

Boundedness of Gegenbauer fractional maximal commutator on generalized Gegenbauer-Morrey spaces

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Abstract. *In this paper, we obtain results of the Spanne-Guliyev, Adams-Guliyev, Adams-Gunavan and Gunawan-Guliyev type on the boundedness of the G -fractional maximal operator M_G^α on generalized Gegenbauer-Morrey (G -generalized Morrey) spaces. In addition, we characterize the boundedness of the k th order Gegenbauer fractional maximal commutator $M_G^{b,\alpha,k}$ on a generalized G -Morrey spaces $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$.*

Keywords. G -fractional maximal operator, G -fractional maximal commutator, generalized G -Morrey spaces, BMO_G space.

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1 Introduction

The classical Morrey spaces were originally introduced by Morrey [34] to study the local behaviour of solutions to second-order elliptic partial differential equations. Later, various problems of harmonic analysis (HA) were studied in these spaces. As is known, such operators as: maximal functions, potentials and singular integrals are important object of HA, play a huge rol and have numerous applications in various HA problems: approximation theory, theory differential equation in various problems, physics and mechanics. On of the main problems of HA is the question of the boundedness of the above operators and their commutators in various functional spaces. Morrey spaces, generalized Morrey spaces, weighted Morrey spaces. Therefore, if is no coincidence that a large number of work are devoted to this theory (see, for example [1-9, 11-14,16,17, 19-26, 33,35-40]). All this indicates the relevance of studuing various kinds of properties in these spaces.

Let $1 \leq p < \infty$, $0 \leq \lambda \leq n$. The classical Morrey space $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ is defined as follows

$$\mathcal{M}_{p,\lambda}(\mathbb{R}^n) := \{f \in L_{loc}^p(\mathbb{R}^n) : \|f\|_{\mathcal{M}_{p,\lambda}} < \infty\},$$

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where

$$\|f\|_{\mathcal{M}_{p,\lambda}} := \sup_B \left(\frac{1}{|B|^{\frac{\lambda}{n}}} \int_B |f(x)|^p dx \right)^{\frac{1}{p}},$$

supremum is taken over all balls $B \subset \mathbb{R}^n$, and $|B|$ is Lebesgue measure. It is known that if $1 \leq p < \infty$, then by $\lambda = 0$, $\mathcal{M}_{p,0}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ and $\lambda = n$, $\mathcal{M}_{p,n}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $\mathcal{M}_{p,\lambda}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to zero on \mathbb{R}^n .

Denote by $W\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space of all functions $f \in WL_{loc}^p(\mathbb{R}^n)$ with the finite norm

$$\|f\|_{W\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{r>0} r \sup_{x \in \mathbb{R}^n, t>0} \left(t^{-\lambda} |\{y \in B(x, t) : |f(y)| > r\}| \right)^{\frac{1}{p}}.$$

The fractional integral operator I_α , $0 < \alpha < n$ is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y) dy}{|x-y|^{n-\alpha}}. \quad (1.1)$$

For locally integrable functions b , the commutator is defined as follows

$$[b, I_\alpha]f(x) := b(x)I_\alpha f(x) - I_\alpha(bf)(x). \quad (1.2)$$

On $\mathcal{M}_{p,\lambda}$ Morrey spaces classical theory of operators (1.1) is based on the Adams theorem [1] and the Spanne's theorem, published in the paper of Peetre [37], but classical $\mathcal{M}_{p,\lambda}$ theory of operators (1.2) is based on the theorem of Komori-Mizuhara [33] and the theorem of Shirai [39] which are given below.

Classical result Hardy-Littlewood-Sobolev's is as follows:

Theorem A. Let $1 < p < q < \infty$, $0 < \alpha < n$ and $1 \leq p < n/\alpha$.

(i) If $1 < p < n/\alpha$, then the condition

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$$

is necessary and sufficient for the boundedness I_α from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$.

(ii) If $p = 1 < q < \infty$, then the condition

$$1 - \frac{1}{q} = \frac{\alpha}{n}$$

is necessary and sufficient for the boundedness I_α from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$.

In [1] for the I_α in the Morrey space, Adams proved the following theorem.

Theorem B. (Adams [1]) Let $0 \leq \alpha < n$, $0 \leq \lambda < n$ and $1 \leq p < (n-\lambda)/\alpha$

(i) If $1 < p < (n-\lambda)/\alpha$, then the condition

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$$

is necessary and sufficient for the boundedness I_α from $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ to $\mathcal{M}_{q,\lambda}(\mathbb{R}^n)$.

(ii) If $p = 1$, then the condition

$$1 - \frac{1}{q} = \frac{\alpha}{n-\lambda}$$

is necessary and sufficient for the boundedness I_α from $\mathcal{M}_{1,\lambda}(\mathbb{R}^n)$ to $W\mathcal{M}_{q,\lambda}(\mathbb{R}^n)$.

Theorema C. (Spanne [37]) Let $0 \leq \alpha < n$, $1 \leq p < n/\alpha$, $0 < \lambda < n - \alpha p$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$. Then:

- (i) If $p > 1$, then I_α is bounded from $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ to $\mathcal{M}_{q,\mu}(\mathbb{R}^n)$ if and only if $\lambda/p = \mu/q$.
- (ii) If $p = 1$, then I_α is bounded from $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ to $W\mathcal{M}_{q,\mu}(\mathbb{R}^n)$ if and only if $\lambda/p = \mu/q$.

Theorem D. (Komori-Mizuhara [33]) Let $0 \leq \alpha < n$, $1 < p < n/\alpha$, $0 < \lambda < n - \alpha p$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$.

Then the following statements are equivalents:

- (a) $b \in BMO(\mathbb{R}^n)$.
- (b) $[b, I_\alpha]$ is bounded from $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ to $L_{q,\mu}(\mathbb{R}^n)$.

Theorem E. (Shirai [39]) Let $0 \leq \alpha < n$, $1 < p < n/\alpha$, $0 < \lambda < n - \alpha p$ and $\lambda/p = \mu/q$.

Then the following conditions are equivalent:

- (a) $b \in BMO(\mathbb{R}^n)$.
- (b) $[b, I_\alpha]$ is bounded from $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ to $\mathcal{M}_{q,\mu}(\mathbb{R}^n)$.

Let $f \in L^{\text{loc}}(\mathbb{R}^n)$. The fractional maximal operator M_α is defined for locally integrable functions in the form

$$M_\alpha f(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{B(x,r)} |f(y)| dy, \quad 0 \leq \alpha < n,$$

and its fractional maximal commutator generated by a locally integrable function b has the form

$$M_{b,\alpha}(f)(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{B(x,r)} |b(x) - b(y)| |f(y)| dy.$$

From the pointwise estimates $M_\alpha f(x) \leq I_\alpha(|f|)(x)$ and $M_{b,\alpha}(f)(x) \leq [b, I_\alpha](|f|)(x)$ follows that Theorems A-E remains strength for the fractional maximal operator and its commutator.

Later these results were obtained for Gegenbauer-Morrey spaces and were reflected in [16,17,29,30].

In [27] we have introduced Riesz potential I_G^α generated by differential operator G in the following form

$$I_G^\alpha f(chx) = \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty \left(\int_0^\infty r^{\frac{\alpha}{2}-1} h_r(cht) dr \right) A_{cht}^\lambda f(chx) sh^{2\lambda} t dt,$$

where

$$h_r(cht) = \int_1^\infty e^{-\nu(\nu+2\lambda)r} P_\nu^\lambda(cht) (\nu^2 - 1)^{\lambda-\frac{1}{2}} d\nu$$

$P_\nu^\lambda(cht)$ is the eigenfunction of G operator and

$$A_{cht}^\lambda f(chx) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda)\Gamma(\frac{1}{2})} \int_0^\pi f(chxcht - shxsht \cos \varphi) (\sin \varphi)^{2\lambda-1} d\varphi,$$

is a generalized shift operator associated with Gegenbauer differential operator G [10]

$$G \equiv G_\lambda = (x^2 - 1)^{\frac{1}{2}-\lambda} \frac{d}{dx} (x^2 - 1)^{\lambda+\frac{1}{2}} \frac{d}{dx}, \quad x \in (1, \infty), \quad \lambda \in \left(0, \frac{1}{2}\right).$$

Denote by, $L_{p,\lambda}(\mathbb{R}_+)$, $1 \leq p \leq \infty$ the space of μ_λ -measurable functions ($\mu_\lambda(x) = sh^{2\lambda}x$) with finite norm

$$\|f\|_{L_{p,\lambda}(\mathbb{R}_+)} = \left(\int_0^\infty |f(chx)|^p sh^{2\lambda}x dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

$$\|f\|_{L_{\infty,\lambda}(\mathbb{R}_+)} = \|f\|_{L_\infty(\mathbb{R}_+)} = \operatorname{ess\,sup}_{x \in \mathbb{R}_+} |f(chx)|,$$

and denote by $WL_{p,\lambda}(\mathbb{R}_+)$ the weak $L_{p,\lambda}(\mathbb{R}_+)$ space of μ -measurable functions $f(chx)$, $x \in \mathbb{R}_+$ with the finite norm

$$\begin{aligned} \|f\|_{WL_{p,\lambda}(\mathbb{R}_+)} &= \sup_{r>0} r |\{x \in \mathbb{R}_+ : |f(chx)| > r\}|_\lambda^{\frac{1}{p}} \\ &= \sup_{r>0} r \left(\int_{\{x \in \mathbb{R}_+ : |f(chx)| > r\}} sh^{2\lambda}x dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \end{aligned}$$

In [15] for potential I_G^α the following theorem which is an analogue of Theorem A was proved.

Theorem F. Let $0 < \lambda < \frac{1}{2}$, $0 < \alpha < 2\lambda + 1$ and $1 \leq p < \frac{2\lambda+1}{\alpha}$.

(a) If $1 < p < \frac{2\lambda+1}{\alpha}$, then the condition

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda + 1}$$

is necessary and sufficient for the boundedness for I_G^α from $L_{p,\lambda}(\mathbb{R}_+)$ to $L_{q,\lambda}(\mathbb{R}_+)$.

(b) If $p = 1$, then the condition

$$1 - \frac{1}{q} = \frac{\alpha}{2\lambda + 1}$$

is necessary and sufficient for the boundedness of I_G^α from $L_{1,\lambda}(\mathbb{R}_+)$ to $WL_{q,\lambda}(\mathbb{R}_+)$.

In [16] was introduced the concept of Gegenbauer-Morrey space (G -Morrey space) associated with differential operator G on the set of locally integrable functions with the finite norm

$$\|f\|_{L_{p,\lambda,\nu}(\mathbb{R}_+)} = \sup_{x \in \mathbb{R}_+, r>0} \left(r^{-\nu} \int_0^r A_{cht}^\lambda |f(chx)|^p sh^{2\lambda}t dt \right)^{\frac{1}{p}}, \quad 0 \leq \nu \leq 2\lambda + 1,$$

and also weak $WL_{p,\lambda,\nu}(\mathbb{R}_+)$ space with the finite norm

$$\begin{aligned} \|f\|_{WL_{p,\lambda,\nu}(\mathbb{R}_+)} &= \sup_{r>0} r \sup_{x \in \mathbb{R}_+, t>0} \left(t^{-\nu} \left| \left\{ y \in (0, t) : A_{chy}^\lambda |f(chx)| > r \right\} \right|_\lambda \right)^{\frac{1}{p}} \\ &= \sup_{r>0} r \sup_{x \in \mathbb{R}_+, t>0} \left(t^{-\nu} \int_{\{y \in (0,t) : A_{chy}^\lambda |f(chx)| > r\}} sh^{2\lambda}y dy \right)^{\frac{1}{p}}. \end{aligned}$$

For potential I_G^α on G -Morrey space we have the following theorem ([16, §2 Theorem 2.1]), which is an analogue of Theorem B.

Theorem G. Let $0 < \alpha < 2\lambda + 1$, $0 < \nu < 2\lambda + 1 - \alpha p$ and $1 \leq p < \frac{2\lambda+1}{\alpha}$

(i) If $1 < p < \frac{2\lambda+1-\nu}{\alpha}$, then the condition

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda + 1 - \nu}$$

is necessary and sufficient for the boundedness I_G^α from $L_{p,\lambda,\nu}(\mathbb{R}_+)$ to $L_{q,\lambda,\nu}(\mathbb{R}_+)$.

(ii) If $p = 1 < \frac{2\lambda+1-\nu}{\alpha}$, then the condition

$$1 - \frac{1}{q} = \frac{\alpha}{2\lambda + 1 - \nu}$$

is necessary and sufficient for the boundedness I_G^α from $L_{1,\lambda,\nu}(\mathbb{R}_+)$ to $WL_{q,\lambda,\nu}(\mathbb{R}_+)$.

In [28] Muckenhoupt type weighted class $A_p^\lambda(\mathbb{R}_+)$ associated and with differential operator G was introduced and in this class for operators M_G^α and I_G^α , on conditions $0 < \alpha < 2\lambda + 1$, $1 < p < \frac{2\lambda+1}{\alpha}$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda+1}$, the strong and weak types weighted $(L_{p,\lambda,\omega}, L_{q,\lambda,\omega})$ inequalities ([28, § 4.1, Theorems 4.1 and 4.2 and also § 4.2, Theorems 4.4 and 4.5]) were proved.

In this paper on pair (ω_1, ω_2) the necessary and sufficient conditions for the boundedness of fractional maximal operator M_G^α , $0 \leq \alpha < \gamma$, from one generalized G -Morrey space $\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)$ to another $\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$, $1 < p \leq q < \infty$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma}$ and from the space $\mathcal{M}_{1,\gamma,\omega_1}(\mathbb{R}_+)$ to the weak space $W\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$, $1 < q < \infty$, $1 - \frac{1}{q} = \frac{\alpha}{\gamma}$ were obtained. Also find necessary and sufficient conditions on the ω , which ensure the Adams type boundedness of the M_G^α from $\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega^{\frac{1}{q}}}(\mathbb{R}_+)$ for $1 < p < q < \infty$ and from $\mathcal{M}_{1,\gamma,\omega}(\mathbb{R}_+)$ to $W\mathcal{M}_{q,\gamma,\omega^{\frac{1}{q}}}(\mathbb{R}_+)$ for $1 = p < q < \infty$.

In the case $b \in BMO_G$ on pair (ω_1, ω_2) , the necessary and sufficient conditions for the boundedness of the commutator $M_G^{b,\alpha}$ from $\mathcal{M}_{1,\gamma,\omega_1}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$ by $1 - \frac{1}{q} = \frac{\alpha}{\gamma}$ and also the necessary and sufficient conditions for the boundedness of commutator $M_G^{b,\alpha}$ from $\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega^{\frac{1}{q}}}(\mathbb{R}_+)$ for $1 < p < q < \infty$.

Later we will consider $\omega(x, r)$, $\omega_1(x, r)$ and $\omega_2(x, r)$ as nonnegative Lebesgue measurable functions on $\mathbb{R}_+ = (0, \infty)$.

Note that all result obtained in the paper are the future development of Gegenbauer harmonic analysis theory, foundations of which were laid in [32]. This theory was later developed in different directions: approximation and embedding theory, transformation theory, theory of singulars integrals, maximal functions theory, theory of potentials and its commutator.

2 Definitions, notations and auxiliary results

Throughout the paper, we will denote by shx , chx the hyperbolic functions.

In what follows, the expression $A \lesssim B$ mean that there exists a constant C such that $0 < A \leq CB$, where C may depend on some inessential parameters. If $A \lesssim B$ and $B \lesssim A$ then we write $A \approx B$ and say that A and B are equivalent.

Let $f \in L_{1,\lambda}^{loc}(\mathbb{R}_+)$. Denote by $H_r = (0, r)$ and $r \in (0, \infty)$.

Later we will need the following relation (see [15, §1, formula (1.2)])

$$|H_r|_\lambda = \int_0^r sh^{2\lambda} t dt \approx \left(sh \frac{r}{2} \right)^\gamma \quad (2.1)$$

where $0 < \lambda < \frac{1}{2}$ and

$$\gamma = \gamma_\lambda(r) = \begin{cases} 2\lambda + 1, & \text{if } 0 < r < 2, \\ 4\lambda, & \text{if } 2 \leq r < \infty \end{cases}$$

and $|H_r|_\lambda$ is an absolutely continuous measure of the interval H_r .

According to formula (2.1) Gegenbauer maximal operator (G -maximal operator) M_G , fractional maximal operator M_G^α and Gegenbauer fractional integral J_G^α for any $x \in \mathbb{R}_+$ are defined as follows:

$$M_G f(chx) = \sup_{r>0} \frac{1}{|H_r|_\lambda} \int_{H_r} A_{cht}^\lambda |f(chx)| sh^{2\lambda} t dt,$$

$$M_G^\alpha f(chx) = \sup_{r>0} \frac{1}{|H_r|_\lambda^{1-\frac{\alpha}{\gamma}}} \int_{H_r} A_{cht}^\lambda |f(chx)| sh^{2\lambda} t dt, \quad 0 \leq \alpha < \gamma.$$

It's obvious that $M_G^0 f(chx) \equiv M_G f(chx)$.

$$J_G^\alpha f(chx) = \int_0^\infty |H_t|_\lambda^{\frac{\alpha}{\gamma}-1} A_{cht}^\lambda f(chx) sh^{2\lambda} t dt, \quad 0 < \alpha < \gamma,$$

where

$$|H_t|_\lambda^{\frac{\alpha}{\gamma}-1} = \begin{cases} |H_t|_\lambda^{\frac{\alpha}{2\lambda+1}-1}, & 0 < t < 2, \quad 0 < \alpha < 2\lambda + 1, \\ |H_t|_\lambda^{\frac{\alpha}{4\lambda}-1}, & 2 \leq t < \infty, \quad 0 < \alpha < 4\lambda. \end{cases}$$

In this section we present some generalization of Gegenbauer-Morrey space (G -Morrey space).

In [29] by analogy to Nakai [36] introduced the concept of generalized G -Morrey space on the set locally integrable functions $f(chx)$, $x \in \mathbb{R}_+$ which the finite norm

$$\|f\|_{M_{p,\lambda,\omega}(\mathbb{R}_+)} = \sup_{x \in \mathbb{R}_+, r > 0} \left(\frac{1}{\omega(r)} \int_{H_r} A_{cht}^\lambda |f(chx)|^p sh^{2\lambda} t dt \right)^{\frac{1}{p}}$$

and weak G -Morrey space $WM_{p,\lambda,\omega}(\mathbb{R}_+)$ with the finite norm

$$\|f\|_{WM_{p,\lambda,\omega}(\mathbb{R}_+)} = \sup_{r>0} r \sup_{x \in \mathbb{R}_+, t > 0} \left(\frac{1}{\omega(r)} \left| \{y \in H_t : A_{chy}^\lambda |f(cht)| > r\} \right|_\lambda \right)^{\frac{1}{p}},$$

where $\omega(r)$ nonnegative Lebesgue measurable function on \mathbb{R}_+ , $1 \leq p < \infty$.

Note that, when $\omega(r) \equiv 1$ the space $M_{p,\lambda,\omega}(\mathbb{R}_+)$ goes above considered space $L_{p,\lambda}(\mathbb{R}_+)$, and Theorem F is a consequence of the Theorem H given below.

Let $0 < \delta \leq 1$. Suppose that $\omega(r)$ satisfies the conditions

$$r \leq t \leq 2r \Rightarrow \omega(t) \approx \omega(r), \quad (a)$$

$$\int_r^\infty \frac{\omega(t)}{t^{\nu\delta+1}} dt \lesssim \begin{cases} r^{-(2\lambda+1)\delta} \omega(r), & \nu = 2\lambda + 1, \quad 0 < r < 2, \\ r^{-4\lambda\delta} \omega(r), & \nu = 4\lambda, \quad 2 \leq r < \infty. \end{cases} \quad (b)$$

Theorem H. [29] Let $0 < \lambda < \frac{1}{2}$, $0 < \alpha < 2\lambda + 1$, $1 \leq p < \frac{\alpha}{2\lambda+1}$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda+1}$. Suppose that ω satisfies the conditions (a) and (b). Then:

(i) if $p > 1$ and $f \in M_{p,\lambda,\omega}(\mathbb{R}_+)$, then the inequality

$$\|I_G^\alpha f\|_{M_{q,\lambda,\omega}^{\frac{q}{p}}} \lesssim \|f\|_{M_{p,\lambda,\omega}}$$

is true.

(ii) if $p = 1$ and $f \in M_{1,\lambda,\omega}(\mathbb{R}_+)$ then the inequality

$$\|I_G^\alpha f\|_{WM_{q,\lambda,\omega}^q} \lesssim \|f\|_{M_{1,\lambda,\omega}}$$

is true.

Theorem L. [29] Let the conditions (a) and (b) be satisfied. Then:

(i) for $f \in M_{p,\lambda,\omega}(\mathbb{R}_+)$ and $1 \leq q < p < \infty$

$$\|M_G^q f\|_{M_{p,\lambda,\omega}} \lesssim \|f\|_{M_{p,\lambda,\omega}},$$

(ii) for $f \in M_{p,\lambda,\omega}(\mathbb{R}_+)$, $1 \leq p < \infty$ and any $t > 0$

$$\|M_G^p f\|_{WM_{p,\lambda,\omega}} \lesssim \|f\|_{M_{p,\lambda,\omega}}.$$

In accordance with the formula (2.1), we give the following generalization of G -Morrey space.

Definition 2.1 Let $1 \leq p < \infty$. Generalized G -Morrey space $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ associated with Gegenbauer differential operator G is defined as a set of locally integrable functions $f(chx)$, $x \in \mathbb{R}_+$ with the finite norm

$$\begin{aligned} \|f\|_{\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)} &= \sup_{x,r \in \mathbb{R}_+} \omega(x,r)^{-1} \left(sh \frac{r}{2} \right)^{-\frac{\gamma}{p}} \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_r)} \\ &= \max \left\{ \sup_{\substack{x \in \mathbb{R}_+ \\ 0 < r < 2}} \omega(x,r)^{-1} \left(sh \frac{r}{2} \right)^{-\frac{2\lambda+1}{p}} \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_r)}, \right. \\ &\quad \left. \sup_{\substack{x \in \mathbb{R}_+ \\ 2 \leq r < \infty}} \omega(x,r)^{-1} \left(sh \frac{r}{2} \right)^{-\frac{4\lambda}{p}} \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_r)} \right\}, \end{aligned}$$

where

$$\left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_r)} = \left(\int_0^r A_{cht}^\lambda |f(chx)|^p sh^{2\lambda} t dt \right)^{\frac{1}{p}}.$$

Also the weak generalized G -Morrey space $W\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ of locally integrable functions $f(chx)$, $x \in \mathbb{R}_+$ with the finite norm

$$\begin{aligned} \|f\|_{W\mathcal{M}_{p,\lambda,\omega}(\mathbb{R}_+)} &= \sup_{x,r \in \mathbb{R}_+} \omega(x,r)^{-1} \left(sh \frac{r}{2} \right)^{-\frac{\gamma}{p}} \left\| A_{chx}^\lambda f \right\|_{WL_{p,\lambda}(H_r)} \\ &= \max \left\{ \sup_{\substack{x \in \mathbb{R}_+ \\ 0 < r < 2}} \omega(x,r)^{-1} \left(sh \frac{r}{2} \right)^{-\frac{2\lambda+1}{p}} \left\| A_{chx}^\lambda f \right\|_{WL_{p,\lambda}(H_r)}, \right. \\ &\quad \left. \sup_{\substack{x \in \mathbb{R}_+ \\ 2 \leq r < \infty}} \omega(x,r)^{-1} \left(sh \frac{r}{2} \right)^{-\frac{4\lambda}{p}} \left\| A_{chx}^\lambda f \right\|_{WL_{p,\lambda}(H_r)} \right\}, \end{aligned}$$

where

$$\left\| A_{chx}^\lambda f \right\|_{WL_{p,\lambda}(H_r)} = \sup_{t>0} \sup_{x,r \in \mathbb{R}_+} t \left| \left\{ y \in H_r : A_{chy}^\lambda |f(chx)| > t \right\} \right|_\lambda^{\frac{1}{p}}.$$

We will prove some auxiliary statements, which we will need later.

Lemma 2.2. *Let $\omega(x, r)$ be a positive measurable function on \mathbb{R}_+ .*

(i) *If*

$$\sup_{t < r < \infty} \left(sh \frac{r}{2} \right)^{-\frac{\gamma}{p}} \omega(x, r)^{-1} = \infty \quad (2.2)$$

is true for some $t > 0$ and any $x \in \mathbb{R}_+$, then $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+) = \Theta$.

(ii) *If*

$$\sup_{0 < r < t} \omega(x, r)^{-1} = \infty \quad (2.3)$$

is true for some $t > 0$ and any $x \in \mathbb{R}_+$, then $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+) = \Theta$, where Θ is a all functions that are equivalent to zero on \mathbb{R}_+ .

Proof. Suppose (2.2) is true and f is not equivalent to zero. Then $\sup_{x \in \mathbb{R}_+} \|f\|_{L_{p,\lambda}(H_t)} > 0$,

hence

$$\begin{aligned} \|f\|_{\mathcal{M}_{p,\gamma,\omega}} &\geq \sup_{x,r \in \mathbb{R}_+} \sup_{t < r < \infty} \omega(x, r)^{-1} \left(sh \frac{r}{2} \right)^{-\frac{\gamma}{p}} \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_r)} \geq \\ &\geq \sup_{x \in \mathbb{R}_+} \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_t)} \sup_{t < r < \infty} \omega(x, r)^{-1} \left(sh \frac{r}{2} \right)^{-\frac{\gamma}{p}}. \end{aligned}$$

Hence $\|f\|_{\mathcal{M}_{p,\gamma,\omega}} = \infty$.

(ii) Let $f \in \mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ and (2.2) be satisfied, then there are two possibilities:

Case 1: $\sup_{0 < r < t} \omega(x, r)^{-1} = \infty$ for all $t > 0$.

Case 2: $\sup_{0 < r < t} \omega(x, r)^{-1} < \infty$ for some $t \in (0, s)$.

For Case 1, by Lebesgue differentiation theorem (see [27], Corollary 2.1 from Theorem 2.2), for almost $x \in \mathbb{R}_+$

$$\lim_{r \rightarrow 0^+} \frac{\left\| A_{chx}^\lambda f \chi_{H_r} \right\|_{L_{p,\lambda}}}{\left\| \chi_{H_r} \right\|_{L_{p,\lambda}}} = |f(chx)|, \quad (2.4)$$

where χ_{H_r} is a characteristic function of a set H_r . We require that $f(chx) = 0$ for all those x . Indeed, fix x and assume $|f(chx)| > 0$. Then from (2.1) and (2.4) there exists $t_0 > 0$ such that

$$\frac{\left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}}}{\left(sh \frac{r}{2} \right)^{\frac{\gamma}{p}}} \gtrsim |f(chx)|,$$

for all $0 < r \leq t_0$. Consequently,

$$\begin{aligned} \|f\|_{\mathcal{M}_{p,\gamma,\omega}} &\geq \sup_{0 < r \leq t_0} \omega(x, r)^{-1} \left(sh \frac{r}{2} \right)^{-\frac{\gamma}{p}} \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_r)} \\ &\gtrsim \sup_{0 < r \leq t_0} \omega(x, r)^{-1} |f(chx)|. \end{aligned}$$

Hence $\|f\|_{\mathcal{M}_{p,\gamma,\omega}} = \infty$, so $f \notin \mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ and we have arrived at a contradiction.

Note that Case 2 implies that $\sup_{t < r < s} \omega(x, r)^{-1} = \infty$, consequently

$$\begin{aligned} \sup_{t < r < \infty} \omega(x, r)^{-1} \left(sh \frac{r}{2} \right)^{-\frac{\gamma}{p}} &\geq \sup_{t < r < s} \omega(x, r)^{-1} \left(sh \frac{r}{2} \right)^{-\frac{\gamma}{p}} \\ &\geq \left(sh \frac{s}{2} \right)^{-\frac{\gamma}{p}} \sup_{t < r < s} \omega(x, r)^{-1} = \infty, \end{aligned}$$

which is the case in (i).

Denote by $L_{\infty, v}(0, \infty)$ the space of all functions $g(cht)$, $t > 0$ with the finite norm

$$\|g\|_{L_{\infty, v}(0, \infty)} = \operatorname{esssup}_{t > 0} v(cht)g(cht)$$

and

$$L_{\infty}(0, \infty) \equiv L_{\infty, 1}(0, \infty).$$

Remark 2.3. Denote by Ω_p^γ a set off all positive measurable functions ω on \mathbb{R}_+ such that for all $r > 0$

$$\sup_{x \in \mathbb{R}_+} \left\| \frac{\left(sh \frac{r}{2} \right)^{-\frac{\gamma}{p}}}{\omega(x, r)} \right\|_{L_{\infty}(t, \infty)} < \infty \quad \text{and} \quad \sup_{x \in \mathbb{R}_+} \|\omega(x, r)^{-1}\|_{L_{\infty}(0, t)} < \infty,$$

respectively. Lemma 2.1 shows, that it makes sense to consider only functions ω , from Ω_p^γ , which we will assume in what follows.

A function $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be almost increasing (resp. almost decreasing) if $\omega(r) \lesssim \omega(s)$ (resp $\omega(r) \gtrsim \omega(s)$) for $r \leq s$. Let $1 \leq p < \infty$. Denote by Φ_p^γ a set of all almost decreasing functions $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $t \in \mathbb{R}_+ \rightarrow \left(sh \frac{t}{2} \right)^{\frac{\gamma}{p}} \omega(t) \in \mathbb{R}_+$ is almost increasing.

Lemma 2.4. Let $\omega \in \Phi_p^\gamma$, $1 \leq p < \infty$, $H_0 = (0, r_0)$ and χ_{H_0} is the characteristic function of the characteristic function of the interval H_0 , then $\chi_{H_0} \in \mathcal{M}_{p, \gamma, \omega}(\mathbb{R}_+)$.

Moreover,

$$\frac{1}{\omega(r_0)} \leq \|\chi_{H_0}\|_{W\mathcal{M}_{p, \gamma, \omega}(\mathbb{R}_+)} \leq \|\chi_{H_0}\|_{\mathcal{M}_{p, \gamma, \omega}(\mathbb{R}_+)} \lesssim \frac{1}{\omega(r_0)}.$$

Proof. Let $\omega \in \Phi_p^\gamma$, $1 \leq p < \infty$, $H_0 = (0, r_0)$ be any interval on \mathbb{R}_+ . It is easy to see that

$$\begin{aligned} \|\chi_{H_0}\|_{W\mathcal{M}_{p, \gamma, \omega}(\mathbb{R}_+)} &= \sup_{x \in \mathbb{R}_+} \frac{1}{\omega(r)} \left(\frac{|H_r \cap H_0|_\lambda}{|H_r|_\lambda} \right)^{\frac{1}{p}} \\ &\geq \frac{1}{\omega(r_0)} \left(\frac{|H_0 \cap H_0|_\lambda}{|H_0|_\lambda} \right)^{\frac{1}{p}} = \frac{1}{\omega(r_0)}. \end{aligned}$$

Now, if $r \leq r_0$ then $\omega(r_0) \lesssim \omega(r)$ and

$$\frac{1}{\omega(r)} \left(\frac{|H_r \cap H_0|_\lambda}{|H_r|_\lambda} \right)^{\frac{1}{p}} \lesssim \frac{1}{\omega(r_0)} \left(\frac{|H_r \cap H_r|_\lambda}{|H_r|_\lambda} \right)^{\frac{1}{p}} \lesssim \frac{1}{\omega(r_0)}.$$

On the other hand, if $r_0 \leq r$ then by (2.1)

$$\omega(r_0) \left(sh \frac{r}{2} \right)^{\frac{\gamma}{p}} \lesssim \omega(r) \left(sh \frac{r}{2} \right)^{\frac{\gamma}{p}},$$

then

$$\begin{aligned} \frac{1}{\omega(r)} \left(\frac{|H_r \cap H_0|_\lambda}{|H_r|_\lambda} \right)^{\frac{1}{p}} &\lesssim \frac{(|H_0 \cap H_0|_\lambda)^{\frac{1}{p}}}{\omega(r) (sh \frac{r}{2})^{\frac{\gamma}{p}}} = \frac{(|H_0|_\lambda)^{\frac{1}{p}}}{\omega(r) (sh \frac{r}{2})^{\frac{\gamma}{p}}} \\ &\lesssim \frac{(sh \frac{r_0}{2})^{\frac{\gamma}{p}}}{\omega(r) (sh \frac{r}{2})^{\frac{\gamma}{p}}} \lesssim \frac{1}{\omega(r_0)}. \end{aligned}$$

This completes the proof.

Lemma 2.5. [32] *If $f \in L_{1,\lambda}^{loc}(\mathbb{R}_+)$, $1 \leq p \leq \infty$. Then for any $0 \leq t < \infty$ the following inequality*

$$\left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(\mathbb{R}_+)} \leq \|f\|_{L_{p,\lambda}(\mathbb{R}_+)}$$

holds.

Denote

$$M_{G_1}^\alpha f(chx) = \sup_{r \in (0,2)} \frac{1}{|H_r|_\lambda^{1-\frac{\alpha}{2\lambda+1}}} \int_{H_r} A_{cht}^\lambda |f(chx)| sh^{2\lambda} t dt.$$

Lemma 2.6. (1) *Let $1 \leq p < \infty$, $0 \leq \alpha < \frac{2\lambda+1}{p}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda+1}$. Then for $p > 1$ and any interval $H_r = (0, r)$, $0 < r < 2$ the following inequality*

$$\begin{aligned} \left\| A_{chx}^\lambda M_{G_1}^\alpha f \right\|_{L_{q,\lambda}(H_r)} &\lesssim \left\| A_{chx}^\lambda f \right\|_{L_{q,\lambda}(H_r)} \\ &+ \left(sh \frac{r}{2} \right)^{\frac{2\lambda+1}{q}} \sup_{r < s < 2} \left(sh \frac{s}{2} \right)^{\alpha-2\lambda-1} \left\| A_{chx}^\lambda f \right\|_{L_{q,\lambda}(H_s)} \end{aligned} \quad (2.5)$$

holds for all $f \in L_{p,\lambda}^{loc}(\mathbb{R}_+)$.

Moreover, for $p = 1$ the following inequality

$$\begin{aligned} \left\| A_{chx}^\lambda M_{G_1}^\alpha f \right\|_{WL_{q,\lambda}(H_r)} &\lesssim \left\| A_{chx}^\lambda f \right\|_{L_{1,\lambda}(H_r)} \\ &+ \left(sh \frac{r}{2} \right)^{\frac{2\lambda+1}{q}} \sup_{r < s < 2} \left(sh \frac{s}{2} \right)^{\alpha-2\lambda-1} \left\| A_{chx}^\lambda f \right\|_{L_{1,\lambda}(H_s)}, \end{aligned} \quad (2.6)$$

holds for all $f \in L_{1,\lambda}^{loc}(\mathbb{R}_+)$.

Proof. Let $1 < p < q < \infty$. For the interval $H_r = (0, 2)$, where $r \in (0, 2)$ let $f = f_1 + f_2$, where $f_1 = f\chi_{H_r}$ and $f_2 = f\chi_{(H_r)^c} = f\chi_{(r,2)}$, then

$$\left\| M_{G_1}^\alpha f \right\|_{L_{q,\lambda}(H_r)} \lesssim \left\| M_{G_1}^\alpha f_1 \right\|_{L_{q,\lambda}(H_r)} + \left\| M_{G_1}^\alpha f_2 \right\|_{L_{q,\lambda}(H_r)}. \quad (2.7)$$

From continuity of the operator $M_G^\alpha : L_{p,\lambda}(\mathbb{R}_+) \rightarrow L_{q,\lambda}(\mathbb{R}_+)$, (see [18, Corollary 5.6]) follows that

$$\left\| M_{G_1}^\alpha f_1 \right\|_{L_{q,\lambda}(H_r)} \lesssim \|f_1\|_{L_{p,\lambda}(H_r)} \lesssim \|f\|_{L_{p,\lambda}(H_r)}. \quad (2.8)$$

If $H_s \cap (H_r)^c \neq \emptyset$, then $s > r$ and, we have

$$\begin{aligned}
M_{G_2}^\alpha f_1(chx) &= \sup_{s>0} |H_s|_\lambda^{\frac{\alpha}{\gamma}-1} \int_{H_s \cap (H_r)^c} A_{cht}^\lambda |f_1(chx)| sh^{2\lambda} t dt \\
&\leq \sup_{r<s<2} |H_s|_\lambda^{\frac{\alpha}{2\lambda+1}-1} \int_r^s A_{cht}^\lambda |f_1(chx)| sh^{2\lambda} t dt \\
&\lesssim \sup_{r<s<2} |H_s|_\lambda^{\frac{\alpha}{2\lambda+1}-1} \int_0^s A_{cht}^\lambda |f(chx)| sh^{2\lambda} t dt \\
&= \sup_{r<s<2} |H_s|_\lambda^{\frac{\alpha}{2\lambda+1}-1} \int_{H_s} A_{cht}^\lambda |f(chx)| sh^{2\lambda} t dt. \tag{2.9}
\end{aligned}$$

From (2.9) and Lemma 2.5, we get

$$\|M_{G_2}^\alpha f_1\|_{L_{q,\lambda}(H_r)} \lesssim |H_r|_\lambda^{\frac{1}{q}} \sup_{r<s<2} |H_s|_\lambda^{\frac{\alpha}{2\lambda+1}-1} \|f\|_{L_{p,\lambda}(H_s)}. \tag{2.10}$$

Taking into account (2.8) and (2.10) in (2.7), we obtain

$$\|M_{G_1}^\alpha f\|_{L_{q,\lambda}(H_r)} \lesssim \|f\|_{L_{p,\lambda}(H_s)} + |H_r|_\lambda^{\frac{1}{q}} \sup_{r<s<2} |H_s|_\lambda^{\frac{\alpha}{2\lambda+1}-1} \|f\|_{L_{p,\lambda}(H_s)}$$

Using equality (see[16], proof of Theorem 1.4) $M_G A_{cht}^\lambda f(chx) = A_{cht}^\lambda M_G f(chx)$, and (2.1), we have (2.5).

Indeed

$$\|A_{chx}^\lambda M_{G_1}^\alpha f\|_{L_{q,\lambda}(H_r)} \lesssim \|A_{chx}^\lambda f\|_{L_{p,\lambda}(H_r)} + \left(sh\frac{r}{2}\right)^{\frac{2\lambda+1}{q}} \sup_{r<s<2} \left(sh\frac{s}{2}\right)^{\alpha-2\lambda-1} \|A_{chx}^\lambda f\|_{L_{p,\lambda}(H_s)}.$$

Let $p = 1$. Obviously that for interval H_r

$$\|M_{G_1}^\alpha f_1\|_{WL_{q,\lambda}(H_r)} \lesssim \|M_{G_1}^\alpha f_1\|_{WL_{q,\lambda}(H_r)} + \|M_{G_1}^\alpha f_2\|_{WL_{q,\lambda}(H_r)}.$$

From continuity of the operator $M_G^\alpha : L_{1,\lambda}(\mathbb{R}_+) \rightarrow WL_{q,\lambda}(\mathbb{R}_+)$ (see[18, Corollary 5.6]) we get

$$\|M_{G_1}^\alpha f_1\|_{WL_{q,\lambda}(H_r)} \lesssim \|f\|_{L_{1,\lambda}(H_r)}. \tag{2.11}$$

Similar to the previous from (2.10) and (2.11), we obtain (2.6). The proof of the Lemma is complete.

Denote

$$M_{G_2}^\alpha f(chx) = \sup_{r \in [2, \infty)} \frac{1}{|H_r|_\lambda^{1-\frac{\alpha}{4\lambda}}} \int_{H_r} A_{cht}^\lambda |f(chx)| sh^{2\lambda} t dt.$$

Lemma 2.6. (2). Let $1 \leq p < \infty$, $0 \leq \alpha < \frac{4\lambda}{p}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{4\lambda}$. Then for $p > 1$ and any interval $H_r = (0, r)$, $r \in [2, \infty)$ the following inequality

$$\|A_{chx}^\lambda M_{G_2}^\alpha f\|_{L_{q,\lambda}(H_r)} \lesssim \|A_{chx}^\lambda f\|_{L_{p,\lambda}(H_r)}$$

$$+ \left(sh \frac{r}{2} \right)^{\frac{4\lambda}{q}} \sup_{s>r \geq 2} \left(sh \frac{s}{2} \right)^{\alpha-4\lambda} \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)} \quad (2.12)$$

holds for all $f \in L_{p,\lambda}^{loc}(\mathbb{R}_+)$.

Moreover, for $p = 1$ the following inequality

$$\begin{aligned} & \left\| A_{chx}^\lambda M_{G_2}^\alpha f \right\|_{WL_{q,\lambda}(H_r)} \lesssim \left\| A_{chx}^\lambda f \right\|_{L_{1,\lambda}(H_r)} \\ & + \left(sh \frac{r}{2} \right)^{\frac{4\lambda}{q}} \sup_{s>r \geq 2} \left(sh \frac{s}{2} \right)^{\alpha-4\lambda} \left\| A_{chx}^\lambda f \right\|_{L_{1,\lambda}(H_s)}, \end{aligned} \quad (2.13)$$

holds for all $f \in L_{1,\lambda}^{loc}(\mathbb{R}_+)$.

Proof. Let $1 < p < q < \infty$. For the interval $H_r = (0, r)$, $r \in [2, \infty)$ let $f = f_1 + f_2$, where $f_1 = f\chi_{H_r}$ and $f_2 = f\chi_{(H_r)^c} = f\chi_{(r,\infty)}$, then

$$\left\| M_{G_2}^\alpha f \right\|_{L_{q,\lambda}(H_r)} \leq \left\| M_{G_2}^\alpha f_1 \right\|_{L_{q,\lambda}(H_r)} + \left\| M_{G_2}^\alpha f_2 \right\|_{L_{q,\lambda}(H_r)}. \quad (2.14)$$

From continuity of the operator $M_G^\alpha : L_{p,\lambda}(\mathbb{R}_+) \rightarrow L_{q,\lambda}(\mathbb{R}_+)$, follows that

$$\left\| M_{G_2}^\alpha f \right\|_{L_{q,\lambda}(H_r)} \lesssim \|f\|_{L_{p,\lambda}(H_r)}. \quad (2.15)$$

If $H_s \cap (H_r)^c \neq \emptyset$, then $s > r$ and, we get

$$\begin{aligned} M_{G_2}^\alpha f_2(chx) &= \sup_{s>0} |H_s|_\lambda^{\frac{\alpha}{4\lambda}-1} \int_{H_s \cap (H_r)^c} A_{cht}^\lambda |f_2(chx)| sh^{2\lambda} t dt \\ &\leq \sup_{2 \leq r < s < \infty} |H_s|_\lambda^{\frac{\alpha}{4\lambda}-1} \int_r^s A_{cht}^\lambda |f_2(chx)| sh^{2\lambda} t dt \\ &\lesssim \sup_{2 \leq r < s < \infty} |H_s|_\lambda^{\frac{\alpha}{4\lambda}-1} \int_0^s A_{cht}^\lambda |f(chx)| sh^{2\lambda} t dt = \\ &= \sup_{2 \leq r < s < \infty} |H_s|_\lambda^{\frac{\alpha}{4\lambda}-1} \int_{H_s} A_{cht}^\lambda |f(chx)| sh^{2\lambda} t dt. \end{aligned} \quad (2.16)$$

By Hölder's inequality from (2.16) and (2.1), we have

$$\left\| M_{G_2}^\alpha f_2 \right\|_{L_{q,\lambda}(H_r)} \lesssim |H_r|_\lambda^{\frac{1}{q}} \sup_{2 \leq r < s < \infty} |H_s|_\lambda^{\frac{\alpha}{4\lambda}-1} \|f\|_{L_{p,\lambda}(H_s)}. \quad (2.17)$$

Taking into account (2.15) and (2.17) in (2.14), we obtain

$$\left\| M_{G_2}^\alpha f \right\|_{L_{q,\lambda}(H_r)} \lesssim \|f\|_{L_{p,\lambda}(H_r)} + |H_r|_\lambda^{\frac{1}{q}} \sup_{2 \leq r < s < \infty} |H_s|_\lambda^{\frac{\alpha}{4\lambda}-1} \|f\|_{L_{p,\lambda}(H_s)}.$$

Arguing as above, we get (2.12).

Now let $p = 1$. Clear that

$$\left\| M_{G_2}^\alpha f \right\|_{WL_{q,\lambda}(H_r)} \lesssim \left\| M_{G_2}^\alpha f_1 \right\|_{WL_{q,\lambda}(H_r)} + \left\| M_{G_2}^\alpha f_2 \right\|_{WL_{q,\lambda}(H_r)}.$$

From continuity of the operator $M_G^\alpha : L_{1,\lambda}(\mathbb{R}_+) \rightarrow WL_{q,\lambda}(\mathbb{R}_+)$, we have

$$\|M_{G_2}^\alpha f_1\|_{WL_{q,\lambda}(H_r)} \lesssim \|f\|_{L_{1,\lambda}(H_r)}. \quad (2.18)$$

From (2.17) and (2.18), we obtain (2.13).

Lemma 2.7. *Let $1 \leq p < \infty$, and $0 \leq \alpha < \frac{\gamma}{p}$.*

- (a) Let $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma}$, $\gamma = 2\lambda + 1$, if $r \in (0, 2)$,
 (b) Let $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma}$, $\gamma = 4\lambda$, if $r \in [2, \infty)$.

Then for any interval $H_r = (0, r) \subset \mathbb{R}_+$ the following inequalities is valid:

$$\|A_{chx}^\lambda M_G^\alpha f\|_{L_{q,\lambda}(H_r)} \lesssim \|A_{chx}^\lambda f\|_{L_{p,\lambda}(H_r)} + \left(sh \frac{r}{2} \right)^{\frac{\gamma}{q}} \sup_{s>r} \left(sh \frac{r}{2} \right)^{\alpha-\gamma} \|A_{chx}^\lambda f\|_{L_{p,\lambda}(H_s)}$$

for any $p > 1$ and all $f \in L_{p,\lambda}^{loc}(\mathbb{R}_+)$ and

$$\|A_{chx}^\lambda M_G^\alpha f\|_{WL_{q,\lambda}(H_r)} \lesssim \|A_{chx}^\lambda f\|_{L_{1,\lambda}(H_r)} + \left(sh \frac{r}{2} \right)^{\frac{\gamma}{q}} \sup_{s>r} \left(sh \frac{s}{2} \right)^{\alpha-\gamma} \|A_{chx}^\lambda f\|_{L_{1,\lambda}(H_s)}$$

for $p = 1$ and all $f \in L_{1,\lambda}^{loc}(\mathbb{R}_+)$.

Proof. By the definition and (2.1), we have

$$\begin{aligned} M_G^\alpha f(chx) &= \sup_{s>r} |H_r|_{\lambda}^{\frac{\alpha}{\gamma}-1} \int_{H_s} A_{cht}^\lambda |f(chx)| sh^{2\lambda} t dt \\ &= M_{G_1}^\alpha f(chx) + M_{G_2}^\alpha f(chx). \end{aligned}$$

Taking into account (2.5),(2.6) also (2.12) and (2.13) we will receive our approval.

Let

$$\begin{aligned} &\left(sh \frac{r}{2} \right)^{\frac{\gamma}{q}} \sup_{s>r} \left(sh \frac{s}{2} \right)^{-\frac{\gamma}{q}} \|A_{chx}^\lambda f\|_{L_{p,\lambda}(H_s)} \\ &= \max \left\{ \left(sh \frac{r}{2} \right)^{\frac{2\lambda+1}{q}} \sup_{r<s<2} \left(sh \frac{s}{2} \right)^{-\frac{2\lambda+1}{q}} \|A_{chx}^\lambda f\|_{L_{p,\lambda}(H_s)}, \right. \\ &\quad \left. \left(sh \frac{r}{2} \right)^{\frac{4\lambda}{q}} \sup_{s>r \geq 2} \left(sh \frac{s}{2} \right)^{-\frac{4\lambda}{q}} \|A_{chx}^\lambda f\|_{L_{p,\lambda}(H_s)} \right\}. \end{aligned}$$

Lemma 2.8. *Suppose $1 \leq p < \infty$, and let the conditions (2.19) holds.*

Then for any interval $H_r = (0, r) \subset \mathbb{R}_+$ the following inequalities is valid:

$$\|A_{chx}^\lambda M_G^\alpha f\|_{L_{q,\lambda}(H_r)} \lesssim \left(sh \frac{r}{2} \right)^{\frac{\gamma}{q}} \sup_{s>r} \left(sh \frac{s}{2} \right)^{-\frac{\gamma}{q}} \|A_{chx}^\lambda f\|_{L_{p,\lambda}(H_s)}$$

for any $p > 1$ and all $f \in L_{p,\lambda}^{loc}(\mathbb{R}_+)$, and

$$\|A_{chx}^\lambda M_G^\alpha f\|_{WL_{q,\lambda}(H_r)} \lesssim \left(sh \frac{r}{2} \right)^{\frac{\gamma}{q}} \sup_{s>r} \left(sh \frac{s}{2} \right)^{-\frac{\gamma}{q}} \|A_{chx}^\lambda f\|_{L_{p,\lambda}(H_s)}$$

for any $p = 1$ and all $f \in L_{1,\lambda}^{loc}(\mathbb{R}_+)$.

Proof. Denote,

$$A := |H_r|_\lambda^{\frac{1}{q}} \left(\sup_{s>r} |H_s|_\lambda^{\frac{\alpha}{\gamma}-1} \int_{H_s} A_{cht}^\lambda |f(chx)| sh^{2\lambda} t dt \right),$$

$$B := \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_r)}.$$

Using the Hölder's inequality and (2.1), we get

$$A \lesssim |H_r|_\lambda^{\frac{1}{q}} \left(\sup_{s>r} |H_s|_\lambda^{\frac{1}{q}} |H_s|_\lambda^{\frac{\alpha}{\gamma}-1} \left(\int_{H_s} A_{cht}^\lambda |f(chx)|^p sh^{2\lambda} t dt \right)^{\frac{1}{p}} \right)$$

$$= |H_r|_\lambda^{\frac{1}{q}} \left(\sup_{s>r} |H_s|_\lambda^{-\frac{1}{q}} \right) \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)} \approx \left(sh \frac{r}{2} \right)^{\frac{\gamma}{q}} \sup_{s>r} \left(sh \frac{s}{2} \right)^{-\frac{\gamma}{q}} \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)}.$$

On the other hand

$$|H_r|_\lambda^{\frac{1}{q}} \left(\sup_{s>r} |H_s|_\lambda^{-\frac{1}{q}} \left(\int_{H_s} A_{cht}^\lambda |f(chx)|^p sh^{2\lambda} t dt \right)^{\frac{1}{p}} \right)$$

$$\gtrsim |H_r|_\lambda^{\frac{1}{q}} \left(\sup_{s>r} |H_s|_\lambda^{-\frac{1}{q}} \right) \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_r)} \approx B.$$

By Lemma 2.7

$$\left\| A_{chx}^\lambda M_G^\alpha f \right\|_{L_{q,\lambda}(H_r)} \lesssim A + B.$$

From this it follows the approval of lemma.

3 Boundedness of fractional maximal operator M_G^α on the space $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$

In the following theorem (see [24, Theorem 3.2]) the Spanne-Guliyev result was obtained.

Theorem 3.1. Suppose $1 \leq p < \infty$, $\omega_1 \in \Omega_p^\gamma$, $\omega_2 \in \Omega_q^\gamma$. Moreover, let the conditions (2.19) holds, also the pair (ω_1, ω_2) satisfy the condition

$$\sup_{s>r} \left(sh \frac{s}{2} \right)^\alpha \omega_1(x, s) \lesssim \omega_2(x, r). \quad (3.1)$$

Then M_G^α is bounded from $\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$ for $p > 1$ and \mathcal{M}_G^α is bounded from $\mathcal{M}_{1,\gamma,\omega_1}(\mathbb{R}_+)$ to $W\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$ for $p = 1$.

Proof. Denote

$$E_\gamma = \begin{cases} (0, 2), & \text{if } \gamma = 2\lambda + 1 \\ [2, \infty), & \text{if } \gamma = 4\lambda. \end{cases}$$

Then by Lemma 2.8, we have

$$\|M_G^\alpha f\|_{\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)} = \sup_{\substack{x \in \mathbb{R}_+ \\ r \in E_\gamma}} \omega_2(x, r)^{-1} \left(sh \frac{r}{2} \right)^{-\frac{\gamma}{q}} \left\| A_{chx}^\lambda M_G^\alpha f \right\|_{L_{q,\gamma}(H_r)}$$

$$\begin{aligned}
& \lesssim \sup_{\substack{x \in \mathbb{R}_+ \\ r \in E_\gamma}} \omega_2(x, r)^{-1} \sup_{s > r} \left(sh \frac{s}{2} \right)^{-\frac{\gamma}{q}} \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)} \\
& \lesssim \sup_{\substack{x \in \mathbb{R}_+ \\ r \in E_\gamma}} \omega_2(x, r)^{-1} \sup_{s > r} \left(sh \frac{s}{2} \right)^{\frac{\gamma}{p} - \frac{\gamma}{q}} \omega_1(x, s) \left(\frac{\omega_1(x, s)^{-1} \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)}}{\left(sh \frac{s}{2} \right)^{\frac{\gamma}{p}}} \right) \\
& \lesssim \|f\|_{\mathcal{M}_{q,\gamma,\omega_1}(\mathbb{R}_+)} \sup_{x,r \in \mathbb{R}_+} \omega_2(x, r)^{-1} \sup_{s > r} \left(sh \frac{s}{2} \right)^\alpha \omega_1(x, s) \\
& \lesssim \|f\|_{\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)}, \text{ if } p \in (1, \infty).
\end{aligned}$$

If $p = 1$, then

$$\begin{aligned}
\|M_G^\alpha f\|_{W\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)} & \lesssim \sup_{\substack{x \in \mathbb{R}_+ \\ r \in E_\gamma}} \omega_2(x, r)^{-1} \left(\sup_{s > r} \left(sh \frac{s}{2} \right)^{-\frac{\gamma}{q}} \left\| A_{chx}^\lambda f \right\|_{L_{1,\lambda}(H_s)} \right) \\
& \lesssim \|f\|_{\mathcal{M}_{1,\gamma,\omega_1}(\mathbb{R}_+)} \sup_{x,r \in \mathbb{R}_+} \omega_2(x, r)^{-1} \sup_{s > r} \left(sh \frac{s}{2} \right)^\alpha \omega_1(x, s) \lesssim \|f\|_{\mathcal{M}_{1,\gamma,\omega_1}(\mathbb{R}_+)}.
\end{aligned}$$

In the case, then $\alpha = 0$ and $p = q$ from Theorem 3.1, we get the following corollary.

Corollary 3.2. *Let $1 \leq p < \infty$ and (ω_1, ω_2) satisfy the condition*

$$\sup_{s > r} \omega_1(x, s) \lesssim \omega_2(x, r). \quad (3.2)$$

Then M_G is bounded from $\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$ for $p > 1$ and M_G is bounded from $\mathcal{M}_{1,\gamma,\omega_1}(\mathbb{R}_+)$ to $W\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$ for $p = 1$.

Later we will need the following lemma.

Lemma 3.3. *The following inequality $t \leq sht \leq e^A t$, $A > 0$ is true for any $t \in [0, A] \subset \mathbb{R}_+$.*

Proof. Consider the function $f(t) = sht - t$. From the inequality $f'(t) = cht - 1 \geq 0$ it follows that $f(t)$ is increasing on \mathbb{R}_+ . Therefore, the functions $f(t)$ takes smallest value when $t = 0$, and $f(t) = 0$, hence, $f(t) \geq 0 \Leftrightarrow sht \geq t$. We will prove the right side of the inequality

$$sht \leq e^A t \Leftrightarrow \frac{e^t - e^{-t}}{2} \leq e^A t \Leftrightarrow e^{2t} \leq 2e^A t e^t + 1.$$

We will find the minimum of the function $\varphi(t) = 2e^A t e^t + 1 - e^{2t}$.

$$\varphi'(t) = 2e^A(e^t + t e^t) - 2e^{2t} = 2e^t(e^A + e^A t - e^t) \geq 0 \Leftrightarrow e^A(t+1) \geq e^t,$$

hence $\varphi(t)$ takes the smallest value at the point $t = 0$ and $\varphi(0) = 0$. Then $\varphi(t) \geq 0 \Leftrightarrow 2e^A t e^t + 1 \geq e^{2t} \Leftrightarrow sht \leq e^A t$, for any $A > 0$.

Lemma 3.4. *Let $H_0 = (0, r_0)$ then the estimate*

$$\left(sh \frac{r_0}{2} \right)^\alpha \lesssim M_G^\alpha \chi_{H_0}(chx)$$

is true for any $x \in \mathbb{R}_+$.

Proof. We will choose c_0 so large, that the inequality $r \leq c_0 r_0$ would hold. Then by Lemma 3.3, when $0 \leq t < c_0$, the inequality $t \leq sht \leq e^{c_0 t}$ is true and by according to (2.1), we get

$$\begin{aligned} |H_r|_\lambda^{1-\frac{\alpha}{\gamma}} &= \left(\int_0^r sh^{2\lambda} t dt \right)^{1-\frac{\alpha}{\gamma}} \leq \left(\int_0^{c_0 r_0} sh^{2\lambda} t dt \right)^{1-\frac{\alpha}{\gamma}} \\ &\approx \left(sh \frac{c_0 r_0}{2} \right)^{\gamma-\alpha} \lesssim \left(e^{c_0} \frac{c_0 r_0}{2} \right)^{\gamma-\alpha} \lesssim (c_0 e^{c_0})^{\gamma-\alpha} \left(sh \frac{r_0}{2} \right)^{\gamma-\alpha}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_{H_r} A_{cht}^\lambda \chi_{H_0}(chx) sh^{2\lambda} t dt &\geq \int_{H_r \cap H_0} A_{cht}^\lambda \chi_{H_0}(chx) sh^{2\lambda} t dt \\ &\geq \int_{H_0} sh^{2\lambda} t dt \approx \left(sh \frac{r_0}{2} \right)^\gamma. \end{aligned}$$

Thus,

$$M_G^\alpha \chi_{H_0}(chx) = \sup_{r>0} |H_r|_\lambda^{\frac{\alpha}{\gamma}-1} \int_{H_r} A_{cht}^\lambda \chi_{H_0}(chx) sh^{2\lambda} t dt \gtrsim \left(sh \frac{r_0}{2} \right)^\alpha.$$

The following theorem is an analogue of Theorem 4.3 from [25] and this is one of the main results of this paper, (Gunawan-Guliyev type result) (see [13, Theorem 2.3]).

Theorem 3.5. Let $0 \leq \alpha < \gamma, p, q \in [1, \infty), \omega_1 \in \Omega_p^\gamma, \omega_2 \in \Omega_q^\gamma$.

(i) If the conditions (2.19) holds and $1 \leq p < \frac{\gamma}{\alpha}$, then the condition (3.1) is sufficient for the boundedness M_G^α from $\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)$ to $W\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$ for $p = 1$. Moreover, if $1 < p < \frac{\gamma}{\alpha}$, then the condition (3.1) is sufficient for the boundedness M_G^α from $\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$.

(ii) If the function $\omega_1 \in \Phi_p^\gamma$, then the condition

$$\left(sh \frac{r}{2} \right)^\alpha \omega_1(r) \lesssim \omega_2(r), \quad r > 0 \quad (3.3)$$

is necessary for the boundedness M_G^α from $\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)$ to $W\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$ for $p = 1$, and $\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$ for $p > 1$.

(iii) Let the condition (2.19) holds and $1 \leq p < \frac{\gamma}{\alpha}$. If $\omega_1 \in \Omega_p^\gamma$, then condition (3.3) is necessary and sufficient for the boundedness of M_G^α from $\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)$ to $W\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$ for $p = 1$. Moreover, if $1 < p < \frac{\gamma}{\alpha}$, then the condition (3.3) is necessary and sufficient for the boundedness M_G^α from $\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$.

Proof. The first part of the theorem follows from Theorem 3.1 when $s = r, \omega_1(x, r) = \omega_1(r)$ and $\omega_2(x, r) = \omega_2(r)$.

We will prove the second part of the theorem.

Let $H_0 = (0, r_0)$ and $x \in H_0$. By lemma 3.4, we have $(sh \frac{r}{2})^\alpha \lesssim M_G^\alpha \chi_{H_0}(chx)$. Therefore, by Lemma 2.4 and 3.4, we obtain

$$\begin{aligned} \left(sh \frac{r_0}{2} \right)^\alpha &\lesssim |H_0|_\lambda^{-\frac{1}{q}} \|M_G^\alpha \chi_{H_s}\|_{L_{q,\lambda}(H_0)} \lesssim \omega_2(r_0) \|M_G^\alpha \chi_{H_0}\|_{\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)} \\ &\lesssim \omega_2(r_0) \|\chi_{H_0}\|_{\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)} \lesssim \frac{\omega_2(r_0)}{\omega_1(r_0)} \end{aligned}$$

or

$$\left(sh \frac{r_0}{2} \right)^\alpha \lesssim \frac{\omega_2(r_0)}{\omega_1(r_0)} \Leftrightarrow \left(sh \frac{r_0}{2} \right)^\alpha \omega_1(r_0) \lesssim \omega_2(r_0)$$

when $r_0 > 0$.

Since, this is true for any the $r_0 > 0$, then the second part is proved.

The third part follows from the first and second parts of the theorem.

Remark 3.6. If we take $\omega_1(r) = \left(sh \frac{r}{2} \right)^{\frac{\nu-\gamma}{p}}$ and $\omega_2(r) = \left(sh \frac{r}{2} \right)^{\frac{\mu-\gamma}{q}}$ at Theorem 3.5 then the condition (3.3) are equivalent to $0 < \nu < \gamma - \alpha p$ and $\frac{\nu}{p} = \frac{\mu}{q}$ respectively.

Therefore, we get the following analogue of Theorem C .

Corollary 3.7. Let $0 \leq \alpha < \gamma$, $1 \leq p < \frac{\gamma}{\alpha}$, $0 < \nu < \gamma - \alpha p$ and decides:

$$\begin{aligned} \text{(a) Let } & \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma - \nu}, \gamma = 2\lambda + 1, \text{ if } r \in (0, 2), \\ \text{(b) Let } & \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma - \nu}, \quad \gamma = 4\lambda, \text{ if } r \in [2, \infty). \end{aligned} \quad (3.4)$$

Then for $p = 1$ M_G^α is bounded from $L_{p,\lambda,\nu}(\mathbb{R}_+)$ to $WL_{q,\lambda,\mu}(\mathbb{R}_+)$ if and only if $\frac{\nu}{p} = \frac{\mu}{q}$.

Moreover, for $p > 1$ M_G^α is bounded from $L_{p,\lambda,\nu}(\mathbb{R}_+)$ to $L_{q,\lambda,\nu}(\mathbb{R}_+)$ if and only if $\frac{\nu}{p} = \frac{\mu}{q}$.

4 Adams-Guliyev and Adams-Gunavan type result

The following statement is an analogue of Theorem 4.5 from [25] and plays an important role in the proofs of our main results.

Theorem 4.1. Let $1 \leq p < \infty$, $0 \leq \alpha < \gamma$ and $f \in L_{p,\lambda}^{loc}(\mathbb{R}_+)$. Then the following inequality

$$M_G^\alpha f(chx) \lesssim \left(sh \frac{r}{2} \right)^\alpha M_G f(chx) + \sup_{s>r} \left(sh \frac{s}{2} \right)^{\alpha - \frac{\gamma}{p}} \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)}. \quad (4.1)$$

is true for all $x \in \mathbb{R}_+$.

Proof. Let $1 \leq p < \infty$ and $f \in L_{p,\lambda}^{loc}(\mathbb{R}_+)$. We split the function f as $f = f_1 + f_2$, where $f_1(chx) = f\chi_{(H_r)}(chx)$ and $f_2(chx) = f\chi_{(H_r)^c}(chx)$. Then

$$M_G^\alpha f(chx) \leq M_G^\alpha f_1(chx) + M_G^\alpha f_2(chx).$$

First let's show

$$M_G^\alpha f_1(chx) \lesssim \left(sh \frac{r}{2} \right)^\alpha M_G f(chx). \quad (4.2)$$

Valid according to the formula (2.1), we can write

$$\begin{aligned} \int_0^r A_{cht}^\lambda |f_1(chx)| sh^{2\lambda} t dt & \lesssim \sum_{k=0}^{\infty} \int_{2^{-k-1}r}^{2^{-k}r} \frac{A_{cht}^\lambda |f(chx)| \left(sh \frac{t}{2} \right)^{\gamma-\alpha} sh^{2\lambda} t}{\left(sh \frac{t}{2} \right)^{\gamma-\alpha}} dt \\ & \lesssim \sum_{k=0}^{\infty} \left(sh \frac{r}{2^{k+1}} \right)^{\gamma-\alpha} \int_{2^{-k-1}r}^{2^{-k}r} \frac{A_{cht}^\lambda |f(chx)| sh^{2\lambda} t}{\left(sh \frac{t}{2} \right)^{\gamma-\alpha}} dt \end{aligned}$$

$$\lesssim \left(sh \frac{r}{2} \right)^{\gamma-\alpha} J_G^\alpha f(chx) \approx |H_r|_\lambda^{1-\frac{\alpha}{\gamma}} J_G^\alpha f(chx). \quad (4.3)$$

From (4.3), we have

$$M_G^\alpha f(chx) \lesssim J_G^\alpha f(chx). \quad (4.4)$$

We will consider $J_G^\alpha f(chx)$.

$$J_G^\alpha f(chx) \lesssim \left(\int_0^r + \int_r^\infty \right) \frac{A_{cht}^\lambda |f(chx)| sh^{2\lambda t}}{(sh \frac{t}{2})^{\gamma-\alpha}} dt = J_1 + J_2. \quad (4.5)$$

Let's estimate J_1 . Applying Lemma 3.3, we obtain

$$\begin{aligned} J_1 &= \sum_{k=0}^{\infty} \int_{2^{-k-1}r}^{2^{-k}r} \frac{A_{cht}^\lambda |f(chx)| sh^{2\lambda t}}{(sh \frac{t}{2})^{\gamma-\alpha}} dt \lesssim \sum_{k=0}^{\infty} \frac{\left(sh \frac{r}{2^{k+1}} \right)^\alpha 2^{-k} r}{\left(sh \frac{r}{2^{k+2}} \right)^\gamma} \int_0^{2^{-k}r} A_{cht}^\lambda |f(chx)| sh^{2\lambda t} dt \\ &\lesssim 2^\gamma \left(sh \frac{r}{2} \right)^\alpha \sum_{k=0}^{\infty} \frac{2^{-k\alpha}}{\left(\frac{r}{2^{k+1}} \right)^\gamma} \int_0^{2^{-k}r} A_{cht}^\lambda |f(chx)| sh^{2\lambda t} dt \\ &\lesssim \left(sh \frac{r}{2} \right)^\alpha M_G f(chx) \sum_{k=0}^{\infty} 2^{-k\alpha} \lesssim \left(sh \frac{r}{2} \right)^\alpha M_G f(chx). \end{aligned} \quad (4.6)$$

We will estimate J_2 . By Hölder's inequality, we have

$$\begin{aligned} J_2 &= \int_r^\infty \frac{A_{cht}^\lambda |f(chx)| sh^{2\lambda t}}{(sh \frac{t}{2})^{\gamma-\alpha}} dt \\ &\lesssim \left(\int_r^\infty \frac{A_{cht}^\lambda |f(chx)| sh^{2\lambda t}}{(sh \frac{t}{2})^{(\gamma-\alpha)p'}} dt \right)^{\frac{1}{p'}} \left(\int_r^\infty \frac{A_{cht}^\lambda |f(chx)| sh^{2\lambda t}}{(sh \frac{t}{2})^{(\gamma-\alpha)p}} dt \right)^{\frac{1}{p}} = J_{21}^{\frac{1}{p'}} J_{22}^{\frac{1}{p}}. \end{aligned} \quad (4.7)$$

Consider J_{21} . Again use the Lemma 3.3, we get

$$\begin{aligned} J_{21} &= \sum_{k=0}^{\infty} \int_{2^k r}^{2^{k+1} r} \frac{A_{cht}^\lambda |f(chx)| sh^{2\lambda t}}{(sh \frac{t}{2})^{(\gamma-\alpha)q}} dt \\ &\lesssim \sum_{k=0}^{\infty} \frac{\left(sh 2^k \frac{r}{2} \right)^{\gamma-(\gamma-\alpha)q} 2^{k+1} r}{\left(sh 2^k r \right)^\gamma} \int_0^{2^{k+1} r} A_{cht}^\lambda |f(chx)| sh^{2\lambda t} dt \\ &\lesssim \left(sh \frac{r}{2} \right)^{\gamma-(\gamma-\alpha)p'} M_G f(chx) \sum_{k=0}^{\infty} \left(2^k \right)^{\gamma-(\gamma-\alpha)q} \lesssim \left(sh \frac{r}{2} \right)^{\gamma-(\gamma-\alpha)p'} M_G f(chx), \end{aligned} \quad (4.8)$$

We will similarly the following estimate for J_{22}

$$J_{22} \lesssim \left(sh \frac{r}{2} \right)^{\gamma-(\gamma-\alpha)p} M_G f(chx). \quad (4.9)$$

Then from (4.8) and (4.9), we have

$$J_{2k} \lesssim \left(sh \frac{r}{2} \right)^\alpha M_G f(chx), \quad k = 1, 2.$$

From this and (4.2) follows that

$$J_2 \lesssim \left(sh \frac{r}{2} \right)^\alpha M_G f(chx), \quad 0 < sh \frac{r}{2} < 1. \quad (4.10)$$

If $sh \frac{r}{2} \geq 1$, then by using (4.8) and (4.9) in (4.7), we obtain

$$\begin{aligned} J_2 &\lesssim \left(sh \frac{r}{2} \right)^{\frac{\gamma}{q} - \gamma + \alpha + \frac{\gamma}{p} - \gamma + \alpha} M_G f(chx) = \left(sh \frac{r}{2} \right)^{2\alpha - \gamma} M_G f(chx) \\ &= \frac{\left(sh \frac{r}{2} \right)^\alpha}{\left(sh \frac{r}{2} \right)^{\gamma - \alpha}} M_G f(chx) \lesssim \left(sh \frac{r}{2} \right)^\alpha M_G f(chx). \end{aligned} \quad (4.11)$$

Thus, from (4.10) and (4.11), we have

$$J_2 \lesssim \left(sh \frac{r}{2} \right)^\alpha M_G f(chx). \quad (4.12)$$

Using (4.6) and (4.12), we obtain (4.2). If $H_s \cap (H_r)^c \neq \emptyset$ then $s > r$ and by (2.9) and (2.16), we have

$$M_G^\alpha f_2(chx) \lesssim \sup_{s>r} |H_r|_\lambda^{\frac{\alpha}{\gamma} - 1} \int_{H_\lambda} A_{cht}^\lambda |f(chx)| sh^{2\lambda} t dt$$

(we used Hölder's inequality)

$$\begin{aligned} &\lesssim \sup_{s>r} |H_s|_\lambda^{\frac{\alpha}{\gamma} - 1} \left(\int_{H_s} A_{cht}^\lambda |f(chx)|^p sh^{2\lambda} t dt \right)^{\frac{1}{p}} \left(\int_{H_s} sh^{2\lambda} t dt \right)^{\frac{1}{q}} \\ &= \sup_{s>r} \frac{|H_s|_\lambda^{\frac{1}{q}}}{|H_s|_\lambda^{1 - \frac{\alpha}{\gamma}}} \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)} = \sup_{s>r} \frac{|H_s|_\lambda^{\frac{\alpha}{\gamma}}}{|H_s|_\lambda^{\frac{1}{p}}} \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)} \\ &\approx \sup_{s>r} \left(sh \frac{s}{2} \right)^{\alpha - \frac{\gamma}{p}} \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)}, \end{aligned} \quad (4.13)$$

At the end we used (2.1).

We get (4.1) by bringing together (4.2) and (4.13).

In the following theorem we obtain Adams-Guliyev type result (see [24, Theorem 3.3]).

Theorem 4.2. Let $1 \leq p < q < \infty$, $0 < \alpha < \frac{\gamma}{p}$, $\omega \in \Omega_p^\gamma$ and satisfy the conditions:

$$\sup_{s>r} \omega(x, s)^{\frac{1}{p}} \lesssim \omega(x, r)^{\frac{1}{p}} \quad (4.14)$$

and

$$\sup_{s>r} \left(sh \frac{s}{2} \right)^\alpha \omega(x, s)^{\frac{1}{p}} \lesssim \left(sh \frac{r}{2} \right)^{-\frac{\alpha p}{q-p}} \quad (4.15)$$

where $x, r \in \mathbb{R}_+$.

The operator M_G^α is bounded from $\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega^{\frac{1}{q}}}(\mathbb{R}_+)$ when $p > 1$ and from $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ to $W\mathcal{M}_{q,\gamma,\omega^{\frac{1}{q}}}(\mathbb{R}_+)$ when $p = 1$.

Proof. Let $1 \leq p < q < \infty$, $0 < \alpha < \frac{\gamma}{p}$ and $f \in \mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}(\mathbb{R}_+)$. Let's put $f = f_1 + f_2$, where $f_1 = f\chi_{(H_r)}$, $f_2 = f\chi_{(H_r)^c}$.

Then

$$M_G^\alpha f(chx) = M_G^\alpha f_1(chx) + M_G^\alpha f_2(chx).$$

For $M_G^\alpha f_1$ estimate (4.2) is valid. For $M_G^\alpha f_2$ by Hölder's inequality and (2.1), we have

$$\begin{aligned} M_G^\alpha f_2(chx) &\lesssim \sup_{s>r} |H_s|_\lambda^{\frac{\alpha}{\gamma}-1} \int_{H_s} A_{cht}^\lambda |f(chx)| sh^{2\lambda} t dt \\ &\lesssim \sup_{s>r} \left(sh \frac{s}{2} \right)^{\frac{\gamma}{q} + \alpha - \gamma} \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)}. \end{aligned} \quad (4.16)$$

Then from (4.16), also from (4.2) and (4.15), we obtain

$$\begin{aligned} M_G^\alpha f(chx) &\lesssim \left(sh \frac{r}{2} \right)^\alpha M_G f(chx) + \sup_{s>r} \left(sh \frac{s}{2} \right)^{\frac{\gamma}{q} + \alpha - \gamma} \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)} \\ &= \left(sh \frac{r}{2} \right)^\alpha M_G^\alpha f(chx) + \sup_{s>r} \left(sh \frac{s}{2} \right)^{\alpha - \frac{\gamma}{p}} \omega(x, s)^{\frac{1}{p}} \frac{\left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)}}{\omega(x, s)^{\frac{1}{p}}} \\ &\lesssim \left(sh \frac{r}{2} \right)^\alpha M_G f(chx) + \|f\|_{\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}(\mathbb{R}_+)} \sup_{s>r} \left(sh \frac{s}{2} \right)^\alpha \omega(x, s)^{\frac{1}{p}} \\ &\lesssim \left(sh \frac{r}{2} \right)^\alpha M_G f(chx) + \left(sh \frac{r}{2} \right)^{-\frac{\alpha p}{q-p}} \|f\|_{\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}(\mathbb{R}_+)}. \end{aligned}$$

Choosing $sh \frac{r}{2} = \left(\frac{\|f\|_{\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}(\mathbb{R}_+)}}{M_G f(chx)} \right)^{\frac{q-p}{\alpha q}}$, we get

$$|M_G^\alpha f(chx)| \lesssim (M_G f(chx))^{\frac{p}{q}} \|f\|_{\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}(\mathbb{R}_+)}^{1-\frac{p}{q}}$$

for any $x \in \mathbb{R}_+$.

If we take $\omega_1 = \omega_2 = \omega^{\frac{1}{p}}$ into account in condition (3.2) then conditions (4.14) and (3.2) will be equivalent and the statement of this theorem will follow from the boundedness of maximal operator M_G on the space $\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}(\mathbb{R}_+)$, which was proved in Corollary 3.2.

Indeed,

$$\begin{aligned} \|M_G^\alpha f\|_{\mathcal{M}_{q,\gamma,\omega^{\frac{1}{q}}}(\mathbb{R}_+)} &= \sup_{x,s \in \mathbb{R}_+} \omega(x, s)^{-\frac{1}{q}} \left(sh \frac{s}{2} \right)^{-\frac{\gamma}{q}} \left\| A_{chx}^\lambda M_G^\alpha f \right\|_{L_{q,\lambda}(H_s)} \\ &\lesssim \|f\|_{\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}(\mathbb{R}_+)}^{1-\frac{p}{q}} \sup_{x,s \in \mathbb{R}_+} \omega(x, s)^{-\frac{1}{q}} \left(sh \frac{s}{2} \right)^{-\frac{\gamma}{q}} \left\| A_{chx}^\lambda M_G f \right\|_{L_{p,\lambda}(H_s)}^{\frac{p}{q}} \\ &\lesssim \|f\|_{\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}(\mathbb{R}_+)} \left(\sup_{x,s \in \mathbb{R}_+} \omega(x, s)^{-\frac{1}{p}} \left(sh \frac{s}{2} \right)^{-\frac{\gamma}{p}} \left\| A_{chx}^\lambda M_G f \right\|_{L_{p,\lambda}(H_s)} \right)^{\frac{p}{q}} \\ &\lesssim \|f\|_{\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}(\mathbb{R}_+)}^{1-\frac{p}{q}} \|M_G f\|_{\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}(\mathbb{R}_+)}^{\frac{p}{q}} \lesssim \|f\|_{\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}(\mathbb{R}_+)}, \end{aligned}$$

if $1 < p < q < \infty$ and

$$\begin{aligned} \|M_G^\alpha f\|_{W\mathcal{M}_{q,\gamma,\omega}^{\frac{1}{q}}} &= \sup_{x,s \in \mathbb{R}_+} \omega(x,s)^{-\frac{1}{q}} \left(sh \frac{s}{2} \right)^{-\frac{\gamma}{q}} \left\| A_{chx}^\lambda M_G^\alpha f \right\|_{WL_{q,\lambda}(H_s)} \\ &\lesssim \|f\|_{\mathcal{M}_{1,\gamma,\omega}^{\frac{1}{q}}} \sup_{x,s \in \mathbb{R}_+} \omega(x,s)^{-\frac{1}{q}} \left(sh \frac{s}{2} \right)^{-\frac{\gamma}{q}} \left\| A_{chx}^\lambda M_G f \right\|_{WL_{1,\lambda}(H_s)}^{\frac{1}{q}} \\ &\lesssim \|f\|_{\mathcal{M}_{1,\gamma,\omega}^{\frac{1}{q}}} \left(\sup_{x,s \in \mathbb{R}_+} \omega(x,s)^{-1} \left(sh \frac{s}{2} \right)^{-\gamma} \left\| A_{chx}^\lambda M_G f \right\|_{WL_{1,\lambda}(H_s)} \right)^{\frac{1}{q}} \\ &\lesssim \|f\|_{\mathcal{M}_{1,\gamma,\omega}^{\frac{1}{q}}} \|M_G f\|_{\mathcal{M}_{1,\gamma,\omega}^{\frac{1}{q}}} \lesssim \|f\|_{\mathcal{M}_{1,\gamma,\omega}}, \end{aligned}$$

if $1 < q < \infty$.

In case $\omega(x,r) = \left(sh \frac{r}{2} \right)^{\nu-\gamma}$, $0 < \nu < \gamma$ from Theorem 4.2 for fractional maximal operator M_G^α , we obtain the following Adams type result [1].

Corollary 4.3. *Let $0 < \alpha < \gamma$, $1 \leq p < \frac{\gamma}{\alpha}$, $0 < \nu < \gamma - \alpha p$ and conditions (3.4) holds. Then when $p > 1$ operator M_G^α is bounded from $L_{p,\lambda,\nu}(\mathbb{R}_+)$ to $L_{q,\lambda,\nu}(\mathbb{R}_+)$ and when $p = 1$ operator M_G^α is bounded from $L_{1,\lambda,\nu}(\mathbb{R}_+)$ to $WL_{q,\lambda,\nu}(\mathbb{R}_+)$.*

From this in particular when $\gamma = 2\lambda + 1$ we get Theorem G.

Theorem 4.4. *Let $0 < \alpha < \gamma$, $1 \leq p < q < \infty$ and $\omega \in \Omega_p^\gamma$.*

(i) *If $\omega(x,r)$ satisfies the condition (3.2), then (4.15) is a sufficient condition for the boundedness of M_G^α from $\mathcal{M}_{1,\gamma,\omega}(\mathbb{R}_+)$ to $W\mathcal{M}_{q,\gamma,\omega}^{\frac{1}{q}}(\mathbb{R}_+)$.*

If $1 < p < q < \infty$, then the condition (4.15) is a sufficient for the boundedness M_G^α from $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega}^{\frac{p}{q}}(\mathbb{R}_+)$.

(ii) *If $\omega \in \Phi_p^\gamma$, then the condition*

$$\left(sh \frac{r}{2} \right)^\alpha \omega(r) \lesssim \left(sh \frac{r}{2} \right)^{-\frac{\alpha p}{q-p}}, \quad r > 0 \quad (4.17)$$

is necessary for the boundedness M_G^α from $\mathcal{M}_{1,\gamma,\omega}(\mathbb{R}_+)$ to $W\mathcal{M}_{q,\gamma,\omega}^{\frac{1}{q}}(\mathbb{R}_+)$, and from $\mathcal{M}_{p,\gamma,\omega}^{\frac{1}{q}}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega}^{\frac{p}{q}}(\mathbb{R}_+)$ if $1 < p < q < \infty$.

(iii) *If $\omega \in \Phi_p^\gamma$, then the condition (4.17) is necessary and sufficient for the boundedness M_G^α from $\mathcal{M}_{1,\gamma,\omega}(\mathbb{R}_+)$ to $W\mathcal{M}_{q,\gamma,\omega}^{\frac{1}{q}}(\mathbb{R}_+)$.*

Moreover, if $1 < p < q < \infty$, then the condition (4.17) is necessary and sufficient for the boundedness M_G^α from $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega}^{\frac{p}{q}}(\mathbb{R}_+)$.

Proof. The first part of the theorem follows from Theorem 4.2 when $\omega(x,r) = \omega^p(r)$. We will now prove the second part of this theorem. Let $H_0 = (0, r_0)$ and $x \in H_0$. By Lemma 3.4 $\left(sh \frac{r_0}{2} \right)^\alpha \lesssim M_G^\alpha \chi_{H_0}(chx)$. Moreover, by Lemma 2.4 and Lemma 3.4, we obtain

$$\begin{aligned} \left(sh \frac{r_0}{2} \right)^\alpha &\lesssim |H_0|_\lambda^{-\frac{1}{q}} \|M_G^\alpha \chi_{H_0}\|_{L_{q,\lambda}(H_0)} \lesssim \omega(r_0)^{\frac{p}{q}} \|M_G^\alpha \chi_{H_0}\|_{\mathcal{M}_{q,\gamma,\omega}^{\frac{p}{q}}(\mathbb{R}_+)} \\ &\lesssim \omega(r_0)^{\frac{p}{q}} \|\chi_{H_0}\|_{\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)} \lesssim \omega(r_0)^{\frac{p}{q}-1} \end{aligned}$$

or

$$\left(sh \frac{r_0}{2} \right)^\alpha \omega(r_0)^{1-\frac{p}{q}} \lesssim 1 \Leftrightarrow \omega(r_0) \lesssim \left(sh \frac{r_0}{2} \right)^{-\frac{\alpha p}{q-p}}$$

$$\Leftrightarrow \left(sh \frac{r_0}{2} \right)^\alpha \omega(r_0) \lesssim \left(sh \frac{r_0}{2} \right)^{-\frac{\alpha p}{q-p}}.$$

The third part of this theorem follows from the first and second parts of theorem.

The following theorem is an analogue of Adams -Gunawan type result, (see [25, Theorem 4.8]).

Theorem 4.5. *Let $0 < \alpha < \gamma$, $1 \leq p < q < \infty$, $\omega \in \Omega_p^\gamma$ and satisfy (3.2), and*

$$\left(sh \frac{r}{2} \right)^\alpha \omega(x, r) + \sup_{s>r} \left(sh \frac{s}{2} \right)^\alpha \omega(x, s) \lesssim \omega(x, r)^{\frac{p}{q}}, r > 0. \quad (4.18)$$

Then M_G^α is bounded from $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\lambda,\omega^{\frac{p}{q}}}(\mathbb{R}_+)$ when $p > 1$ and from $\mathcal{M}_{1,\gamma,\omega}(\mathbb{R}_+)$ to $W\mathcal{M}_{q,\gamma,\omega^{\frac{1}{q}}}(\mathbb{R}_+)$ when $p = 1$.

Proof. Let $1 \leq p < \infty$ and $f \in \mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$. By Theorem 4.1, we have (4.1). Then from (4.18), we get

$$\begin{aligned} M_G^\alpha f(chx) &\lesssim \left(sh \frac{r}{2} \right)^\alpha M_G f(chx) + \sup_{s>r} \left(sh \frac{s}{2} \right)^{\alpha - \frac{\gamma}{p}} \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)} \\ &\lesssim \left(sh \frac{r}{2} \right)^\alpha M_G f(chx) + \sup_{s>r} \left(sh \frac{s}{2} \right)^\alpha \omega(x, s) \|f\|_{\mathcal{M}_{p,\lambda,\omega}} \\ &\lesssim \min \left\{ \omega(x, r)^{\frac{p}{q}-1} M_G f(chx), \omega(x, r)^{\frac{p}{q}} \|f\|_{\mathcal{M}_{p,\gamma,\omega}} \right\}. \end{aligned}$$

If $\omega(x, r)^{\frac{p}{q}-1} M_G f(chx) \leq \omega(x, r)^{\frac{p}{q}} \|f\|_{\mathcal{M}_{p,\gamma,\omega}}$, then $\omega(x, r) \geq M_G f(chx) \|f\|_{\mathcal{M}_{p,\gamma,\omega}}^{-1}$ and, we have

$$\begin{aligned} M_G^\alpha f(chx) &\lesssim \omega(x, r)^{\frac{p}{q}-1} M_G f(chx) \\ &\lesssim M_G f(chx) \left(\frac{\|f\|_{\mathcal{M}_{p,\lambda,\omega}}}{M_G f(chx)} \right)^{1-\frac{p}{q}} \lesssim (M_G f(chx))^{\frac{p}{q}} \|f\|_{\mathcal{M}_{p,\gamma,\omega}}^{1-\frac{p}{q}}. \end{aligned}$$

If $\omega(x, r)^{\frac{p}{q}-1} M_G^\alpha f(chx) \geq \omega(x, r)^{\frac{p}{q}} \|f\|_{\mathcal{M}_{p,\gamma,\omega}}$ then $\omega(x, r) \leq M_G f(chx) \|f\|_{\mathcal{M}_{p,\gamma,\omega}}^{-1}$ and we have

$$M_G^\alpha f(chx) \lesssim \omega(x, r)^{\frac{p}{q}} \|f\|_{\mathcal{M}_{p,\gamma,\omega}} \lesssim \left(\frac{M_G f(chx)}{\|f\|_{\mathcal{M}_{p,\gamma,\omega}}} \right)^{\frac{p}{q}} \|f\|_{\mathcal{M}_{p,\gamma,\omega}} \lesssim (M_G f(chx))^{\frac{p}{q}} \|f\|_{\mathcal{M}_{p,\gamma,\omega}}^{1-\frac{p}{q}}.$$

Thus, we obtain

$$M_G^\alpha f(chx) \lesssim (M_G f(chx))^{\frac{p}{q}} \|f\|_{\mathcal{M}_{p,\gamma,\omega}}^{1-\frac{p}{q}}. \quad (4.19)$$

From Corollary 3.2 and (4.19), we have

$$\begin{aligned} \|M_G^\alpha f\|_{\mathcal{M}_{q,\gamma,\omega^{\frac{p}{q}}}} &\lesssim \|f\|_{\mathcal{M}_{p,\gamma,\omega}}^{1-\frac{p}{q}} \left\| (M_G f)^{\frac{p}{q}} \right\|_{\mathcal{M}_{p,\gamma,\omega^{\frac{p}{q}}}} \\ &\lesssim \|f\|_{\mathcal{M}_{p,\gamma,\omega}}^{1-\frac{p}{q}} \|M_G f\|_{\mathcal{M}_{q,\gamma,\omega}}^{\frac{p}{q}} \lesssim \|f\|_{\mathcal{M}_{p,\gamma,\omega}}, \end{aligned}$$

if $1 < p < q < \infty$ and

$$\|M_G^\alpha f\|_{W\mathcal{M}_{q,\gamma,\omega^{\frac{1}{q}}}} \lesssim \|f\|_{\mathcal{M}_{1,\gamma,\omega}}^{1-\frac{1}{q}} \|M_G f\|_{\mathcal{M}_{1,\gamma,\omega}}^{\frac{1}{q}} \lesssim \|f\|_{\mathcal{M}_{1,\gamma,\omega}}$$

if $p = 1$.

The following result is an analogue of Theorem 4.9 from [25].

Theorem 4.6. *Let $0 < \alpha < \gamma$, $1 \leq p < q < \infty$, and $\omega \in \Omega_p^\gamma$.*

(i) *If $\omega(x, r)$ satisfies the condition (4.2), then the condition (4.18) is sufficient for the boundedness M_G^α from $\mathcal{M}_{1,\gamma,\omega}(\mathbb{R}_+)$ to $W\mathcal{M}_{q,\gamma,\omega^{\frac{1}{q}}}(\mathbb{R}_+)$.*

Moreover, if $p > 1$, then the condition (4.18) is sufficient for the boundedness M_G^α from $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega^{\frac{p}{q}}}(\mathbb{R}_+)$.

(ii) *If $\omega \in \Phi_p^\gamma$, then the condition*

$$\left(sh \frac{r}{2} \right)^\alpha \omega(r) \lesssim \omega(r)^{\frac{p}{q}}, \quad r > 0 \quad (4.20)$$

is necessary for the boundedness M_G^α from $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega^{\frac{p}{q}}}(\mathbb{R}_+)$ if $p > 1$, also from $\mathcal{M}_{1,\gamma,\omega}(\mathbb{R}_+)$ to $W\mathcal{M}_{q,\gamma,\omega^{\frac{1}{q}}}(\mathbb{R}_+)$ if $p = 1$.

(iii) *If $\omega \in \Phi_p^\gamma$ then the condition (4.20) is necessary and sufficient for the boundedness M_G^α from $\mathcal{M}_{1,\lambda,\omega}(\mathbb{R}_+)$ to $W\mathcal{M}_{q,\lambda,\omega^{\frac{1}{q}}}(\mathbb{R}_+)$.*

Moreover, if $p > 1$ the condition (4.20) is necessary and sufficient for the boundedness M_G^α from $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega^{\frac{p}{q}}}(\mathbb{R}_+)$.

Proof. The first part of this theorem follows from Theorem 4.5.

We will prove the second part. Let $H_0 = (0, r_0)$ and $x \in H_0$. By Lemma 2.4 and Lemma 3.3, we get

$$\begin{aligned} \left(sh \frac{r_0}{2} \right)^\alpha &\lesssim |H_0|_\lambda^{-\frac{1}{q}} \|M_G^\alpha \chi_{H_0}\|_{L_{q,\lambda}(H_0)} \\ &\lesssim \omega(r_0)^{\frac{p}{q}} \|M_G^\alpha \chi_{H_0}\|_{\mathcal{M}_{q,\gamma,\omega^{\frac{p}{q}}}(\mathbb{R}_+)} \lesssim \omega(r_0)^{\frac{p}{q}} \|\chi_{H_0}\|_{\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)} \lesssim \omega(r_0)^{\frac{p}{q}-1} \end{aligned}$$

or

$$\left(sh \frac{r_0}{2} \right)^\alpha \omega(r_0)^{1-\frac{p}{q}} \lesssim 1 \Leftrightarrow \left(sh \frac{r_0}{2} \right)^\alpha \omega(r_0) \lesssim \omega(r_0)^{\frac{p}{q}}, \quad r_0 > 0.$$

Since, this true for any $x \in \mathbb{R}_+$ and $r_0 > 0$, then our statement is proved.

The third part follows from the first and second parts of the theorem.

5 Fractional maximal commutator on generalized G -Morrey spaces $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$

In this section we characterise the boundedness of fractional maximal commutator on generalized G -Morrey spaces $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$.

The result obtained are analogues of relevant Spanne-Guliyev results, which were obtained in [24,25] for fractional maximal operator

$$M_\alpha f(x) = \sup_{r>0} |B(x,r)|^{\frac{\alpha}{Q}-1} \int_{B(x,r)} |f(y)| dy, \quad 0 \leq \alpha < Q,$$

and k -th ($k = 1, 2, \dots$) order fractional maximal commutator

$$M_{b,\alpha,k} f(x) = \sup_{r>0} |B(x,r)|^{\frac{\alpha}{Q}-1} \int_{B(x,r)} |b(x) - b(y)|^k |f(y)| dy,$$

where $B(x,r)$ is an open ball centered at the point x with radius $r > 0$, and $|B(x,r)|$ is its Lebesgue measure.

51 Spanne-Guliyev type result

Firstly we will consider the BMO_G Gegenbauer space, which was introduced in [31].

Definition 5.1. Denote by $BMO_G(\mathbb{R}_+)$ the Gegenbauer- BMO space (G - BMO space) as the set of locally integrable functions on $\mathbb{R}_+ = (0, \infty)$ such that

$$\|f\|_{BMO_G(\mathbb{R}_+)} = \sup_{x,r \in \mathbb{R}_+} \frac{1}{|H_r|} \int_{H_r} \left| A_{chy}^\lambda f(chx) - f_{H_r}(chx) \right| sh^{2\lambda} y dy < \infty,$$

where

$$f_{H_r}(chx) = \frac{1}{|H_r|} \int_{H_r} A_{chy}^\lambda f(chx) sh^{2\lambda} y dy$$

and $H_r = (0, r)$.

According to the definition

$$BMO_G(\mathbb{R}_+) := \left\{ f \in L_{1,\lambda}^{loc}(\mathbb{R}_+) : \|f\|_{BMO_G(\mathbb{R}_+)} < \infty \right\}.$$

In [31] for the operator M_G^α commutator k -th ($k = 1, 2, \dots$) order is defined as follows:

$$M_G^{b,\alpha,k} f(chx) = \sup_{r>0} |H_r|^{\frac{\alpha}{\gamma}-1} \int_{H_r} \left| A_{chy}^\lambda f(chx) - b_{H_r}(chx) \right|^k A_{chy}^\lambda |f(chx)| sh^{2\lambda} y dy,$$

where $b \in L_{1,\lambda}^{loc}(\mathbb{R}_+)$ and $0 < \alpha < \gamma$.

We will later need the following preliminary statement.

Lemma 5.2. Let $1 < p < \infty$, $0 < \alpha < \frac{\gamma}{p}$, (2.19) holds and $b \in BMO_G(\mathbb{R}_+)$. Then for any $f \in L_{1,\lambda}^{loc}(\mathbb{R}_+)$ and any interval $H_r = (0, r)$, $r > 0$ the following inequality

$$\left\| A_{chx}^\lambda M_G^{b,\alpha,k} f \right\|_{L_{q,\lambda}(H_r)} \lesssim \left(sh \frac{r}{2} \right)^{\frac{\gamma}{q}} \|b\|_{BMO_G(\mathbb{R}_+)}^k \sup_{s>r} \left(sh \frac{s}{2} \right)^{-\frac{\gamma}{q}} \left(1 + \left(\frac{sh \frac{s}{2}}{sh \frac{r}{2}} \right)^{k\gamma} \right) \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)}$$

is true.

Proof. Write $f = f_1 + f_2$ where $f_1 = f \chi_{H_r}$ and $f_2 = f \chi_{(H_r)^c}$ where χ_{H_s} -is a characteristic function of the set H_s . Then we obtain

$$\left\| A_{chx}^\lambda M_G^{b,\alpha,k} f \right\|_{L_{q,\lambda}(H_r)} \leq \left\| A_{chx}^\lambda M_G^{b,\alpha,k} f_1 \right\|_{L_{q,\lambda}(H_r)} + \left\| A_{chx}^\lambda M_G^{b,\alpha,k} f_2 \right\|_{L_{q,\lambda}(H_r)}. \quad (5.1)$$

By the boundedness of $M_G^{b,\alpha,k}$ from $L_{p,\lambda}(\mathbb{R}_+)$ to $L_{q,\lambda}(\mathbb{R}_+)$ (see [31, Theorem 4.1]), we have (see prof of Lemma 2.6)

$$\left\| A_{chx}^\lambda M_G^{b,\alpha,k} f_1 \right\|_{L_{q,\lambda}(H_r)} \lesssim \|b\|_{BMO_G}^k \left\| A_{chx}^\lambda f_1 \right\|_{L_{p,\lambda}(H_r)} \lesssim \|b\|_{BMO_G}^k \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_r)}.$$

Let $x \in \mathbb{R}_+$ and $H_s \cap (H_r)^c \neq \emptyset$ then $s > r$ and we have (see the proof of Lemma 2.6)

$$M_G^{b,\alpha,k} f(chx) \lesssim \sup_{s>r} |H_r|^{\frac{\alpha}{\gamma}-1} \int_{H_s} \left| A_{chy}^\lambda b(chx) - b_{H_s}(chx) \right|^k A_{chy}^\lambda |f(chx)| sh^{2\lambda} y dy. \quad (5.2)$$

From (5.2), Lemma 2.5, we obtain

$$\begin{aligned}
& \left\| A_{chx}^\lambda M_G^{b,\alpha,k} f \right\|_{L_{q,\lambda}(H_r)} \leq \left\| M_G^{b,\alpha,k} f_2 \right\|_{L_{q,\lambda}(H_r)} \\
& \lesssim \left\{ \int_{H_r} \left[\sup_{s>r} |H_s|^{\frac{\alpha}{\gamma}-1} \int_{H_s} \left| A_{chy}^\lambda b(chx) - b_{H_s}(chx) \right|^k A_{chy}^\lambda |f(chx)| sh^{2\lambda} y dy \right]^q sh^{2\lambda} x dx \right\}^{\frac{1}{q}} \\
& \lesssim \left\{ \int_{H_r} \left[\sup_{s>r} |H_s|^{\frac{\alpha}{\gamma}-1} \int_{H_s} \left| A_{chy}^\lambda b(chx) - b_{H_r}(chx) \right|^k A_{chy}^\lambda |f(chx)| sh^{2\lambda} y dy \right]^q sh^{2\lambda} x dx \right\}^{\frac{1}{q}} \\
& + \left\{ \int_{H_r} \left[\sup_{s>r} |H_s|^{\frac{\alpha}{\gamma}-1} \int_{H_s} |b_{H_r}(chx) - b_{H_s}(chx)|^k A_{chy}^\lambda |f(chx)| sh^{2\lambda} y dy \right]^q sh^{2\lambda} x dx \right\}^{\frac{1}{q}} \\
& = A_1 + A_2. \tag{5.3}
\end{aligned}$$

We will estimate A_1 . By applying Holder's inequality and considering the following relations (see [31, Corollary 2.8])

$$\sup_{x,r \in \mathbb{R}_+} \left(\frac{1}{|H_s|_\lambda} \int_{H_s} \left| A_{chy}^\lambda f(chx) - b_{H_s}(chx) \right|^p sh^{2\lambda} y dy \right)^{\frac{1}{p}} \approx \|f\|_{BMO_G}, \tag{5.4}$$

we have

$$\begin{aligned}
& \int_{H_s} \left| A_{chy}^\lambda b(chx) - b_{H_s}(chx) \right|^k A_{chy}^\lambda |f(chx)| sh^{2\lambda} y dy \\
& \lesssim \left(\int_{H_s} \left| A_{chy}^\lambda b(chx) - b_{H_s}(chx) \right|^{kp'} sh^{2\lambda} y dy \right)^{\frac{1}{p'}} \left(\int_{H_s} A_{chy}^\lambda |f(chx)|^p sh^{2\lambda} y dy \right)^{\frac{1}{p}} \\
& \lesssim |H_s|^{\frac{1}{p'}} \|b\|_{BMO_G}^k \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)}.
\end{aligned}$$

From this, and (2.1), we obtain

$$\begin{aligned}
A_1 & \leq \|b\|_{BMO_G}^k |H_r|^{\frac{1}{q}} \sup_{s>r} |H_s|^{\frac{\alpha}{\gamma}-\frac{1}{p}} \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)} \\
& \approx \|b\|_{BMO}^k \left(sh \frac{r}{2} \right)^{\frac{\gamma}{q}} \sup_{s>r} \left(sh \frac{s}{2} \right)^{\alpha-\frac{\gamma}{p}} \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)}. \tag{5.5}
\end{aligned}$$

For estimate A_2 , again using Holder's inequality and the inequality (see [31])

$$|f_{H_1}(chx) - b_{H_2}(chx)| \leq \frac{|H_2|_\lambda}{|H_1|_\lambda} \|f\|_{BMO_G}, H_1 \subset H_2,$$

and (2.1), we get

$$A_2 \lesssim \left(sh \frac{r}{2} \right)^{\frac{\gamma}{q}} \sup_{s>r} \left(sh \frac{s}{2} \right)^{\alpha-\frac{\gamma}{p}} |b_{H_s}(chx) - b_{H_r}(chx)|^k \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)}$$

$$\begin{aligned}
&\lesssim \left(sh \frac{r}{2} \right)^{\frac{\gamma}{q}} \|b\|_{BMO_G}^k \sup_{s>r} \left(sh \frac{s}{2} \right)^{\alpha - \frac{\gamma}{p}} \left(\frac{|H_s|_\lambda}{|H_r|_\lambda} \right)^k \|A_{chx}^\lambda f\|_{L_{p,\lambda}(H_s)} \\
&\lesssim \left(sh \frac{r}{2} \right)^{\frac{\gamma}{q}} \|b\|_{BMO_G}^k \sup_{s>r} \left(sh \frac{s}{2} \right)^{\alpha - \frac{\gamma}{p}} \left(\frac{sh \frac{s}{2}}{sh \frac{r}{2}} \right)^{k\gamma} \|A_{chx}^\lambda f\|_{L_{p,\lambda}(H_s)}. \tag{5.6}
\end{aligned}$$

Summing (5.4) and (5.5), we obtain

$$\begin{aligned}
&\left\| A_{chx}^\lambda M_G^{b,\alpha,k} f_2 \right\|_{L_{q,\lambda}(H_r)} \\
&\lesssim \left(sh \frac{r}{2} \right)^{\frac{\gamma}{q}} \|b\|_{BMO_G}^k \sup_{s>r} \left(sh \frac{s}{2} \right)^{\alpha - \frac{\gamma}{p}} \left(1 + \left(\frac{sh \frac{s}{2}}{sh \frac{r}{2}} \right)^{k\gamma} \right) \|A_{chx}^\lambda f\|_{L_{p,\lambda}(H_s)} \\
&= \left(sh \frac{r}{2} \right)^{\frac{\gamma}{q}} \|b\|_{BMO_G}^k \sup_{s>r} \left(sh \frac{s}{2} \right)^{-\frac{\gamma}{q}} \left(1 + \left(\frac{sh \frac{s}{2}}{sh \frac{r}{2}} \right)^{k\gamma} \right) \|A_{chx}^\lambda f\|_{L_{p,\lambda}(H_s)}.
\end{aligned}$$

Finally from (5.1), we have

$$\begin{aligned}
&\left\| A_{chx}^\lambda M_G^{b,\alpha,k} f \right\|_{L_{q,\lambda}(H_r)} \lesssim \|b\|_{BMO_G}^k \|A_{chx}^\lambda f\|_{L_{p,\lambda}(H_r)} \\
&+ \left(sh \frac{r}{2} \right)^{\frac{\gamma}{q}} \|b\|_{BMO_G}^k \sup_{s>r} \left(sh \frac{s}{2} \right)^{-\frac{\gamma}{q}} \left(1 + \left(\frac{sh \frac{s}{2}}{sh \frac{r}{2}} \right)^{k\gamma} \right) \|A_{chx}^\lambda f\|_{L_{p,\lambda}(H_s)} \\
&\lesssim \left(sh \frac{r}{2} \right)^{\frac{\gamma}{q}} \|b\|_{BMO_G}^k \sup_{s>r} \left(sh \frac{s}{2} \right)^{-\frac{\gamma}{q}} \left(1 + \left(\frac{sh \frac{s}{2}}{sh \frac{r}{2}} \right)^{k\gamma} \right) \|A_{chx}^\lambda f\|_{L_{p,\lambda}(H_s)}.
\end{aligned}$$

Lemma is proved.

The following theorem is Spanne-Guliyev type result (see [24, Theorem 4.1]).

Theorem 5.3. Let $1 < p < \infty$, $0 \leq \alpha < \frac{\gamma}{p}$, (2.19) holds, $b \in BMO_G(\mathbb{R}_+)$ and the pair (ω_1, ω_2) satisfy the condition

$$\sup_{s>r} \left(sh \frac{s}{2} \right)^{\alpha - \frac{\gamma}{p}} \left(1 + \left(\frac{sh \frac{s}{2}}{sh \frac{r}{2}} \right)^{k\gamma} \right) \omega_1(x, s) \lesssim \omega_2(x, r). \tag{5.7}$$

Then the operator $M_G^{b,\alpha,k}$ is bounded from $\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$.

Moreover,

$$\left\| M_G^{b,\alpha,k} f \right\|_{\mathcal{M}_{q,\gamma,\omega_2}} \lesssim \|b\|_{BMO_G}^k \|f\|_{\mathcal{M}_{p,\gamma,\omega_1}}.$$

is true.

Proof. By Lemma 5.2 we can write

$$\begin{aligned}
&\left\| M_G^{b,\alpha,k} f \right\|_{\mathcal{M}_{q,\gamma,\omega_2}} = \sup_{x,r \in \mathbb{R}_+} \omega_2(x, r)^{-1} \left(sh \frac{r}{2} \right)^{-\frac{\gamma}{q}} \left\| A_{chx}^\lambda M_G^{b,\alpha,k} f \right\|_{L_{q,\lambda}(H_r)} \\
&\lesssim \|b\|_{BMO_G}^k \sup_{x,r \in \mathbb{R}_+} \omega_2(x, r)^{-1} \left(sh \frac{r}{2} \right)^{-\frac{\gamma}{q}} \sup_{s>r} \left(sh \frac{s}{2} \right)^{-\frac{\gamma}{q}} \left(1 + \left(\frac{sh \frac{s}{2}}{sh \frac{r}{2}} \right)^{k\gamma} \right) \|A_{chx}^\lambda f\|_{L_{p,\lambda}(H_s)} \\
&\lesssim \|b\|_{BMO_G}^k \sup_{x,r \in \mathbb{R}_+} \omega_1(x, r)^{-1} \left(sh \frac{r}{2} \right)^{-\frac{\gamma}{q}} \|A_{chx}^\lambda f\|_{L_{p,\lambda}(H_r)} \lesssim \|f\|_{\mathcal{M}_{p,\gamma,\omega_1}}.
\end{aligned}$$

Theorem is proved.

In the case when $\alpha = 0$ and $p = q$ we get the following result.

Corollary 5.4. *Let $1 < p < \infty$, $b \in BMO_G(\mathbb{R}_+)$ and the pair (ω_1, ω_2) satisfy the condition*

$$\sup_{s>r} \left(sh \frac{s}{2} \right)^{-\frac{\gamma}{q}} \left(1 + \left(\frac{sh \frac{s}{2}}{sh \frac{r}{2}} \right)^{k\gamma} \right) \omega_1(x, s) \lesssim \omega_2(x, r). \quad (5.8)$$

Then the operator $M_G^{b,k}$ is bounded from $\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$.

Moreover, we have

$$\left\| M_G^{b,k} f \right\|_{\mathcal{M}_{q,\gamma,\omega_2}} \lesssim \|b\|_{BMO_G}^k \|f\|_{\mathcal{M}_{p,\gamma,\omega_1}}.$$

52 Adams-Guliyev type result

The following theorem is Adams-Guliyev type result (see [24, Theorem 4.2]).

Theorem 5.5. *Let $1 < p < q < \infty$, $0 < \alpha < \frac{\gamma}{p}$, $b \in BMO_G(\mathbb{R}_+)$ and $\omega(x, r)$ satisfy the conditions:*

$$\sup_{s>r} \left(sh \frac{s}{2} \right)^{-\gamma} \left(1 + \left(\frac{sh \frac{s}{2}}{sh \frac{r}{2}} \right)^{k\gamma} \right)^p \omega(x, s) \lesssim \omega(x, r), \quad (5.9)$$

$$\sup_{s>r} \left(sh \frac{s}{2} \right)^\alpha \left(1 + \left(\frac{sh \frac{s}{2}}{sh \frac{r}{2}} \right)^{k\gamma} \right) \omega(x, s)^{\frac{1}{p}} \lesssim \left(sh \frac{r}{2} \right)^{-\frac{\alpha p}{q-p}}. \quad (5.10)$$

Then the operator $M_G^{b,\alpha,k}$ is bounded from $\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega^{\frac{1}{q}}}(\mathbb{R}_+)$.

Proof. Let $1 < p < q < \infty$, $0 < \alpha < \frac{\gamma}{p}$ and $f \in \mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}(\mathbb{R}_+)$. Let $f = f_1 + f_2$ where $f_1 = f\chi_{H_r}$ and $f_2 = f\chi_{(H_r)^c}$.

By doing the same as in Section 5.1, for $p \in (1, \infty)$, we can write

$$\begin{aligned} M_G^{b,\alpha,k} f_2(chx) &\lesssim \sup_{s>r} \left(sh \frac{s}{2} \right)^{\alpha-\gamma} \int_{H_s} \left| A_{chy}^\lambda b(chx) - b_{H_s}(chx) \right|^k A_{chy}^\lambda |f_2(chx)| sh^{2\lambda} y dy \\ &\lesssim \|b\|_{BMO_G}^k \sup_{s>r} \left(sh \frac{s}{2} \right)^{\alpha-\frac{\gamma}{p}} \left(1 + \left(\frac{sh \frac{s}{2}}{sh \frac{r}{2}} \right)^{k\gamma} \right) \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)}. \end{aligned} \quad (5.11)$$

From (4.2) follows that

$$M_G^{b,\alpha,k} f_1(chx) \lesssim \left(sh \frac{r}{2} \right)^\alpha M_G^{b,k} f(chx). \quad (5.12)$$

From (5.9) - (5.11) we obtain

$$\begin{aligned} M_G^{b,\alpha,k} f(chx) &\lesssim \left(sh \frac{r}{2} \right)^\alpha M_G^{b,k} f(chx) \\ &+ \|b\|_{BMO_G}^k \sup_{s>r} \left(sh \frac{s}{2} \right)^{\alpha-\frac{\gamma}{p}} \left(1 + \left(\frac{sh \frac{s}{2}}{sh \frac{r}{2}} \right)^{k\gamma} \right) \left\| A_{chx}^\lambda f \right\|_{L_{p,\lambda}(H_s)} \\ &\lesssim \left(sh \frac{r}{2} \right)^\alpha M_G^{b,k} f(chx) + \|b\|_{BMO_G}^k \sup_{s>r} \omega(x, s)^{\frac{1}{p}} \left(sh \frac{s}{2} \right)^\alpha \left(1 + \left(\frac{sh \frac{s}{2}}{sh \frac{r}{2}} \right)^{k\gamma} \right) \|f\|_{\mathcal{M}_{p,\gamma,\omega^{\frac{1}{p}}}} \end{aligned} \quad (5.13)$$

$$\lesssim \left(sh \frac{r}{2} \right)^\alpha M_G^{b,k} f(chx) + \|b\|_{BMO_G}^k \left(sh \frac{r}{2} \right)^{-\frac{\alpha p}{q-p}} \|f\|_{\mathcal{M}_{p,\gamma,\omega}^{\frac{1}{p}}}. \quad (5.14)$$

Soosing

$$sh \frac{r}{2} = \left(\frac{\|b\|_{BMO_G}^k \|f\|_{\mathcal{M}_{p,\gamma,\omega}^{\frac{1}{p}}}}{M_G^{b,k} f(chx)} \right)^{\frac{q-p}{\alpha q}},$$

for any $x \in \mathbb{R}_+$, we have

$$M_G^{b,\alpha,k} f(chx) \lesssim \|b\|_{BMO_G}^{k(1-\frac{p}{q})} \left(M_G^{b,k} f(chx) \right) \|f\|_{\mathcal{M}_{p,\gamma,\omega}^{\frac{1}{p}}}^{1-\frac{p}{q}}.$$

Now the statement of the theorem follows from the the boundedness of commutator $M_G^{b,k}$ on $\mathcal{M}_{p,\gamma,\omega}^{\frac{1}{p}}(\mathbb{R}_+)$ which was proved in Corollary 5.4 by condition (5.8). Indeed,

$$\begin{aligned} \left\| M_G^{b,\alpha,k} f \right\|_{\mathcal{M}_{q,\gamma,\omega}^{\frac{1}{q}}} &= \sup_{x,r \in \mathbb{R}_+} \omega(x,r)^{-\frac{1}{q}} \left(sh \frac{r}{2} \right)^{-\frac{\gamma}{q}} \left\| A_{chx}^\lambda M_G^{b,\alpha,k} f \right\|_{L_{q,\lambda}(H_r)} \\ &\lesssim \|b\|_{BMO_G}^{k(1-\frac{p}{q})} \|f\|_{\mathcal{M}_{p,\gamma,\omega}^{\frac{1}{p}}}^{1-\frac{p}{q}} \sup_{x,r \in \mathbb{R}_+} \omega(x,r)^{-\frac{1}{q}} \left(sh \frac{r}{2} \right)^{-\frac{\gamma}{q}} \left\| A_{chx}^\lambda M_G^{b,k} f \right\|_{L_{p,\lambda}(H_r)} \\ &\lesssim \|b\|_{BMO_G}^{k(1-\frac{p}{q})} \|f\|_{\mathcal{M}_{p,\gamma,\omega}^{\frac{1}{p}}}^{1-\frac{p}{q}} \left(\sup_{x,r \in \mathbb{R}_+} \omega(x,r)^{-\frac{1}{p}} \left(sh \frac{r}{2} \right)^{-\frac{\gamma}{p}} \left\| A_{chx}^\lambda M_G^{b,k} f \right\|_{L_{p,\lambda}(H_r)} \right)^{\frac{p}{q}} \\ &\lesssim \|b\|_{BMO_G}^{k(1-\frac{p}{q})} \left\| M_G^{b,k} f \right\|_{\mathcal{M}_{p,\gamma,\omega}^{\frac{1}{p}}}^{\frac{p}{q}} \|f\|_{\mathcal{M}_{p,\gamma,\omega}^{\frac{1}{p}}}^{1-\frac{p}{q}} \lesssim \|b\|_{BMO_G}^{k(1-\frac{p}{q})} \|f\|_{\mathcal{M}_{p,\gamma,\omega}^{\frac{1}{p}}}. \end{aligned}$$

In case when $\omega(x,r) = \left(sh \frac{r}{2} \right)^{\nu-\gamma}$, $0 < \nu < \gamma$, from the Theorem 5.5 follows Adams-Guliyev type result for commutator of Gegenbauer fractional maximal operator M_G^α .

Corollary 5.6. *Let $0 < \alpha < \gamma$, $1 < p < \frac{\gamma}{\alpha}$, $0 < \nu < \gamma - \alpha p$, condition (3.4) holds and $b \in BMO_G(\mathbb{R}_+)$. Then the operator $M_G^{b,\alpha,k}$ is bounded from $L_{p,\lambda}(\mathbb{R}_+)$ to $L_{q,\lambda}(\mathbb{R}_+)$.*

We will use the following estimates to prove the main results.

Lemma 5.7. Let $b \in L_{1,\lambda}^{loc}(\mathbb{R}_+)$ and $H_0 = (0, r_0) \subset H_r = (0, r)$. Then the estimate

$$|b_{H_r}(chx) - b_{H_0}(chx)| \left(sh \frac{r}{2} \right)^\alpha \lesssim M_G^{b,\alpha} \chi_{H_0}(chx)$$

is true for any $x \in H_0$.

Proof. We choose c_0 large enough to satisfy the inequality $r \leq c_0 r_0$. Then by Lemma 3.3 when $0 \leq t < c_0$ the inequality $t \leq sht \leq e^{c_0 t}$ is true and by (2.1), we get (see proof of Lemma 3.4)

$$|H_r|_\lambda^{1-\frac{\alpha}{\gamma}} \lesssim |H_0|_\lambda^{1-\frac{\alpha}{\gamma}}. \quad (5.15)$$

From the definition and the estimate (5.14), we get

$$M_G^{b,\alpha} \chi_{H_0}(chx) = \sup_{r>0} |H_r|_\lambda^{\frac{\alpha}{\gamma}-1} \int_{H_r} \left| A_{chy}^\lambda b(chx) - b_{H_r}(chx) \right| A_{chy}^\lambda \chi_{H_0}(chx) sh^{2\lambda} y dy$$

$$\begin{aligned}
&\geq |H_r|_\lambda^{\frac{\alpha}{\gamma}-1} \int_{H_r \cap H_0} \left| A_{chy}^\lambda b(chx) - b_{H_r}(chx) \right| sh^{2\lambda} y dy \\
&\gtrsim |H_0|_\lambda^{\frac{\alpha}{\gamma}-1} \int_{H_0} \left| A_{chy}^\lambda b(chx) - b_{H_r}(chx) \right| sh^{2\lambda} y dy \\
&\gtrsim |H_0|_\lambda^{\frac{\alpha}{\gamma}-1} \left| \int_{H_0} \left[A_{chy}^\lambda b(chx) - b_{H_r}(chx) \right] sh^{2\lambda} y dy \right| \\
&= |H_0|_\lambda^{\frac{\alpha}{\gamma}-1} \left| \int_{H_0} A_{chy}^\lambda b(chx) sh^{2\lambda} y dy - b_{H_r}(chx) |H_0|_\lambda \right| \\
&= |H_0|_\lambda^{\frac{\alpha}{\gamma}} \left| \frac{1}{|H_0|_\lambda} \int_{H_0} A_{chy}^\lambda b(chx) sh^{2\lambda} y dy - b_{H_r}(chx) \right| \\
&\approx \left(sh \frac{r}{2} \right)^\alpha |b_{H_0}(chx) - b_{H_r}(chx)|.
\end{aligned}$$

The following theorem is an analogue of Theorem 5.5 from [25] and one of the main results of this paper.

Theorem 5.8. *Let $b \in BMO_G(\mathbb{R}_+) \setminus \{\text{const}\}$, $p, q \in [1, \infty)$, $0 \leq \alpha < \gamma$, $\omega_1 \in \Omega_p^\gamma$ and $\omega_2 \in \Omega_q^\gamma$.*

(i) *Let $1 < p < \frac{\gamma}{\alpha}$, and (2.19) holds, then the condition*

$$\sup_{s>r} \left(sh \frac{s}{2} \right)^{-\frac{\gamma}{q}} \left(1 + \left(\frac{sh \frac{s}{2}}{sh \frac{r}{2}} \right)^\gamma \right) \omega_1(x, s) \lesssim \omega_2(x, r), r > 0$$

is sufficient for the boundedness $M_G^{b,\alpha}$ from $\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$.

(ii) *If $\omega_1 \in \Phi_p^\gamma$, then the condition*

$$\left(sh \frac{r}{2} \right)^\alpha \omega_1(r) \lesssim \omega_2(r), \quad s > 0 \quad (5.16)$$

is necessary for the boundedness $M_G^{b,\alpha}$ from $\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$

(iii) *Let $1 < p < \frac{\gamma}{\alpha}$ and (2.19) holds. If $\omega_1 \in \Phi_p^\gamma$ satisfies the condition*

$$\sup_{s>r} \left(sh \frac{s}{2} \right)^\alpha \left(1 + \left(\frac{sh \frac{s}{2}}{sh \frac{r}{2}} \right)^\gamma \right) \omega_1(s) \lesssim \left(sh \frac{r}{2} \right)^\alpha \omega_1(r),$$

for any $r > 0$, then the condition (5.15) is necessary and sufficient for the boundedness $M_G^{b,\alpha}$ from $\mathcal{M}_{p,\gamma,\omega_1}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega_2}(\mathbb{R}_+)$.

Proof. The first part of our theorem follows from Theorem 5.3 when $k = 1$.

We will prove the second part of this theorem. Note that $\|b\|_{BMO_G}$ is a seminorm and $\|b\|_{BMO_G} = 0$ is and only if b is a constant almost everywhere (see [31, §2]). Therefore, if $b \in BMO_G(\mathbb{R}_+) \setminus \{\text{const}\}$ then $\|b\|_{BMO_G} > 0$. For any $r > 0$ there exists a point $x_0 \in H_0 \subset H_r$ such that $\|b_{H_r} - b_{H_0}\|_{L_{1,\lambda}(H_0)} > 0$ but in other cases it is a constant.

Let $b \in BMO_G(\mathbb{R}_+) \setminus \{const\}$ then by Hölder's inequality, we get

$$\begin{aligned} \|b_{H_r} - b_{H_0}\|_{L_{1,\lambda}(H_0)} &= \int_0^{r_0} A_{chy}^\lambda |b_{H_r}(chx) - b_{H_0}(chx)| sh^{2\lambda} t dt \\ &\leq \|b_{H_r} - b_{H_0}\|_{L_{p,\lambda}(H_0)} \left(\int_0^{r_0} sh^{2\lambda} t dt \right)^{\frac{1}{q}} = \|b_{H_r} - b_{H_0}\|_{L_{p,\lambda}(H_0)} |H_0|_\lambda^{\frac{1}{q}}. \end{aligned}$$

According to Lemma 5.4

$$\left(sh \frac{r_0}{2} \right)^\alpha |b_{H_r}(chx) - b_{H_0}(chx)| \lesssim M_G^{b,\alpha} \chi_{H_0}(chx).$$

Therefore, taking into account the relation (5.3) and Lemma 2.4, we can write

$$\begin{aligned} \left(sh \frac{r_0}{2} \right)^\alpha &\lesssim \frac{\|M_G^{b,\alpha} \chi_{H_0}\|_{L_{q,\lambda}(H_0)}}{\|b_{H_r} - b_{H_0}\|_{L_{q,\lambda}(H_0)}} \lesssim \|M_G^{b,\alpha} \chi_{H_0}\|_{L_{q,\lambda}(H_0)} |H_0|_\lambda^{-\frac{1}{q}} \\ &\lesssim \omega_2(r_0) \|M_G^{b,\alpha} \chi_{H_0}\|_{\mathcal{M}_{q,\gamma,\omega_2}} \lesssim \omega_2(r_0) \|\chi_{H_0}\|_{\mathcal{M}_{p,\gamma,\omega_1}} \lesssim \frac{\omega_2(r_0)}{\omega_1(r_0)}. \end{aligned}$$

The third statement of the theorem follows from the first and the second parts.

53 Adams-Guliyev and Adams-Gunavan type results

In this section we characterize the boundedness of the operator $M_G^{b,\alpha}$ on generalized G -Morrey spaces.

The following result is an analogue of Theorem 5.7 from [25].

Theorem 5.9. (Adams-Guliyev type results) *Let $1 < p < q < \infty$, $0 < \alpha < \frac{\gamma}{p}$, $b \in BMO_G(\mathbb{R}_+)$ and let $\varphi \in \Omega_p^\gamma$ satisfy the conditions:*

$$\sup_{s>r} \left(sh \frac{s}{2} \right)^{-\frac{\gamma}{p}} \left(1 + \left(\frac{sh \frac{s}{2}}{sh \frac{r}{2}} \right)^\gamma \right) \varphi(x, s) \lesssim \varphi(x, r), \quad (5.17)$$

$$\sup_{s>r} \left(sh \frac{s}{2} \right)^\alpha \left(1 + \left(\frac{sh \frac{s}{2}}{sh \frac{r}{2}} \right)^\gamma \right) \varphi(x, s) \lesssim \left(sh \frac{r}{2} \right)^{-\frac{\alpha p}{q-p}}. \quad (5.18)$$

Then the operator $M_G^{b,\alpha}$ is bounded from $\mathcal{M}_{p,\gamma,\varphi}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\varphi^{\frac{p}{q}}}(\mathbb{R}_+)$.

Proof. We get the statement of this theorem if we take into account $\omega(x, r)^{\frac{1}{p}} = \varphi(x, r)$ in Theorem 5.5.

The following theorem is an analogue of the Theorem 5.8 from [25] and this is one the most important results of this paper.

Theorem 5.10. *Let $1 < p < q < \infty$, $0 < \alpha < \frac{\gamma}{p}$, $b \in BMO_G(\mathbb{R}_+)$ and $\omega \in \Omega_p^\gamma$*

(i) *If $\omega(x, r)$ satisfies the condition (5.16), then condition (5.17) is sufficient for the boundedness $M_G^{b,\alpha}$ from $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega^{\frac{p}{q}}}(\mathbb{R}_+)$.*

(ii) If $\omega \in \Phi_p^\gamma$, then the condition

$$\left(sh \frac{s}{2} \right)^\alpha \omega(r) \lesssim \left(sh \frac{r}{2} \right)^{-\frac{\alpha p}{q-p}} \quad (5.19)$$

is necessary for the boundedness $M_G^{b,\alpha}$ from $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega^{\frac{p}{q}}}(\mathbb{R}_+)$.

(iii) If $\omega \in \Phi_p^\gamma$ satisfies the condition

$$\sup_{s>r} \left(sh \frac{s}{2} \right)^\alpha \left(1 + \left(\frac{sh \frac{s}{2}}{sh \frac{r}{2}} \right)^\gamma \right) \omega(s) \lesssim \left(sh \frac{r}{2} \right)^\alpha \omega(r), \quad (5.20)$$

for all $r > 0$, then the conditions (5.18) necessary and sufficient for the on of $M_G^{b,\alpha}$ from $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega^{\frac{p}{q}}}(\mathbb{R}_+)$.

Proof. The first part of the theorem follows from Theorem 5.6. We will prove the second part of this theorem. Let $b \in BMO_G(\mathbb{R}_+) \setminus \{const\}$, $H_0 = (0, r_0)$, $x \in H_0 \subset H_r = (0, r)$, then $\|b_{H_r} - b_{H_0}\|_{L_{q,\lambda}(H_0)} > 0$.

By Lemma 5.7

$$|b_{H_r}(chx) - b_{H_0}(chx)| \left(sh \frac{r_0}{2} \right)^\alpha \lesssim M_G^{b,\alpha} \chi_{H_0}(chx).$$

Therefore, by Lemma 2.4 and relation (5.3) we have

$$\begin{aligned} \left(sh \frac{r_0}{2} \right)^\alpha &\lesssim \frac{\|M_G^{b,\alpha} \chi_{H_0}\|_{L_{q,\lambda}(H_0)}}{\|b_{H_r} - b_{H_0}\|_{L_{q,\lambda}(H_0)}} \lesssim \|M_G^{b,\alpha} \chi_{H_0}\|_{L_{q,\lambda}(H_0)} |H_0|_\lambda^{-\frac{1}{q}} \\ &\lesssim \omega_r(r_0)^{\frac{p}{q}} \|M_G^{b,\alpha} \chi_{H_0}\|_{\mathcal{M}_{q,\gamma,\omega^{\frac{p}{q}}}} \lesssim \omega(r_0)^{\frac{p}{q}} \|\chi_{H_0}\|_{\mathcal{M}_{p,\gamma,\omega}} \lesssim \omega_r(r_0)^{\frac{p}{q}-1} \end{aligned}$$

or

$$\left(sh \frac{r_0}{2} \right)^\alpha \omega(r_0)^{1-\frac{p}{q}} \lesssim 1 \Leftrightarrow \left(sh \frac{r_0}{2} \right)^\alpha \omega(r_0) \lesssim \left(sh \frac{r_0}{2} \right)^{-\frac{\alpha p}{q-p}},$$

for any $r_0 > 0$.

Since the last relation is true for every $r \in \mathbb{R}_+$, we are done.

The third statement of the theorem follows from first a second parts of the theorem.

The following result is an analogue of Theorem 5.9 in [25].

Theorem 5.11. (Adams-Gunavan type result). Let $1 < p < q < \infty$, $0 < \alpha < \frac{\gamma}{p}$, $b \in BMO_G(\mathbb{R}_+)$ and $\omega(x, r)$ satisfy condition (5.7) and

$$\left(sh \frac{r}{2} \right)^\alpha \omega(x, r) + \sup_{s>r} \left(1 + \left(\frac{sh \frac{s}{2}}{sh \frac{r}{2}} \right)^\gamma \right) \left(sh \frac{r}{2} \right)^\alpha \omega(x, s) \lesssim \omega(x, r)^{\frac{p}{q}}. \quad (5.21)$$

Then $M_G^{b,\alpha}$ is bounded from $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega^{\frac{p}{q}}}(\mathbb{R}_+)$.

Proof. Let $1 < p < q < \infty$, $0 < \alpha < \frac{\gamma}{p}$ and $f \in \mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$. We write $f = f_1 + f_2$, where $f_1 = f\chi_{H_r}$ and $f_2 = f\chi_{(H_r)^c}$. From (5.12) when $k = 1$, we have

$$M_G^{b,\alpha} f(chx) \lesssim \left(sh \frac{r}{2} \right)^\alpha M_G^b f(chx)$$

$$+ \|b\|_{BMO_G} \sup_{s>r} \left(sh \frac{s}{2} \right)^{\alpha - \frac{\gamma}{p}} \left(1 + \left(\frac{sh \frac{s}{2}}{sh \frac{r}{2}} \right)^\gamma \right) \|f\|_{L_{p,\lambda}(H_s)}. \quad (5.22)$$

Then from (5.20) and (5.21) we get

$$\begin{aligned} M_G^{b,\alpha} f(chx) &\lesssim \left(sh \frac{r}{2} \right)^\alpha M_G^b f(chx) \\ + \|b\|_{BMO_G} \|f\|_{\mathcal{M}_{p,\gamma,\omega}} \sup_{s>r} \left(sh \frac{s}{2} \right)^\alpha \left(1 + \left(\frac{sh \frac{s}{2}}{sh \frac{r}{2}} \right)^\gamma \right) \omega(x, s) \\ &\lesssim \left(sh \frac{r}{2} \right)^\alpha M_G^b f(chx) + \|b\|_{BMO_G} \|f\|_{\mathcal{M}_{p,\gamma,\omega}} \omega(x, r)^{\frac{p}{q}} \\ &\lesssim \min \left\{ \omega(x, r)^{\frac{p}{q}-1} M_G^b f(chx), \omega(x, r)^{\frac{p}{q}} \|f\|_{\mathcal{M}_{p,\gamma,\omega}} \right\} \\ &\lesssim \left(M_G^b f(chx) \right)^{\frac{p}{q}} \|f\|_{\mathcal{M}_{p,\gamma,\omega}}^{1-\frac{p}{q}}. \end{aligned} \quad (5.23)$$

Taking in (5.7) $\omega_1(x, s) = \omega(x, s) \left(sh \frac{r}{2} \right)^{-\frac{\gamma}{q}}$, $\omega_2(x, r) = \omega(x, r) \left(sh \frac{r}{2} \right)^{-\frac{\gamma}{q}}$ and taking into account (5.22), obtain

$$\begin{aligned} \left\| M_G^{b,\alpha} f \right\|_{\mathcal{M}_{q,\gamma,\omega}^{\frac{p}{q}}} &\lesssim \|f\|_{\mathcal{M}_{p,\gamma,\omega}}^{1-\frac{p}{q}} \left\| \left(M_G^b f \right)^{\frac{p}{q}} \right\|_{\mathcal{M}_{q,\gamma,\omega}^{\frac{p}{q}}} \\ &\lesssim \|f\|_{\mathcal{M}_{p,\gamma,\omega}}^{1-\frac{p}{q}} \left\| M_G^b f \right\|_{\mathcal{M}_{p,\gamma,\omega}}^{\frac{p}{q}} \lesssim \|f\|_{\mathcal{M}_{p,\gamma,\omega}}. \end{aligned}$$

The following theorem is an analogue of the Theorem 5.10 in [25].

Theorem 5.12. Let $1 < p < q < \infty$, $0 < \alpha < \gamma$, $b \in BMO_G(\mathbb{R}_+) \setminus \{const\}$ and $\omega \in \Omega_p^\gamma$.

(i) If $1 < p < \infty$ and $\omega(s)$ satisfies the condition

$$\sup_{s>r} \left(sh \frac{s}{2} \right)^{-\frac{\gamma}{p}} \left(1 + \left(\frac{sh \frac{s}{2}}{sh \frac{r}{2}} \right)^\gamma \right) \omega_1(x, s) \lesssim \omega_2(x, r), \quad r > 0,$$

then the condition

$$\left(sh \frac{r}{2} \right)^\alpha \omega(r) + \sup_{s>r} \left(1 + \left(\frac{sh \frac{s}{2}}{sh \frac{r}{2}} \right)^\gamma \right) \left(sh \frac{s}{2} \right)^\alpha \omega(s) \lesssim \omega(r)^{\frac{p}{q}}, \quad r > 0$$

is sufficient for the boundedness $M_G^{b,\alpha}$ from $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega}^{\frac{p}{q}}(\mathbb{R}_+)$.

(ii) If $\omega \in \Phi_p^\gamma$, then the condition

$$\left(sh \frac{r}{2} \right)^\alpha \lesssim \omega(r)^{\frac{p}{q}-1} \quad (5.24)$$

is necessary for the boundedness $M_G^{b,\alpha}$ from $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega}^{\frac{p}{q}}(\mathbb{R}_+)$.

(iii) Let $1 < p < \infty$ If $\omega \in \Phi_p^\gamma$ and satisfies the condition

$$\sup_{s>r} \left(sh \frac{s}{2} \right)^\alpha \left(1 + \left(\frac{sh \frac{s}{2}}{sh \frac{r}{2}} \right)^\gamma \right) \omega(s) \lesssim \left(sh \frac{r}{2} \right)^\alpha \omega(r)$$

for any $r > 0$, then the condition (5.23) is necessary and sufficient for the boundedness $M_G^{b,\alpha}$ from $\mathcal{M}_{p,\gamma,\omega}(\mathbb{R}_+)$ to $\mathcal{M}_{q,\gamma,\omega}^{\frac{p}{q}}(\mathbb{R}_+)$.

Proof. The first part of the theorem follows from the Theorem 5.11.

We shall prove the second part. Let $b \in BMO_G(\mathbb{R}_+) \setminus \{const\}$, $H_0 = (0, r_0)$ and $x \in H_0 \subset H_r = (0, r)$, then $\|b_{H_r} - b_{H_0}\|_{L_{q,\lambda}(H_0)} > 0$. By Lemma 5.7

$$\left(sh \frac{r_0}{2}\right)^\alpha \|b_{H_r}(chx) - b_{H_0}\| \lesssim M_G^{b,\alpha} \chi_{H_0}(chx).$$

Therefore, by Lemma 2.4 and (5.3), we have

$$\begin{aligned} \left(sh \frac{r_0}{2}\right)^\alpha &\lesssim \frac{\|M_G^{b,\alpha} \chi_{H_0}\|_{L_{q,\lambda}(H_0)}}{\|b_{H_r} - b_{H_0}\|_{L_{q,\lambda}(H_0)}} \lesssim \|M_G^{b,\alpha} \chi_{H_0}\|_{L_{q,\lambda}(H_0)} |H_0|_\lambda^{-\frac{1}{q}} \\ &\lesssim \omega(r_0)^{\frac{p}{q}} \|M_G^{b,\alpha} \chi_{H_0}\|_{\mathcal{M}_{q,\gamma,\omega}^{\frac{p}{q}}} \lesssim \omega(r_0)^{\frac{p}{q}} \|\chi_{H_0}\|_{\mathcal{M}_{p,\gamma,\omega}} \lesssim \omega(r_0)^{\frac{p}{q}-1}. \end{aligned}$$

Since this is true for every $r_0 > 0$, we are done. The third statement of the theorem follows from the first and second parts of the theorem.

In the case, when $\omega(x, r) = \left(sh \frac{r}{2}\right)^{\frac{\nu-\gamma}{p}}$, $0 < \nu < \gamma$ from Theorem 5.12, we get the following Adams type result for the G -fractional maximal commutator $M_G^{b,\alpha}$.

Corollary 5.13. *Let $b \in BMO_G(\mathbb{R}_+) \setminus \{const\}$, $0 < \alpha < \gamma$, $1 < p < q < \infty$ $0 < \nu < \gamma - \alpha p$. Then $M_G^{b,\alpha}$ is bounded from $L_{p,\lambda}(\mathbb{R}_+)$ to $L_{q,\lambda}(\mathbb{R}_+)$ if and only if condition (3.4) holds.*

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