesssim r^{n-\beta} \int_{2r}^\infty \frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta}} \frac{dt}{t}.$$

Proof of Theorem 1.1 From Lemma 2.6, we have

$$\frac{1}{\operatorname{ess\,inf}_{t < s < \infty} \varphi_1(x,s) s^{\frac{n}{p}}} = \operatorname{ess\,sup}_{t < s < \infty} \frac{1}{\varphi_1(x,s) s^{\frac{n}{p}}}.$$

Note the fact that $\|f\|_{L_p(B(x_0,t))}$ is a nondecreasing function of t , and $f \in M_{p,\varphi_1}^{\alpha,V}$, then

$$\begin{aligned} & \frac{\left(1 + \frac{t}{\rho(x_0)}\right)^\alpha \|f\|_{L_p(B(x_0,t))}}{\operatorname{ess\,inf}_{t < s < \infty} \varphi_1(x,s) s^{\frac{n}{p}}} \\ & \lesssim \operatorname{ess\,sup}_{t < s < \infty} \frac{\left(1 + \frac{t}{\rho(x_0)}\right)^\alpha \|f\|_{L_p(B(x_0,t))}}{\varphi_1(x,s) s^{\frac{n}{p}}} \\ & \lesssim \sup_{0 < s < \infty} \frac{\left(1 + \frac{s}{\rho(x_0)}\right)^\alpha \|f\|_{L_p(B(x_0,s))}}{\varphi_1(x,s) s^{\frac{n}{p}}} \\ & \lesssim \|f\|_{M_{p,\varphi_1}^{\alpha,V}}. \end{aligned}$$

Since $\alpha \geq 0$, and (φ_1, φ_2) satisfies the condition (1.3), then

$$\begin{aligned} & \int_{2r}^\infty \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \\ & = \int_{2r}^\infty \frac{\left(1 + \frac{t}{\rho(x_0)}\right)^\alpha \|f\|_{L_p(B(x_0,t))}}{\operatorname{ess\,inf}_{t < s < \infty} \varphi_1(x,s) s^{\frac{n}{p}}} \frac{\operatorname{ess\,inf}_{t < s < \infty} \varphi_1(x,s) s^{\frac{n}{p}}}{\left(1 + \frac{t}{\rho(x_0)}\right)^\alpha t^{\frac{n}{q}}} \frac{dt}{t} \\ & \lesssim \|f\|_{M_{p,\varphi_1}^{\alpha,V}} \int_{2r}^\infty \frac{\operatorname{ess\,inf}_{t < s < \infty} \varphi_1(x,s) s^{\frac{n}{p}}}{\left(1 + \frac{t}{\rho(x_0)}\right)^\alpha t^{\frac{n}{q}}} \frac{dt}{t} \\ & \lesssim \|f\|_{M_{p,\varphi_1}^{\alpha,V}} \left(1 + \frac{r}{\rho(x_0)}\right)^{-\alpha} \int_r^\infty \frac{\operatorname{ess\,inf}_{t < s < \infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{q}}} \frac{dt}{t} \\ & \lesssim \|f\|_{M_{p,\varphi_1}^{\alpha,V}} \left(1 + \frac{r}{\rho(x_0)}\right)^{-\alpha} \varphi_2(x_0, r). \end{aligned} \tag{3.5}$$

Then by Theorem 3.1 we get

$$\begin{aligned}
& \|M_{\beta,V}^\theta f\|_{M_{q,\varphi_2}^{\alpha,V}} \\
& \lesssim \sup_{x_0 \in \mathbb{R}^n, r > 0} \left(1 + \frac{r}{\rho(x_0)}\right)^\alpha \varphi_2(x_0, r)^{-1} r^{-n/q} \|\mathcal{I}_\beta f\|_{L_p(B(x_0, r))} \\
& \lesssim \sup_{x_0 \in \mathbb{R}^n, r > 0} \left(1 + \frac{r}{\rho(x_0)}\right)^\alpha \varphi_2(x_0, r)^{-1} \int_{2r}^\infty \frac{\|f\|_{L_p(B(x_0, t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \\
& \lesssim \|f\|_{M_{p,\varphi_1}^{\alpha,V}}.
\end{aligned}$$

Let $q = \frac{n}{n-\beta}$, similar to the estimates of (3.5) we have

$$\int_{2r}^\infty \frac{\|f\|_{L_1(B(x_0, t))}}{t^{n-\beta}} \frac{dt}{t} \lesssim \|f\|_{M_{1,\varphi_1}^{\alpha,V}} \left(1 + \frac{r}{\rho(x_0)}\right)^{-\alpha} \varphi_2(x_0, r).$$

Thus by Theorem 3.1 we get

$$\begin{aligned}
& \|M_{\beta,V}^\theta f\|_{WM_{\frac{n}{n-\beta}, \varphi_2}^{\alpha,V}} \\
& \lesssim \sup_{x_0 \in \mathbb{R}^n, r > 0} \left(1 + \frac{r}{\rho(x_0)}\right)^\alpha \varphi_2(x_0, r)^{-1} r^{\beta-n} \|\mathcal{I}_\beta f\|_{WL_{\frac{n}{n-\beta}}(B(x_0, r))} \\
& \lesssim \sup_{x_0 \in \mathbb{R}^n, r > 0} \left(1 + \frac{r}{\rho(x_0)}\right)^\alpha \varphi_2(x_0, r)^{-1} \int_{2r}^\infty \frac{\|f\|_{L_1(B(x_0, t))}}{t^{n-\beta}} \frac{dt}{t} \\
& \lesssim \|f\|_{M_{1,\varphi_1}^{\alpha,V}}.
\end{aligned}$$

4 Proof of Theorem 1.2

As the proof of Theorem 1.1, it suffices to prove the following result.

Theorem 4.1 *Let $V \in RH_{n/2}$, $b \in BMO_\theta(\rho)$. If $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ then the inequality*

$$\|[b, M_{\beta,V}^\theta f]\|_{L_q(B(x_0, r))} \lesssim [b]_\theta r^{\frac{n}{q}} \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0, t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \quad (4.1)$$

holds for any $f \in L_{loc}^p(\mathbb{R}^n)$.

Proof. We write f as $f = f_1 + f_2$, where $f_1(y) = f(y)\chi_{B(x_0, 2r)}(y)$. Then

$$\|[b, M_{\beta,V}^\theta f]\|_{L_q(B(x_0, r))} \leq \|[b, M_{\beta,V}^\theta](f_1)\|_{L_q(B(x_0, r))} + \|[b, M_{\beta,V}^\theta](f_2)\|_{L_q(B(x_0, r))}.$$

By the boundedness of $[b, M_{\beta,V}^\theta]$ on $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ and (3.1) we get

$$\begin{aligned}
\|[b, M_{\beta,V}^\theta](f_1)\|_{L_q(B(x_0, r))} & \lesssim [b]_\theta \|f\|_{L_p(B(x_0, 2r))} \\
& \lesssim [b]_\theta r^{\frac{n}{q}} \int_{2r}^\infty \frac{\|f\|_{L_p(B(x_0, t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \\
& \lesssim [b]_\theta r^{\frac{n}{q}} \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0, t))}}{t^{\frac{n}{q}}} \frac{dt}{t}. \quad (4.2)
\end{aligned}$$

We now turn to deal with the term $\|[b, M_{\beta, V}^\theta](f_2)\|_{L_q(B(x_0, r))}$. For any given $x \in B(x_0, r)$ we have

$$|[b, M_{\beta, V}^\theta]f_2(x)| \leq |b(x) - b_{2B}| |\mathcal{I}_\beta(f_2)(x)| + |\mathcal{I}_\beta((b - b_{2B})f_2)(x)|.$$

By (2.2), Lemma 2.2 and (3.2) we have

$$\begin{aligned} \sup_{x \in B} |M_{\beta, V}^\theta(f_2)(x)| &\lesssim \int_{(2B)^c} \frac{1}{\left(1 + \frac{|x-y|}{\rho(x)}\right)^N} \frac{|f(y)|}{|x_0 - y|^{n-\beta}} dy \\ &\lesssim \frac{1}{\left(1 + \frac{2r}{\rho(x)}\right)^N} \sum_{k=1}^{\infty} (2^{k+1}r)^{-n+\beta} \int_{2^{k+1}B} |f(y)| dy \\ &\lesssim \frac{1}{\left(1 + \frac{2r}{\rho(x_0)}\right)^{N/(k_0+1)}} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0, t))}}{t^{\frac{n}{q}}} \frac{dt}{t}. \end{aligned}$$

Then by Lemma 2.3, and taking $N \geq (k_0 + 1)\theta$ we get

$$\begin{aligned} &\|(b(x) - b_{2B})M_{\beta, V}^\theta(f_2)\|_{L_q(B(x_0, r))} \\ &\lesssim [b]_\theta r^{\frac{n}{q}} \left(1 + \frac{2r}{\rho(x_0)}\right)^{\theta - N/(k_0+1)} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0, t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \\ &\lesssim [b]_\theta r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0, t))}}{t^{\frac{n}{q}}} \frac{dt}{t}. \end{aligned} \quad (4.3)$$

Finally, let us estimate $\|M_{\beta, V}^\theta((b - b_{2B})f_2)\|_{L_q(B(x_0, r))}$. By (2.2), Lemma 2.2 and (3.2) we have

$$\begin{aligned} &\sup_{x \in \beta} |M_{\beta, V}^\theta((b - b_{2B})f_2)(x)| \\ &\lesssim \int_{(2B)^c} \frac{1}{\left(1 + \frac{|x-y|}{\rho(x)}\right)^N} \frac{|b(y) - b_{2B}| |f(y)|}{|x_0 - y|^{n-\beta}} dy \\ &\lesssim \sum_{k=1}^{\infty} \frac{1}{(2^k r)^{n-\beta} \left(1 + \frac{2^k r}{\rho(x)}\right)^N} \int_{2^{k+1}B} |b(y) - b_{2B}| |f(y)| dy \\ &\lesssim \sum_{k=1}^{\infty} \frac{1}{(2^k r)^{n-\beta} \left(1 + \frac{2^k r}{\rho(x_0)}\right)^{N/(k_0+1)}} \int_{2^{k+1}B} |b(y) - b_{2B}| |f(y)| dy. \end{aligned}$$

Note that

$$\begin{aligned} \int_{2^{k+1}B} |b(y) - b_{2B}| |f(y)| dy &\lesssim \left(\int_{2^{k+1}B} |b(y) - b_{2B}|^{p'} \right)^{1/p'} \|f\|_{L_p(B(x_0, 2^{k+1}r))} \\ &\lesssim [b]_\theta k \left(1 + \frac{2^k r}{\rho(x_0)}\right)^{\theta'} (2^k r)^{\frac{n}{p'}} \|f\|_{L_p(B(x_0, 2^{k+1}r))}. \end{aligned}$$

Then

$$\begin{aligned}
\sup_{x \in B} |M_{\beta, V}^\theta((b - b_{2B})f_2)(x)| &\lesssim [b]_\theta \sum_{k=1}^{\infty} \frac{k(2^k r)^{-\frac{n}{p} + \beta}}{\left(1 + \frac{2^k r}{\rho(x_0)}\right)^{N/(k_0+1) - \theta'}} \|f\|_{L_p(B(x_0, 2^{k+1}r))} \\
&\lesssim [b]_\theta \sum_{k=1}^{\infty} k(2^k r)^{-\frac{n}{q}} \|f\|_{L_p(B(x_0, 2^{k+1}r))} \\
&\lesssim [b]_\theta \sum_{k=1}^{\infty} k \int_{2^k r}^{2^{k+1}r} \frac{\|f\|_{L_p(B(x_0, t))}}{t^{\frac{n}{q}}} \frac{dt}{t}.
\end{aligned}$$

Since $2^k r \leq t \leq 2^{k+1} r$, then $k \approx \ln \frac{t}{r}$. Thus

$$\begin{aligned}
\sup_{x \in B} |M_{\beta, V}^\theta((b - b_{2B})f_2)(x)| &\lesssim [b]_\theta \sum_{k=1}^{\infty} k \int_{2^k r}^{2^{k+1}r} \frac{\|f\|_{L_p(B(x_0, t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \\
&\lesssim [b]_\theta \sum_{k=1}^{\infty} \int_{2^k r}^{2^{k+1}r} \ln \frac{t}{r} \frac{\|f\|_{L_p(B(x_0, t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \\
&\lesssim [b]_\theta \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0, t))}}{t^{\frac{n}{q}}} \frac{dt}{t}.
\end{aligned}$$

Then

$$\|M_{\beta, V}^\theta((b - b_{2B})f_2)\|_{L_q(B(x_0, r))} \lesssim [b]_\theta r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0, t))}}{t^{\frac{n}{q}}} \frac{dt}{t}. \quad (4.4)$$

Combining (4.2), (4.3) and (4.4), the proof of Theorem 4.1 is completed.

Proof of Theorem 1.2. Since $f \in M_{p, \varphi_1}^{\alpha, V}$ and (φ_1, φ_2) satisfies the condition (1.4), by (3.5) we have

$$\begin{aligned}
&\int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0, t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \\
&= \int_{2r}^{\infty} \frac{\left(1 + \frac{t}{\rho(x_0)}\right)^\alpha \|f\|_{L_p(B(x_0, t))}}{\operatorname{ess\,inf}_{t < s < \infty} \varphi_1(x, s) s^{\frac{n}{p}}} \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname{ess\,inf}_{t < s < \infty} \varphi_1(x, s) s^{\frac{n}{p}}}{\left(1 + \frac{t}{\rho(x_0)}\right)^\alpha t^{\frac{n}{q}}} \frac{dt}{t} \\
&\lesssim \|f\|_{M_{p, \varphi_1}^{\alpha, V}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname{ess\,inf}_{t < s < \infty} \varphi_1(x, s) s^{\frac{n}{p}}}{\left(1 + \frac{t}{\rho(x_0)}\right)^\alpha t^{\frac{n}{q}}} \frac{dt}{t} \\
&\lesssim \|f\|_{M_{p, \varphi_1}^{\alpha, V}} \left(1 + \frac{r}{\rho(x_0)}\right)^{-\alpha} \int_r^{\infty} \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname{ess\,inf}_{t < s < \infty} \varphi_1(x, s) s^{\frac{n}{p}}}{t^{\frac{n}{q}}} \frac{dt}{t} \\
&\lesssim \|f\|_{M_{p, \varphi_1}^{\alpha, V}} \left(1 + \frac{r}{\rho(x_0)}\right)^{-\alpha} \varphi_2(x_0, r). \quad (4.5)
\end{aligned}$$

Then from Theorem 4.1 we get

$$\begin{aligned}
 & \| [b, \mathcal{I}_\beta] f \|_{M_{q, \varphi_2}^{\alpha, V}} \\
 & \lesssim \sup_{x_0 \in \mathbb{R}^n, r > 0} \left(1 + \frac{r}{\rho(x_0)} \right)^\alpha \varphi_2(x_0, r)^{-1} r^{-n/q} \| [b, \mathcal{I}_\beta] f \|_{L_q(B(x_0, r))} \\
 & \lesssim [b]_\theta \sup_{x_0 \in \mathbb{R}^n, r > 0} \left(1 + \frac{r}{\rho(x_0)} \right)^\alpha \varphi_2(x_0, r)^{-1} \int_{2r}^\infty \left(1 + \ln \frac{t}{r} \right) \frac{\| f \|_{L_p(B(x_0, t))} dt}{t^{\frac{n}{q}} t} \\
 & \lesssim [b]_\theta \| f \|_{M_{p, \varphi_1}^{\alpha, V}}.
 \end{aligned}$$

5 Proof of Theorem 1.3

The statement is derived from the estimate (3.4). The estimation of the norm of the operator, that is, the boundedness in the non-vanishing space, immediately follows from by Theorem 1.1. So we only have to prove that

$$\lim_{r \rightarrow 0} \sup_{x \in \mathbb{R}^n} \mathfrak{A}_{p, \varphi_1}^{\alpha, V}(f; x, r) = 0 \quad \Rightarrow \quad \lim_{r \rightarrow 0} \sup_{x \in \mathbb{R}^n} \mathfrak{A}_{q, \varphi_2}^{\alpha, V}(M_{\beta, V}^\theta f; x, r) = 0 \quad (5.1)$$

and

$$\lim_{r \rightarrow 0} \sup_{x \in \mathbb{R}^n} \mathfrak{A}_{1, \varphi_1}^{\alpha, V}(f; x, r) = 0 \quad \Rightarrow \quad \lim_{r \rightarrow 0} \sup_{x \in \mathbb{R}^n} \mathfrak{A}_{n/(n-\beta), \varphi_2}^{W, \alpha, V}(M_{\beta, V}^\theta f; x, r) = 0. \quad (5.2)$$

To show that $\sup_{x \in \mathbb{R}^n} \left(1 + \frac{r}{\rho(x)} \right)^\alpha \varphi_2(x, r)^{-1} r^{-n/p} \| M_{\beta, V}^\theta f \|_{L_q(B(x, r))} < \varepsilon$ for small r , we split the right-hand side of (3.4):

$$\left(1 + \frac{r}{\rho(x)} \right)^\alpha \varphi_2(x, r)^{-1} r^{-n/p} \| M_{\beta, V}^\theta f \|_{L_q(B(x, r))} \leq C [I_{\delta_0}(x, r) + J_{\delta_0}(x, r)], \quad (5.3)$$

where $\delta_0 > 0$ (we may take $\delta_0 > 1$), and

$$I_{\delta_0}(x, r) := \frac{\left(1 + \frac{r}{\rho(x)} \right)^\alpha}{\varphi_2(x, r)} \int_r^{\delta_0} t^{-\frac{n}{q}-1} \| f \|_{L_p(B(x, t))} dt$$

and

$$J_{\delta_0}(x, r) := \frac{\left(1 + \frac{r}{\rho(x)} \right)^\alpha}{\varphi_2(x, r)} \int_{\delta_0}^\infty t^{-\frac{n}{q}-1} \| f \|_{L_p(B(x, t))} dt$$

and it is supposed that $r < \delta_0$. We use the fact that $f \in VM_{p, \varphi_1}^{\alpha, V}(\mathbb{R}^n)$ and choose any fixed $\delta_0 > 0$ such that

$$\sup_{x \in \mathbb{R}^n} \left(1 + \frac{t}{\rho(x)} \right)^\alpha \varphi_1(x, t)^{-1} t^{-n/p} \| f \|_{L_p(B(x, t))} < \frac{\varepsilon}{2CC_0}$$

where C and C_0 are constants from (1.5) and (5.3). This allows to estimate the first term uniformly in $r \in (0, \delta_0)$:

$$\sup_{x \in \mathbb{R}^n} CI_{\delta_0}(x, r) < \frac{\varepsilon}{2}, \quad 0 < r < \delta_0.$$

The estimation of the second term now may be made already by the choice of r sufficiently small. Indeed, thanks to the condition (2.6) we have

$$J_{\delta_0}(x, r) \leq c_{\sigma_0} \frac{\left(1 + \frac{r}{\rho(x)}\right)^\alpha}{\varphi_1(x, r)} \|f\|_{VM_{p, \varphi_1}^{\alpha, V}},$$

where c_{σ_0} is the constant from (1.2). Then, by (2.6) it suffices to choose r small enough such that

$$\sup_{x \in \mathbb{R}^n} \frac{\left(1 + \frac{r}{\rho(x)}\right)^\alpha}{\varphi_2(x, r)} \leq \frac{\varepsilon}{2c_{\sigma_0} \|f\|_{VM_{p, \varphi_1}^{\alpha, V}}},$$

which completes the proof of (5.1).

The proof of (5.2) is similar to the proof of (5.1).

6 Proof of Theorem 1.4

The norm inequality having already been provided by Theorem 1.2, we only have to prove the implication

$$\begin{aligned} \lim_{r \rightarrow 0} \sup_{x \in \mathbb{R}^n} \left(1 + \frac{t}{\rho(x)}\right)^\alpha \varphi_1(x, t)^{-1} t^{-n/p} \|f\|_{L_p(B(x, t))} &= 0 \\ \implies \lim_{r \rightarrow 0} \sup_{x \in \mathbb{R}^n} \left(1 + \frac{t}{\rho(x)}\right)^\alpha \varphi_2(x, t)^{-1} t^{-n/p} \|[b, M_{\beta, V}^\theta f]\|_{L_q(B(x, t))} &= 0. \end{aligned}$$

To check that

$$\sup_{x \in \mathbb{R}^n} \left(1 + \frac{t}{\rho(x)}\right)^\alpha \varphi_2(x, t)^{-1} t^{-n/p} \|[b, M_{\beta, V}^\theta f]\|_{L_q(B(x, t))} < \varepsilon \quad \text{for small } r,$$

we use the estimate (4.1):

$$\varphi_2(x, t)^{-1} t^{-n/p} \|[b, M_{\beta, V}^\theta f]\|_{L_q(B(x, t))} \lesssim \frac{[b]_\theta}{\varphi_2(x, r)} \int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0, t))}}{t^{\frac{n}{q}}} \frac{dt}{t}.$$

We take $r < \delta_0$ where δ_0 will be chosen small enough and split the integration:

$$\left(1 + \frac{t}{\rho(x)}\right)^\alpha \varphi_2(x, t)^{-1} t^{-n/p} \|[b, M_{\beta, V}^\theta f]\|_{L_q(B(x, t))} \leq C[I_{\delta_0}(x, r) + J_{\delta_0}(x, r)], \quad (6.1)$$

where

$$I_{\delta_0}(x, r) := \frac{\left(1 + \frac{t}{\rho(x)}\right)^\alpha}{\varphi_2(x, r)} \int_r^{\delta_0} \left(1 + \ln \frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0, t))}}{t^{\frac{n}{q}}} \frac{dt}{t}$$

and

$$J_{\delta_0}(x, r) := \frac{\left(1 + \frac{t}{\rho(x)}\right)^\alpha}{\varphi_2(x, r)} \int_{\delta_0}^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0, t))}}{t^{\frac{n}{q}}} \frac{dt}{t}.$$

We choose a fixed $\delta_0 > 0$ such that

$$\sup_{x \in \mathbb{R}^n} \left(1 + \frac{t}{\rho(x)}\right)^\alpha \varphi_1(x, t)^{-1} t^{-n/p} \|f\|_{L_p(B(x, t))} < \frac{\varepsilon}{2CC_0}, \quad t \leq \delta_0,$$

where C and C_0 are constants from (6.1) and (1.6), which yields the estimate of the first term uniform in $r \in (0, \delta_0)$: $\sup_{x \in \mathbb{R}^n} CI_{\delta_0}(x, r) < \frac{\varepsilon}{2}$, $0 < r < \delta_0$.

For the second term, writing $1 + \ln \frac{t}{r} \leq 1 + |\ln t| + \ln \frac{1}{r}$, we obtain

$$J_{\delta_0}(x, r) \leq \frac{c_{\delta_0} + \widetilde{c}_{\delta_0} \ln \frac{1}{r}}{\varphi_2(x, r)} \|f\|_{M_{p, \varphi_1}^{\alpha, V}},$$

where c_{δ_0} is the constant from (1.8) with $\delta = \delta_0$ and \widetilde{c}_{δ_0} is a similar constant with omitted logarithmic factor in the integrand. Then, by (1.7) we can choose small r such that $\sup_{x \in \mathbb{R}^n} J_{\delta_0}(x, r) < \frac{\varepsilon}{2}$, which completes the proof.

References

1. Akbulut, A., Guliyev, R.V., Celik, S., Omarova, M.N.: *Fractional integral associated with Schrödinger operator on vanishing generalized Morrey spaces*, J. Math. Inequal. **12** (3), 789805 (2018).
2. Akbulut, A., Gadjev, T.S., Serbetci, A., Rustamov, Y.I.: *Regularity of solutions to nonlinear elliptic equations in generalized Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics **43** (4), 1431 (2023).
3. Bongioanni, B., Harboure, E., Salinas, O. : *Commutators of Riesz transforms related to Schrödinger operators*, J. Fourier Anal. Appl. **17** (1), 115-134 (2011).
4. Bui, T. : *Weighted estimates for commutators of some singular integrals related to Schrödinger operators*, Bull. Sci. Math. **138** (2), 270-292 (2014).
5. Chiarenza, F., Frasca, M. : *Morrey spaces and Hardy-Littlewood maximal function*, Rend Mat. **7**, 273-279 (1987).
6. Chen, D., Song, L. : *The boundedness of the commutator for Riesz potential associated with Schrödinger operator on Morrey spaces*, Anal. Theory Appl. **30** (4), 363-368 (2014).
7. Fazio, G. Di, Ragusa, M.A. : *Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients*, J. Funct. Anal. **112**, 241-256 (1993).
8. Fan, D., Lu, S., Yang, D. : *Boundedness of operators in Morrey spaces on homogeneous spaces and its applications*, Acta Math. Sinica (N.S.) **14**, suppl., 625-634 (1998).
9. Guliyev, V.S. Integral operators on function spaces on the homogeneous groups and on domains in \mathbb{R}^n , *Doctoral dissertation, Moscow, Mat. Inst. Steklov* (Russian) 1994, 329 p.
10. Guliyev, V.S. Function spaces, integral operators and two weighted inequalities on homogeneous groups. Some applications, *Baku, Elm*, 332 pp. (1999).
11. Guliyev V.S. : *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces*, J. Inequal. Appl. Art. ID 503948, 20 pp. (2009).
12. Guliyev V.S. : *Function spaces and integral operators associated with Schrödinger operators: an overview*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **40** (2014), 178-202.
13. Guliyev V.S., Guliyev R.V., Omarova, M.N. : *Riesz transforms associated with Schrödinger operator on vanishing generalized Morrey spaces*, Appl. Comput. Math. **17** (1), 56-71 (2018).
14. Guliyev, V.S., Akbulut, A. : *Commutator of fractional integral with Lipschitz functions associated with Schrödinger operator on local generalized Morrey spaces*, Bound. Value Probl. Paper No. 80, 14 pp. (2018).
15. Guliyev, V.S., Isayev, F.A., Serbetci, A. : *Multilinear Calderón-Zygmund operators with kernels of Dini's type and their commutators on generalized local Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics **42** (4), 46-64 (2022).

16. Morrey, C. : *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43**, 126-166 (1938).
17. Mizuhara, T. : *Boundedness of some classical operators on generalized Morrey spaces*, Harmonic Analysis (S.Igari, Ed.) ICM 90 Satellite Proceedings, Springer-Verlag, Tokyo, 183-189 (1991).
18. Nakai, E. : *Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces*, Math. Nachr. **166**, 95-103 (1994).
19. Ragusa, M.A. : *Commutators of fractional integral operators on vanishing-Morrey spaces*, J. Global Optim. **40** (1-3), 361-368 (2008).
20. Sawano, Y.: *A thought on generalized Morrey spaces*, J. Indonesian Math. Soc. **25** (3), 210-281 (2019).
21. Shen, Z. *L_p estimates for Schrödinger operators with certain potentials*, Annales de l'institut Fourier **45** (2), 513-546 (1995).
22. Tang, L., Dong, J. : *Boundedness for some Schrödinger type operator on Morrey spaces related to certain nonnegative potentials*, J. Math. Anal. Appl. **355**, 101-109 (2009).
23. Tang, L. : *Weighted norm inequalities for Schrödinger type operators*, Forum Math. **27** (4), 2491-2532 (2015).
24. Vitanza, C. : *Functions with vanishing Morrey norm and elliptic partial differential equations*, In: Proceedings of methods of real analysis and partial differential equations, Capri, 147-150. Springer (1990).
25. Wheeden, R., Zygmund, A. : *Measure and integral, An introduction to real analysis, Pure and Applied Mathematics, 43, Marcel Dekker, Inc., New York-Basel* (1977).