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Eigenvalue asymptotics of a one-dimensional Schrödinger operator with confining potential

Agil Kh.Khanmamedov⋆**, Malika H. Makhmudova**

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Abstract. *The one-dimensional Schrödinger equation on the entire axis with an exponentially confining potential is considered. The asymptotic behavior of the eigenvalues is found.*

Keywords. Schrödinger equation · confining potential · eigenvalues · modified Bessel function of the second kind

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1 Introduction and main results

Confining potentials are used as a model of coupled systems with strong localization. Among other models, we note the linear potential and the harmonic oscillator potential, which describe confinement with a quadratic and linear force, respectively(see [5, 7, 8, 11]). Recently, exponentially confining potentials have also grown in great interest (see [2]).

Consider a one-dimensional Schrödinger potentials equation of the form

$$-y'' + \left(A_1 e^x + A_2 e^{-x}\right) y = \lambda y, \; -\infty < x < +\infty, \tag{1.1}$$

where A_1 and A_2 are positive constants and λ is the spectral parameter. The left side of equation (1.1) generates a self-adjoint operator $L\left(A_1, A_2\right) = -\frac{d^2}{dx^2} + A_1 e^x + A_2 e^{-x}$ in the space $L_2\left(-\infty, +\infty\right)$. Since $A_1 e^x + A_2 e^{-x} \to +\infty$ for $x \to \pm\infty$, the spectrum of the operator L consists [4] of simple real eigenvalues λ_n, $n = 0, 1, 2, ...$, condensing to $+\infty$, with $\lambda_n \geq \inf\limits_{-\infty<x<+\infty} \left(A_1 e^x + A_2 e^{-x}\right) = 2\sqrt{A_1 A_2} > 0$.

In this paper, we study the asymptotic behavior of the eigenvalues λ_n, $n = 0, 1, 2, ...$ for $n \to \infty$.

Let us formulate the main result of this work.

⋆ Corresponding author

A.Kh. Khanmamedov,
Baku State University, Baku, Azerbaijan
Institute of Mathematics and Mechanics of NAS of Azerbaijan Baku, Azerbaijan
Azerbaijan University, Baku, Azerbaijan
E-mail: agil khanmamedov@yahoo.com

M.H. Makhmudova
Baku State University, Baku, Azerbaijan
E-mail: MLK_Maxmudova@hotmail.com

Theorem 1.1 *For the eigenvalues* λ_n, $n = 0, 1, 2, ...$ *of the operator* $L(A_1, A_2)$ *the asymptotic formula*

$$\lambda_n \sim \frac{\pi n}{2\ln n},\ n \to \infty. \tag{1.2}$$

2 Proof of the theorem

Consider the following boundary value problem

$$-y'' + \alpha e^x y = \lambda y,\ -\infty < x < +\infty, \tag{2.1}$$

$$y(0)\cos\beta + y'(0)\sin\beta = 0, \tag{2.2}$$

where $\alpha > 0$ and β-is a real number. Consider a self-adjoint operator $L_0(\alpha)$ generated in the space $L_2(0, +\infty)$ by the left side of equation (2.1) and boundary condition (2.2). Since $e^x \to +\infty$ for $x \to +\infty$, the spectrum of the operator $L_0(\alpha)$ is discrete [4] and has a unique limit point at infinity. Denote by $\mu_n = \mu_n(\alpha)$, $n = 0, 1, 2, ...$ the eigenvalues of the operator $L_0(\alpha)$. Obviously $\mu_n \geq \inf\limits_{0<x<+\infty} \alpha e^x = \alpha > 0$. It is known [1,3] that equation (2.1) has a solution $\psi(x, \lambda)$, which can be represented as

$$\psi(x, \lambda) = K_{2i\sqrt{\lambda}}\left(2\sqrt{\alpha}e^{\frac{x}{2}}\right),$$

where $K_\nu(z)$ is a modified Bessel function of the second kind (see [1,3]), i.e. solution of the equation

$$z^2u'' + zu' - \left(z^2 + \nu^2\right)u = 0.$$

It is also known [1,3] that for each $z > 0$ the function $K_\nu(z)$ is an entire function of index ν and the integral representation

$$K_{i\lambda}(z) = \int_0^\infty e^{-zcht}\cos\lambda t dt,\ \ |\arg z| < \frac{\pi}{2},\ \lambda \in C$$

is valid. Therefore, for every fixed x, $0 \leq x < +\infty$, the solution $\psi(x, \lambda)$ serves as an entire function with respect to λ. Further, the following asymptotic formula is true [1]

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}}e^{-z}\left(1 + O\left(z^{-1}\right)\right),\ z \to \infty.$$

This implies that for every fixed λ the solution $\psi(x, \lambda)$ belongs to the space $L_2(0, +\infty)$.Whence it follows that the eigenvalues of the operator $L_0(\alpha)$ coincide with the zeros of the function $\Psi(\lambda) = \psi(0, \lambda)\cos\beta + \psi'(0, \lambda)\sin\beta$.

Let us now study the asymptotic behavior of the eigenvalues of the $L_0(\alpha)$. Since the function $q(x) = \alpha e^x$satisfies all conditions of Theorem 7.3 from the monograph [9] (see also [10]), we have

$$\int_0^{\ln\alpha^{-1}\mu_n} \sqrt{\mu_n - \alpha e^x}dx \sim \pi n,\ n \to \infty. \tag{2.3}$$

Further, note that

$$\int_0^{\ln\alpha^{-1}\mu_n} \sqrt{\mu_n - \alpha e^x}dx = \int_a^{\mu_n} t^{-1}\sqrt{\mu_n - t}dt$$

$$= \mu_n \int_a^{\mu_n} \frac{\mu_n}{t}\sqrt{1 - \frac{t}{\mu_n}}d\frac{t}{\mu_n} = \mu_n \int_{\alpha\mu_n^{-1}}^1 u\sqrt{1 - u}du. \tag{2.4}$$

Since the function $G(u) = 2\sqrt{1-u} - \ln\left(1+\sqrt{1-u}\right) + \ln\left(1-\sqrt{1-u}\right)$ serves as an antiderivative function $g(u) = u^{-1}\sqrt{1-u}$, then from formula (2.4) we have

$$\int_0^{\ln \alpha^{-1}\mu_n} \sqrt{\mu_n - \alpha e^x}dx = \mu_n \ln \mu_n \left(1 + O\left(\frac{1}{\ln \mu}\right)\right),\ n \to \infty.$$

Comparing this relation with (2.3), we obtain

$$\mu_n \ln \mu_n = \pi n \left[1 + o(1)\right],\ n \to \infty.$$

If now we are looking for μ_n in the form $\mu_n = \frac{\pi n}{\ln \pi n}(1+\varepsilon_n)$, then from the last equality it is easy to derive the relation $\varepsilon_n = o(1)$, $n \to \infty$. Therefore, for the zeros of the function $\psi(0,\lambda) = K_{2i\sqrt{\lambda}}(2\sqrt{\alpha})$, i.e. for the eigenvalues of the operator $L_0(\alpha)$ is true the following asymptotic equality

$$\mu_n \sim \frac{\pi n}{\ln n},\ n \to \infty. \tag{2.5}$$

We now introduce a self-adjoint operator $L_0(A_1, A_2)$, generated in space $L_2(0,+\infty)$ by the differential expression $l(y) = -y'' + (A_1 e^x + A_2 e^{-x})\,y$ and boundary condition (2.2). Further, setting $\alpha = \alpha_1 = \min\{A_1, A_2\}$ and $\alpha = \alpha_2 = \max\{A_1, A_2\}$, we find that

$$L_0(\alpha_1) \le L_0(A_1, A_2) \le L_0(\alpha_1).$$

Then, by virtue of the minimax principle (see [6]), we find that the eigenvalues λ_n^0 of the operator $L_0(A_1, A_2)$ satisfy the inequality

$$\mu_n(\alpha_1) \le \lambda_n^0 \le \mu_n(\alpha_2).$$

From the last inequality and (2.5) it follows that

$$\lambda_n^0 \sim \frac{\pi n}{\ln n},\ n \to \infty. \tag{2.6}$$

Consider now in the space $L_2(0,+\infty)$ the self-adjoint operators $L_D y = -y'' + A(e^x + e^{-x})\,y$, $y(0) = 0$ and $L_N y = -y'' + A(e^x + e^{-x})\,y$, $y'(0) = 0$, which are special cases of the operator $L_0(A_1, A_2)$. Let $\lambda_n(D)$ and $\lambda_n(N)$, $n = 0, 1, 2, ..$ denote the eigenvalues of the operators L_D and L_N, respectively. Due to (2.6) we have

$$\begin{array}{l} \lambda_n(D) \sim \frac{\pi n}{\ln n},\ n \to \infty, \\ \lambda_n(N) \sim \frac{\pi n}{\ln n},\ n \to \infty. \end{array} \tag{2.7}$$

Since the function $e^x + e^{-x}$ is even, the spectrum of the operator $L(A, A)$ coincides with the eigenvalues of the operators L_D and L_N. Setting $\hat{\lambda}_{2n} = \lambda_n(D)$, $\hat{\lambda}_{2n+1} = \lambda_n(N)$, $n = 0, 1, 2, ...$, from (2.7) we obtain

$$\hat{\lambda}_n \sim \frac{\pi n}{2\ln n},\ n \to \infty.$$

Then from the inequality

$$L(\alpha_1, \alpha_1) \le L(A_1, A_2) \le L(\alpha_2, \alpha_2)$$

and the minimax principle follows (1.2). Thus the theorem is proved.

References

1. Abramowitz M., Stegun I., eds.: Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, *(Natl. Bur. Stand. Appl. Math. Ser.)*, *U.S. Gov. Printing Office, Washington, DC*, **55**, (1964).
2. Alhaydari A.D.: *Exponential confining potential well*, Theoret. and Math. Phys. **206**(1), 84–96 (2021).
3. Bateman H., Erd'elyi A.: Higher Transcendental Functions, *McGraw-Hill, New York*, **2**, (1953).
4. Berezin F.A., Shubin M.A.: The Schrödinger Equation *[in Russian], Moscow State Univ. Press, Moscow* (1983); *English transl., Kluwer, Dordrecht* (1991).
5. Ferreira P. L., Helayel J. A., Zagury N.: *A linear-potential model for quark confinement*, Il Nuovo Cimento A, **55**, 215–226 (1980).
6. Fournais S., Helffer B.: Spectral Methods in Surface Superconductivit. - Progress in Nonlinear Differential Equations and Their Applications, *Birkhauser*, (2010).
7. Makhmudova M.G., Khanmamedov A. Kh.: *On the spectral properties of the one-dimensional Stark operator on the half-line*, Ukrainian Math. J. **71**(11), 1813-1819 (2020).
8. Savchuk A.M., Shkalikov A.A.: *Spectral properties of the complex Airy operator on the half-line*, Funct. Anal. Appl. **51**(1), 66-79 (2017).
9. Titchmarsh E. C.: Eigenfunction Expansions With Second-Order Differential Operators, *Clarendon, Oxford*, **1**, (1946).
10. Titchmarsh E. C.: *On the asymptotic distribution of eigenvalues asymptotic*, Quart. J. Math. Oxford Ser. (2) **5**, 228-240 (1954).
11. Zettili N.: Quantum Mechanics: Concepts and Applications, *Wiley, New York* (2009).