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Boundedness of the anisotropic fractional maximal operator in total anisotropic Morrey spaces

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Abstract. *We shall give necessary and sufficient conditions for the boundedness of the anisotropic fractional maximal operator M^d_α in total anisotropic Morrey spaces $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$.*

Keywords. total anisotropic Morrey spaces, anisotropic fractional maximal function.

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1 Introduction

The aim of this paper is to study anisotropic fractional maximal operator M^d_α in total anisotropic Morrey spaces $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$.

Let $\mathbb{R}^n$ be the n-dimension Euclidean space with the norm $|x|$ for each $x \in \mathbb{R}^n$, S^{n-1} denotes the unit sphere on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ and $r > 0$, let $\mathcal{E}(x,r)$ denote the open ball centered at x of radius r and ${}^{\complement}\mathcal{E}(x,r)$ denote the set $\mathbb{R}^n \backslash \mathcal{E}(x,r)$. Let $d = (d_1, \ldots, d_n)$, $d_i \geq 1$, $i = 1, \ldots, n$, $|d| = \sum_{i=1}^n d_i$ and $t^d x \equiv \big(t^{d_1} x_1, \ldots, t^{d_n} x_n\big)$. By [3,5], the function $F(x,\rho) = \sum_{i=1}^n x_i^2 \rho^{-2d_i}$, considered for any fixed $x \in \mathbb{R}^n$, is a decreasing one with respect to $\rho > 0$ and the equation $F(x,\rho) = 1$ is uniquely solvable. This unique solution will be denoted by $\rho(x)$. It is a simple matter to check that $\rho(x-y)$ defines a distance between any two points x, $y \in \mathbb{R}^n$. Thus $\mathbb{R}^n$, endowed with the metric ρ, defines a homogeneous metric space ([3–5]). The balls with respect to ρ, centered at x of radius r, are just the ellipsoids

$$\mathcal{E}_d(x,r) = \left\{ y \in \mathbb{R}^n : \frac{(y_1 - x_1)^2}{r^{2d_1}} + \cdots + \frac{(y_n - x_n)^2}{r^{2d_n}} < 1 \right\},$$

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with the Lebesgue measure $|\mathcal{E}_d(x,r)| = v_n r^{|d|}$, where v_n is the volume of the unit ball in $\mathbb{R}^n$. Let also $\Pi_d(x,r) = \{y \in \mathbb{R}^n : \max_{1\le i\le n} |x_i - y_i|^{1/d_i} < r\}$ denote the parallelopiped, ${}^{\complement}\mathcal{E}_d(x,r) = \mathbb{R}^n \setminus \mathcal{E}_d(x,r)$ be the complement of $\mathcal{E}_d(0,r)$. If $d = \mathbf{1} \equiv (1,\dots,1)$, then clearly $\rho(x) = |x|$ and $\mathcal{E}_{\mathbf{1}}(x,r) = \mathcal{E}(x,r)$. Note that in the standard parabolic case $d = (1,\dots,1,2)$ we have

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + x_n^2}}{2}}, \qquad x = (x', x_n).$$

Let $f \in L_1^{\rm loc}(\mathbb{R}^n)$. The anisotropic fractional maximal operator M_α^d is given by

$$M_\alpha^d f(x) = \sup_{t>0} |\mathcal{E}(x,t)|^{-1+\frac{\alpha}{|d|}} \int_{\mathcal{E}(x,t)} |f(y)|dy, \ \ 0 \le \alpha < |d|,$$

where $|\mathcal{E}(x,t)|$ is the Lebesgue measure of the ellipsoid $\mathcal{E}(x,t)$. If $\alpha = 0$, then $M^d \equiv M_0^d$ is the anisotropic Hardy-Littlewood maximal operator. If $d = \mathbf{1}$, then $M_\alpha \equiv M_\alpha^d$ is the fractional maximal operator and $M \equiv M^d$ is the classical Hardy-Littlewood maximal operator.

Morrey spaces, introduced by C. B. Morrey [11], play important roles in the regularity theory of PDE, including heat equations and Navier-Stokes equations. In [9] Guliyev introduce a variant of Morrey spaces called total Morrey spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$, $0 < p < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. In [1] the authors was consider the total anisotropic Morrey spaces $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$, give basic properties of the spaces $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ and study some embeddings into the Morrey space $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$. In [10] was find necessary and sufficient conditions for the boundedness of the fractional maximal operator M_α in the total Morrey spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$.

The aim of this paper is to give necessary and sufficient conditions for the boundedness of the anisotropic fractional maximal operator M_α^d on total anisotropic Morrey spaces $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$.

The structure of the paper is as follows. In Section 2 we give a characterization for the strong and weak type Spanne and Adams type boundedness of the anisotropic fractional maximal operator M_α^d on $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$, respectively.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Anisotropic fractional maximal operator in total anisotropic Morrey spaces

In this section we find necessary and sufficient conditions for the boundedness of the anisotropic fractional maximal operator M_α^d in the total anisotropic Morrey spaces $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$.

Definition 2.1 *Let $d = (d_1,\dots,d_n)$, $d_i \ge 1$, $i = 1,\dots,n$. Let also $0 < p < \infty$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $[t]_1 = \min\{1,t\}$, $t > 0$. We denote by $L^d_{p,\lambda}(\mathbb{R}^n)$ the anisotropic Morrey space, by $\widetilde{L}^d_{p,\lambda}(\mathbb{R}^n)$ the modified anisotropic Morrey space [6,8], and by $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ the total anisotropic Morrey space [1,9] the set of all classes of locally integrable functions f with* *the finite norms*

$$\|f\|_{L^d_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} t^{-\frac{\lambda}{p}} \|f\|_{L_p(\mathcal{E}(x,t))},$$

$$\|f\|_{\widetilde{L}^d_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{L_p(\mathcal{E}(x,t))},$$

$$\|f\|_{L^d_{p,\lambda,\mu}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{L_p(\mathcal{E}(x,t))},$$

respectively.

Definition 2.2 *Let $d = (d_1, \dots, d_n)$, $d_i \geq 1$, $i = 1, \dots, n$. Let also $0 < p < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. We define the weak anisotropic Morrey space $WL^d_{p,\lambda}(\mathbb{R}^n)$, the weak modified anisotropic Morrey space $W\widetilde{L}^d_{p,\lambda}(\mathbb{R}^n)$ [6,8] and the weak total anisotropic Morrey space $WL^d_{p,\lambda,\mu}(\mathbb{R}^n)$ [1,9] as the set of all locally integrable functions f with finite norms*

$$\|f\|_{WL^d_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} t^{-\frac{\lambda}{p}} \|f\|_{WL_p(\mathcal{E}(x,t))},$$

$$\|f\|_{W\widetilde{L}^d_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{WL_p(\mathcal{E}(x,t))},$$

$$\|f\|_{WL^d_{p,\lambda,\mu}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{WL_p(\mathcal{E}(x,t))},$$

respectively.

Lemma 2.1 *[9, Lemma 2] If $0 < p < \infty$, $0 \leq \lambda \leq |d|$ and $0 \leq \mu \leq |d|$, then*

$$L^d_{p,\lambda,\mu}(\mathbb{R}^n) = L^d_{p,\min\{\lambda,\mu\}}(\mathbb{R}^n) \cap L^d_{p,\max\{\lambda,\mu\}}(\mathbb{R}^n)$$

and

$$\|f\|_{L^d_{p,\lambda,\mu}(\mathbb{R}^n)} = \max\left\{\|f\|_{L^d_{p,\min\{\lambda,\mu\}}}, \|f\|_{L^d_{p,\max\{\lambda,\mu\}}}\right\} = \|f\|_{L^d_{p,\mu,\lambda}}.$$

Lemma 2.2 *[9, Lemma 3] If $0 < p < \infty$, $0 \leq \lambda \leq |d|$ and $0 \leq \mu \leq |d|$, then*

$$WL^d_{p,\lambda,\mu}(\mathbb{R}^n) = WL^d_{p,\min\{\lambda,\mu\}}(\mathbb{R}^n) \cap WL^d_{p,\max\{\lambda,\mu\}}(\mathbb{R}^n)$$

and

$$\|f\|_{WL^d_{p,\lambda,\mu}(\mathbb{R}^n)} = \max\left\{\|f\|_{WL^d_{p,\min\{\lambda,\mu\}}}, \|f\|_{WL^d_{p,\max\{\lambda,\mu\}}}\right\}.$$

Remark 2.1 Let $0 < p < \infty$. If $\min\{\lambda, \mu\} < 0$ or $\max\{\lambda, \mu\} > |d|$, then

$$L^d_{p,\lambda,\mu}(\mathbb{R}^n) = WL^d_{p,\lambda,\mu}(\mathbb{R}^n) = \Theta(\mathbb{R}^n),$$

where $\Theta \equiv \Theta(\mathbb{R}^n)$ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

The following local estimate is valid (see also [7]).

Lemma 2.3 *[7, Lemma 4.1] Let* $0 \le \alpha < |d|$, $1 \le p < \frac{|d|}{\alpha}$, *and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{|d|}$. *Then, for* $p > 1$ *the inequality*

$$\|M_\alpha^d f\|_{L_q(\mathcal{E}(x,r))} \lesssim r^{\frac{|d|}{q}} \sup_{t>2r} t^{-\frac{|d|}{q}} \|f\|_{L_p(\mathcal{E}(x,t))} \tag{2.1}$$

holds for all $\mathcal{E}(x,r)$ *and for all* $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$.

Moreover if $p = 1$, *then the inequality*

$$\|M_\alpha^d f\|_{WL_q(\mathcal{E}(x,r))} \lesssim r^{\frac{|d|}{q}} \sup_{t>2r} t^{-\frac{|d|}{q}} \|f\|_{L_1(\mathcal{E}(x,t))} \tag{2.2}$$

holds for all $\mathcal{E}(x,r)$ *and for all* $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

The following is Spanne's type result for the anisotropic fractional maximal operators in total anisotropic Morrey spaces (see, for example, [7]).

Theorem 2.1 *(Spanne type result) Let* $1 \le p < \infty$, $0 \le \min\{\lambda,\mu\} \le \max\{\lambda,\mu\} < |d|$, $0 \le \alpha < \frac{|d| - \max\{\lambda,\mu\}}{p}$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{|d|}$.

1. If $p > 1$, $f \in L^d_{p,\lambda,\mu}(\mathbb{R}^n)$, *then* $M_\alpha^d f \in L^d_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}(\mathbb{R}^n)$ *and*

$$\|M_\alpha^d f\|_{L^d_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}} \le C_{p,\lambda,\mu,d} \, \|f\|_{L^d_{p,\lambda,\mu}}, \tag{2.3}$$

where $C_{p,\lambda,\mu,d}$ *depends only on* p,λ,μ *and* n.

2. If $p = 1$, $f \in L^d_{1,\lambda,\mu}(\mathbb{R}^n)$, *then* $Mf \in WL^d_{q,\lambda q,\mu q}(\mathbb{R}^n)$ *and*

$$\|M_\alpha^d f\|_{WL^d_{q,\lambda q,\mu q}} \le C_{1,\lambda,\mu,d} \, \|f\|_{L^d_{1,\lambda,\mu}}, \tag{2.4}$$

where $C_{1,\lambda,\mu,d}$ *is independent of* f.

Proof. Let $1 < p < \infty$. From the inequality (2.1) (see Lemma 2.3) we get

$$\begin{aligned}
\|M_\alpha^d f\|_{L^d_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}} &= \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\frac{\lambda}{p}} \, [1/r]_1^{\frac{\mu}{p}} \, \|M_\alpha^d f\|_{L_q(\mathcal{E}(x,r))} \\
&\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\frac{\lambda}{p}} \, [1/r]_1^{\frac{\mu}{p}} \, r^{\frac{|d|}{q}} \sup_{t>2r} t^{-\frac{|d|}{q}} \, \|f\|_{L_p(\mathcal{E}(x,t))} \\
&\lesssim \|f\|_{L_{p,\lambda,\mu}} \sup_{r>0} [r]_1^{-\frac{\lambda}{p}} \, [1/r]_1^{\frac{\mu}{p}} \, r^{-\alpha+\frac{|d|}{p}} \sup_{t>r} t^{\alpha-\frac{|d|}{p}} \, [t]_1^{\frac{\lambda}{p}} \, [1/t]_1^{-\frac{\mu}{p}} \\
&= \|f\|_{L_{p,\lambda,\mu}} \sup_{r>0} [r]_1^{-\alpha+\frac{|d|-\lambda}{p}} \, [1/r]_1^{\alpha-\frac{|d|-\mu}{p}} \sup_{t>r} [t]_1^{\alpha-\frac{|d|-\lambda}{p}} \, [1/t]_1^{-\alpha+\frac{|d|-\mu}{p}} \\
&= \|f\|_{L_{p,\lambda,\mu}},
\end{aligned}$$

which implies that the operator $M_\alpha^d f$ is bounded from $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ to $L_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}(\mathbb{R}^n)$.

Let $p = 1$. From the inequality (2.2) (see Lemma 2.3) we get

$$\|M_\alpha^d f\|_{WL_{q,\lambda q,\mu q}} = \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\lambda}\, [1/r]_1^{\mu}\, \|M_\alpha^d f\|_{WL_q(\mathcal{E}(x,r))}$$
$$\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\lambda}\, [1/r]_1^{\mu}\, r^{\frac{|d|}{q}} \sup_{t>2r} t^{-\frac{|d|}{q}} \|f\|_{L_1(\mathcal{E}(x,t))}$$
$$\lesssim \|f\|_{L_{1,\lambda,\mu}} \sup_{r>0} [r]_1^{-\lambda}\, [1/r]_1^{\mu}\, r^{-\alpha+n} \sup_{t>r} t^{\alpha-|d|}\, [t]_1^{\lambda}\, [1/t]_1^{-\mu}$$
$$= \|f\|_{L_{1,\lambda,\mu}} \sup_{r>0} [r]_1^{-\alpha+|d|-\lambda}\, [1/r]_1^{\alpha-(|d|-\mu)} \sup_{t>r} [t]_1^{\alpha-(|d|-\lambda)}\, [1/t]_1^{-\alpha+(|d|-\mu)}$$
$$= \|f\|_{L_{1,\lambda,\mu}},$$

which implies that the operator $M_\alpha^d f$ is bounded from $L_{1,\lambda,\mu}(\mathbb{R}^n)$ to $WL_{q,\lambda q,\mu q}(\mathbb{R}^n)$.

From Theorem 2.1 in the case $\alpha = 0$ we get the following corollaries.

Corollary 2.1 *[9, Theorem 1] Let $1 \le p < \infty$, $0 \le \lambda < |d|$ and $0 \le \mu < |d|$.*

1. If $p > 1$, $f \in L_{p,\lambda,\mu}(\mathbb{R}^n)$, then $Mf \in L_{p,\lambda,\mu}(\mathbb{R}^n)$ and

$$\|M^d f\|_{L_{p,\lambda,\mu}} \le C_{p,\lambda,\mu,d}\, \|f\|_{L_{p,\lambda,\mu}},$$

where $C_{p,\lambda,\mu,d}$ depends only on p, λ, μ, d and n.

2. If $f \in L_{1,\lambda,\mu}(\mathbb{R}^n)$, then $Mf \in WL_{1,\lambda,\mu}(\mathbb{R}^n)$ and

$$\|M^d f\|_{WL_{1,\lambda,\mu}} \le C_{1,\lambda,\mu,d}\, \|f\|_{L_{1,\lambda,\mu}},$$

where $C_{1,\lambda,\mu,d}$ depends only on p, λ, μ, d and n.

From Theorem 2.1 in the case $\lambda = \mu$ or $\mu = 0$ we get the following corollaries.

Corollary 2.2 *[12, Theorem 5.4] Let $1 \le p < \infty$, $0 \le \lambda < |d|$, $0 \le \alpha < \frac{|d|-\lambda}{p}$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{|d|}$.*

1. If $p > 1$, $f \in L_{p,\lambda}(\mathbb{R}^n)$, then $M_\alpha^d f \in L_{q,\frac{\lambda q}{p}}(\mathbb{R}^n)$ and

$$\|M_\alpha^d f\|_{L_{q,\frac{\lambda q}{p}}} \le C_{p,\lambda,d}\, \|f\|_{L_{p,\lambda}}, \tag{2.5}$$

where $C_{p,\lambda,d}$ depends only on p, λ, d and n.

2. If $p = 1$, $f \in L_{1,\lambda}(\mathbb{R}^n)$, then $Mf \in WL_{q,\lambda}(\mathbb{R}^n)$ and

$$\|M_\alpha^d f\|_{WL_{q,\lambda q}} \le C_{1,\lambda,d}\, \|f\|_{L_{1,\lambda}}, \tag{2.6}$$

where $C_{1,\lambda,d}$ is independent of f.

Corollary 2.3 *Let $1 \le p < \infty$, $0 \le \lambda < |d|$, $0 \le \alpha < \frac{|d|-\lambda}{p}$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{|d|}$.*

1. If $p > 1$, $f \in \widetilde{L}_{p,\lambda}^d(\mathbb{R}^n)$, then $M_\alpha^d f \in \widetilde{L}^{q,\frac{\lambda q}{p}}(\mathbb{R}^n)$ and

$$\|M_\alpha^d f\|_{\widetilde{L}^d_{q,\frac{\lambda q}{p}}} \le C_{p,\lambda,d}\, \|f\|_{\widetilde{L}^d_{p,\lambda}}, \tag{2.7}$$

where $C_{p,\lambda,d}$ depends only on p, λ and n.

2. If $p = 1$, $f \in \widetilde{L}^{1,\lambda}(\mathbb{R}^n)$, then $M_\alpha^d f \in W\widetilde{L}^{q,\lambda}(\mathbb{R}^n)$ and

$$\|M_\alpha^d f\|_{W\widetilde{L}^{q,\lambda q}} \le C_{1,\lambda,d}\, \|f\|_{\widetilde{L}^{1,\lambda}}, \tag{2.8}$$

where $C_{1,\lambda,d}$ is independent of f.

The following is Adam's type result for the anisotropic fractional maximal operators in total anisotropic Morrey spaces (see, for example, [6]).

Theorem 2.2 *(Adams type result) Let* $1 \le p < \infty$, $0 \le \min\{\lambda,\mu\} \le \max\{\lambda,\mu\} < |d|$, $0 \le \alpha < \frac{|d|-\max\{\lambda,\mu\}}{p}$.

1) If $1 < p < \frac{|d|-\max\{\lambda,\mu\}}{\alpha}$, *then condition* $\frac{\alpha}{|d|-\min\{\lambda,\mu\}} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{|d|-\max\{\lambda,\mu\}}$ *is necessary and sufficient for the boundedness of the operator* M^d_α *from* $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ *to* $L^d_{q,\lambda,\mu}(\mathbb{R}^n)$.

2) If $p = 1 < \frac{|d|-\max\{\lambda,\mu\}}{\alpha}$, *then condition* $\frac{\alpha}{|d|-\min\{\lambda,\mu\}} \le 1 - \frac{1}{q} \le \frac{\alpha}{|d|-\max\{\lambda,\mu\}}$ *is necessary and sufficient for the boundedness of the operator* M^d_α *from* $L^d_{1,\lambda,\mu}(\mathbb{R}^n)$ *to* $WL^d_{q,\lambda,\mu}(\mathbb{R}^n)$.

3) If $\frac{|d|-\max\{\lambda,\mu\}}{\alpha} \le p \le \frac{|d|-\min\{\lambda,\mu\}}{\alpha}$, *then the operator* M^d_α *is bounded from* $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ *to* $L_\infty(\mathbb{R}^n)$.

Proof. *Sufficiency.* Let $1 \le p < \frac{|d|-\max\{\lambda,\mu\}}{\alpha}$, $\frac{\alpha}{|d|-\min\{\lambda,\mu\}} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{|d|-\max\{\lambda,\mu\}}$ and $f \in L_{p,\lambda,\mu}(\mathbb{R}^n)$.

$$\begin{aligned}
M^d_\alpha f(x) &\approx \sup_{r>0} r^{\alpha-|d|}\, \|f\|_{L_1(\mathcal{E}(x,r))} \\
&\le \sup_{r>0} \min\{r^\alpha\, M^d f(x), r^{\alpha-\frac{|d|}{p}}\, \|f\|_{L_p(\mathcal{E}(x,r))}\} \\
&\le \sup_{r>0} \min\{r^\alpha\, M^d f(x), r^{\alpha-\frac{|d|}{p}}\, [r]_1^{\frac{\lambda}{p}}\, [1/r]_1^{-\frac{\mu}{p}}\, \|f\|_{L^d_{p,\lambda,\mu}}\} \\
&\le \sup_{r>0} \min\{r^\alpha\, M^d f(x), [r]_1^{\alpha-\frac{|d|-\lambda}{p}}\, [1/r]_1^{-\alpha+\frac{|d|-\mu}{p}}\, \|f\|_{L_{p,\lambda,\mu}}\} \\
&\le \max\Big\{ \sup_{0<r\le 1} \min\{r^\alpha\, M^d f(x), r^{\alpha-\frac{|d|-\lambda}{p}}\, \|f\|_{L_{p,\lambda,\mu}}\}, \\
&\qquad \sup_{r>1} \min\{r^\alpha\, M^d f(x), r^{\alpha-\frac{|d|-\mu}{p}}\, \|f\|_{L_{p,\lambda,\mu}}\}\Big\}.
\end{aligned}$$

Minimizing with respect to r, at

$$r = \Big(\frac{\|f\|^d_{L_{p,\lambda,\mu}}}{M^d f(x)}\Big)^{\frac{p}{|d|-\min\{\lambda,\mu\}}} \quad \text{and} \quad r = \Big(\frac{\|f\|^d_{L_{p,\lambda,\mu}}}{M^d f(x)}\Big)^{\frac{p}{|d|-\max\{\lambda,\mu\}}}$$

we have

$$\begin{aligned}
M^d_\alpha f(x) \le \max\Big\{ &\big(M^d f(x)\big)^{1-\frac{\alpha p}{|d|-\min\{\lambda,\mu\}}}\, \|f\|_{L_{p,\lambda,\mu}}^{\frac{\alpha p}{|d|-\min\{\lambda,\mu\}}}, \\
&\big(M^d f(x)\big)^{1-\frac{\alpha p}{|d|-\max\{\lambda,\mu\}}}\, \|f\|_{L_{p,\lambda,\mu}}^{\frac{\alpha p}{|d|-\max\{\lambda,\mu\}}}\Big\}, \qquad (2.9)
\end{aligned}$$

where we have used that the supremum is achieved when the minimum parts are balanced. From Corollary 2.1 and inequality (2.9), we get

$$\begin{aligned}
\|M^d_\alpha f\|_{L^d_{q,\lambda,\mu}} &\lesssim \|f\|_{L^d_{p,\lambda,\mu}}^{1-\frac{p}{q}}\, \|(M^d f)^{\frac{p}{q}}\|_{L^d_{q,\lambda,\mu}} \\
&= \|f\|_{L_{p,\lambda,\mu}}^{1-\frac{p}{q}}\, \|M^d f\|_{L^d_{p,\lambda,\mu}}^{\frac{p}{q}} \lesssim \|f\|_{L^d_{p,\lambda,\mu}},
\end{aligned}$$

if $1 < p < q < \infty$ and

$$\|M_\alpha^d f\|_{WL^d_{q,\lambda,\mu}} \lesssim \|f\|_{L^d_{1,\lambda,\mu}}^{1-\frac{1}{q}} \|M^d f\|_{WL^d_{1,\lambda,\mu}}^{\frac{1}{q}} \lesssim \|f\|_{L^d_{1,\lambda,\mu}},$$

if $p = 1 < q < \infty$.

Necessity. Let $1 < p < \frac{|d|-\max\{\lambda,\mu\}}{\alpha}$, $\frac{\alpha}{|d|-\min\{\lambda,\mu\}} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{|d|-\max\{\lambda,\mu\}}$, $f \in L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ and assume that M_α^d is bounded from $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ to $L^d_{q,\lambda,\mu}(\mathbb{R}^n)$.

Define $f_{t^d}(x) =: f(t^d x)$, $[t]_{1,+} = \max\{1,t\}$. Then

$$\begin{aligned}
\left\|f_{t^d}\right\|_{L^d_{p,\lambda,\mu}} &= \sup_{x\in\mathbb{R}^n, r>0} [r]_1^{-\frac{\lambda}{p}} \, [1/r]_1^{\frac{\mu}{p}} \, \|f_{t^d}\|_{L_p(\mathcal{E}(x,r))} \\
&= t^{-\frac{|d|}{p}} \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\frac{\lambda}{p}} \, [1/r]_1^{\frac{\mu}{p}} \, \|f\|_{L_p(\mathcal{E}(x,tr))} \\
&= t^{-\frac{|d|}{p}} \sup_{r>0} \Big(\frac{[tr]_1}{[r]_1}\Big)^{\frac{\lambda}{p}} \sup_{r>0} \Big(\frac{[1/r]_1}{[1/(tr)]_1}\Big)^{\frac{\mu}{p}} \sup_{x\in\mathbb{R}^n,\, r>0} [tr]_1^{-\frac{\lambda}{p}} \, [1/(tr)]_1^{\frac{\mu}{p}} \, \|f\|_{L_p(\mathcal{E}(x,tr))} \\
&= t^{-\frac{|d|}{p}} \, [t]_{1,+}^{\frac{\lambda}{p}} \, [1/t]_{1,+}^{-\frac{\mu}{p}} \, \|f\|_{L_{p,\lambda,\mu}},
\end{aligned}$$

and

$$M_\alpha^d f_{t^d}(x) = t^{-\alpha} \, M_\alpha^d f(t^d x),$$

$$\begin{aligned}
\left\|M_\alpha^d f_{t^d}\right\|_{L_{q,\lambda,\mu}} &= t^{-\alpha} \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\frac{\lambda}{p}} \, [1/r]_1^{\frac{\mu}{p}} \, \|M_\alpha^d f(t^d \cdot)\|_{L_q(\mathcal{E}(x,r))} \\
&= t^{-\alpha-\frac{|d|}{q}} \sup_{r>0} \Big(\frac{[tr]_1}{[r]_1}\Big)^{\lambda/q} \sup_{r>0} \Big(\frac{[1/r]_1}{[1/(tr)]_1}\Big)^{\mu/q} \sup_{x\in\mathbb{R}^n,\, r>0} [tr]_1^{-\frac{\lambda}{p}} \, [1/(tr)]_1^{\frac{\mu}{p}} \, \|M_\alpha^d f\|_{L_q(\mathcal{E}(t^d x,tr))} \\
&= t^{-\alpha-\frac{|d|}{q}} \, [t]_{1,+}^{\frac{\lambda}{q}} \, [1/t]_{1,+}^{-\frac{\mu}{q}} \left\|M_\alpha^d f\right\|_{L_{q,\lambda,\mu}}.
\end{aligned}$$

By the boundedness of M_α^d from $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ to $L^d_{q,\lambda,\mu}(\mathbb{R}^n)$ we have

$$\begin{aligned}
\left\|M_\alpha^d f\right\|_{L^d_{q,\lambda,\mu}} &= t^{\alpha+\frac{|d|}{q}} \, [t]_{1,+}^{-\frac{\lambda}{q}} \, [1/t]_{1,+}^{\frac{\mu}{q}} \left\|M_\alpha^d f_{t^d}\right\|_{L^d_{q,\lambda,\mu}} \\
&\lesssim t^{\alpha+\frac{|d|}{q}} \, [t]_{1,+}^{-\frac{\lambda}{q}} \, [1/t]_{1,+}^{\frac{\mu}{q}} \left\|f_{t^d}\right\|_{L_{p,\lambda,\mu}} \\
&= t^{\alpha+\frac{|d|}{q}-\frac{|d|}{p}} \, [t]_{1,+}^{\frac{\lambda}{p}-\frac{\lambda}{q}} \, [1/t]_{1,+}^{-\frac{\mu}{p}+\frac{\mu}{q}} \, \|f\|_{L_{p,\lambda,\mu}} \\
&= t^{\alpha} \, [t]_{1,+}^{-\frac{|d|-\lambda}{p}+\frac{|d|-\lambda}{q}} \, [1/t]_{1,+}^{\frac{|d|-\mu}{p}-\frac{|d|-\mu}{q}} \, \|f\|_{L^d_{p,\lambda,\mu}}.
\end{aligned}$$

Since $L^d_{p,\lambda,\mu}(\mathbb{R}^n) = L^d_{p,\mu,\lambda}(\mathbb{R}^n)$, we can assume that $\lambda < \mu$, and then $\min\{\lambda,\mu\} = \lambda$, $\max\{\lambda,\mu\} = \mu$.

If $\frac{1}{p} < \frac{1}{q} + \frac{\alpha}{|d|-\max\{\lambda,\mu\}}$, then by letting $t \to 0$ we have $\left\|M_\alpha^d f\right\|_{L_{q,\lambda,\mu}} = 0$ for all $f \in L^d_{p,\lambda,\mu}(\mathbb{R}^n)$.

As well as if $\frac{1}{p} > \frac{1}{q} + \frac{\alpha}{|d|-\min\{\lambda,\mu\}}$, then at $t \to \infty$ we obtain $\left\|M_\alpha^d f\right\|_{L^d_{q,\lambda,\mu}} = 0$ for all $f \in L^d_{p,\lambda,\mu}(\mathbb{R}^n)$.

Therefore $\frac{\alpha}{|d|-\min\{\lambda,\mu\}} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{|d|-\max\{\lambda,\mu\}}$.

Let $p = 1 < \frac{|d|-\max\{\lambda,\mu\}}{\alpha}$, $f \in L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ and assume that M^d_α is bounded from $L^d_{1,\lambda,\mu}(\mathbb{R}^n)$ to $WL^d_{q,\lambda,\mu}(\mathbb{R}^n)$. Then

$$\left\|f_{t^d}\right\|_{L^d_{1,\lambda,\mu}} = t^{-|d|}\,[t]^{\lambda}_{1,+}\,[1/t]^{-\mu}_{1,+}\,\|f\|_{L^d_{1,\lambda,\mu}}$$

and

$$\left\|M^d_\alpha f_{t^d}\right\|_{WL^d_{q,\lambda,\mu}} = t^{-\alpha}\sup_{x\in\mathbb{R}^n,\,r>0}[r]_1^{-\frac{\lambda}{p}}\,[1/r]_1^{\frac{\mu}{p}}\,\|M^d_\alpha f(t^d\,\cdot)\|_{WL_q(\mathcal{E}(x,r))}$$
$$= t^{-\alpha-\frac{|d|}{q}}\sup_{r>0}\Big(\frac{[tr]_1}{[r]_1}\Big)^{\lambda/q}\sup_{r>0}\Big(\frac{[1/r]_1}{[1/(tr)]_1}\Big)^{\mu/q}\sup_{x\in\mathbb{R}^n,\,r>0}[tr]_1^{-\frac{\lambda}{p}}\,[1/(tr)]_1^{\frac{\mu}{p}}\,\|M^d_\alpha f\|_{WL_q(\mathcal{E}(t^dx,tr))}$$
$$= t^{-\alpha-\frac{n}{q}}\,[t]^{\frac{\lambda}{q}}_{1,+}\,[1/t]^{-\frac{\mu}{q}}_{1,+}\left\|M^d_\alpha f\right\|_{WL^d_{q,\lambda,\mu}}.$$

By the boundedness of M^d_α from $L^d_{1,\lambda,\mu}(\mathbb{R}^n)$ to $WL^d_{q,\lambda,\mu}(\mathbb{R}^n)$ we have

$$\begin{aligned}\left\|M^d_\alpha f\right\|_{WL^d_{q,\lambda,\mu}} &= t^{\alpha+\frac{|d|}{q}}\,[t]^{-\frac{\lambda}{q}}_{1,+}\,[1/t]^{\frac{\mu}{q}}_{1,+}\left\|M^d_\alpha f_{t^d}\right\|_{WL^d_{q,\lambda,\mu}}\\ &\lesssim t^{\alpha+\frac{|d|}{q}}\,[t]^{-\frac{\lambda}{q}}_{1,+}\,[1/t]^{\frac{\mu}{q}}_{1,+}\left\|f_{t^d}\right\|_{L^d_{1,\lambda,\mu}}\\ &= t^{\alpha+\frac{|d|}{q}-|d|}\,[t]^{\lambda-\frac{\lambda}{q}}_{1,+}\,[1/t]^{-\mu+\frac{\mu}{q}}_{1,+}\,\|f\|_{L^d_{1,\lambda,\mu}}\\ &= t^{\alpha}\,[t]^{-|d|+\lambda+\frac{|d|-\lambda}{q}}_{1,+}\,[1/t]^{|d|-\mu-\frac{|d|-\mu}{q}}_{1,+}\,\|f\|_{L^d_{1,\lambda,\mu}}.\end{aligned}$$

If $1 < \frac{1}{q} + \frac{\alpha}{|d|-\min\{\lambda,\mu\}}$, then by letting $t \to 0$ we have $\left\|M^d_\alpha f\right\|_{WL^d_{q,\lambda,\mu}} = 0$ for all $f \in L^d_{1,\lambda,\mu}(\mathbb{R}^n)$.

As well as if $1 > \frac{1}{q} + \frac{\alpha}{|d|-\max\{\lambda,\mu\}}$, then at $t \to \infty$ we obtain $\left\|M^d_\alpha f\right\|_{WL^d_{q,\lambda,\mu}} = 0$ for all $f \in L^d_{1,\lambda,\mu}(\mathbb{R}^n)$.

Therefore $\frac{\alpha}{|d|-\min\{\lambda,\mu\}} \le 1 - \frac{1}{q} \le \frac{\alpha}{|d|-\max\{\lambda,\mu\}}$.

3) Let us show that, if $\frac{|d|-\max\{\lambda,\mu\}}{\alpha} \le p \le \frac{|D|-\min\{\lambda,\mu\}}{\alpha}$, then the operator M^d_α is bounded from $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ to $L_\infty(\mathbb{R}^n)$.

Let $\frac{|d|-\max\{\lambda,\mu\}}{\alpha} \le p \le \frac{|d|-\min\{\lambda,\mu\}}{\alpha}$ and $f \in L^d_{p,\lambda,\mu}(\mathbb{R}^n)$.

Since $L^d_{p,\lambda,\mu}(\mathbb{R}^n) = L^d_{p,\mu,\lambda}(\mathbb{R}^n)$, we can assume that $\lambda > \mu$, and then $\max\{\lambda,\mu\} = \lambda$, $\min\{\lambda,\mu\} = \mu$.

$$\begin{aligned} M^d_\alpha f(x) &\approx \sup_{r>0} r^{\alpha-|d|}\,\|f\|_{L_1(\mathcal{E}(x,r))} \le \sup_{r>0} r^{\alpha-\frac{|d|}{p}}\,\|f\|_{L_p(\mathcal{E}(x,r))} \\ &\le \sup_{r>0} r^{\alpha-\frac{|d|}{p}}\,[r]_1^{\frac{\lambda}{p}}\,[1/r]_1^{-\frac{\mu}{p}}\,\|f\|_{L_{p,\lambda,\mu}} \le \sup_{r>0} [r]_1^{\alpha-\frac{|d|-\lambda}{p}}\,[1/r]_1^{-\alpha+\frac{|d|-\mu}{p}}\,\|f\|_{L_{p,\lambda,\mu}} \\ &\le \max\Big\{\sup_{0<r\le 1} r^{\alpha-\frac{|d|-\lambda}{p}}\,\|f\|_{L_{p,\lambda,\mu}}, \sup_{r>1} r^{\alpha-\frac{|d|-\mu}{p}}\,\|f\|_{L_{p,\lambda,\mu}}\Big\} \lesssim \|f\|_{L_{p,\lambda,\mu}} \\ &\Longleftrightarrow \frac{|d|-\lambda}{p} \le \alpha \le \frac{|d|-\mu}{p} \quad \Longleftrightarrow \frac{|d|-\lambda}{\alpha} \le p \le \frac{|d|-\mu}{\alpha} \\ &\Longleftrightarrow \frac{|d|-\max\{\lambda,\mu\}}{\alpha} \le p \le \frac{|d|-\min\{\lambda,\mu\}}{\alpha}, \end{aligned}$$

which implies that the operator M^d_α is bounded from $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ to $L_\infty(\mathbb{R}^n)$.

From Theorem 2.2 in the case $\lambda = \mu$ or $\mu = 0$ we get the following corollaries.

Corollary 2.4 *[2, Theorem 3.1] (Adams result) Let* $1 \le p < \infty$, $0 \le \lambda < |d|$, $0 \le \alpha < \frac{|d|-\lambda}{p}$.

1) If $1 < p < \frac{|d|-\lambda}{\alpha}$, *then condition* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{|d|-\lambda}$ *is necessary and sufficient for the boundedness of the operator* M^d_α *from* $L^d_{p,\lambda}(\mathbb{R}^n)$ *to* $L^d_{q,\lambda}(\mathbb{R}^n)$.

2) If $p = 1 < \frac{|d|-\lambda}{\alpha}$, *then condition* $1 - \frac{1}{q} = \frac{\alpha}{|d|-\lambda}$ *is necessary and sufficient for the boundedness of the operator* M^d_α *from* $L^d_{1,\lambda}(\mathbb{R}^n)$ *to* $WL^d_{q,\lambda}(\mathbb{R}^n)$.

3) If $p = \frac{|d|-\lambda}{\alpha}$, *then the operator* M^d_α *is bounded from* $L^d_{p,\lambda}(\mathbb{R}^n)$ *to* $L_\infty(\mathbb{R}^n)$.

Corollary 2.5 *[6, Corollary 1] Let* $1 \le p < \infty$, $0 \le \lambda < n$, $0 \le \alpha < \frac{|d|-\lambda}{p}$.

1) If $1 < p < \frac{|d|-\lambda}{\alpha}$, *then condition* $\frac{\alpha}{n} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{|d|-\lambda}$ *is necessary and sufficient for the boundedness of the operator* M^d_α *from* $\widetilde{L}^d_{p,\lambda}(\mathbb{R}^n)$ *to* $\widetilde{L}^d_{q,\lambda}(\mathbb{R}^n)$.

2) If $p = 1 < \frac{|d|-\lambda}{\alpha}$, *then condition* $\frac{\alpha}{|d|} \le 1 - \frac{1}{q} \le \frac{\alpha}{|d|-\lambda}$ *is necessary and sufficient for the boundedness of the operator* M^d_α *from* $\widetilde{L}^d_{1,\lambda}(\mathbb{R}^n)$ *to* $W\widetilde{L}^d_{q,\lambda}(\mathbb{R}^n)$.

3) If $\frac{|d|-\lambda}{\alpha} \le p \le \frac{|d|}{\alpha}$, *then the operator* M^d_α *is bounded from* $\widetilde{L}^d_{p,\lambda}(\mathbb{R}^n)$ *to* $L_\infty(\mathbb{R}^n)$.

Remark 2.2 Note that in the case of $d = \mathbf{1} \equiv (1,\ldots,1)$ from Theorem 2.1 we get [10, Theorem 2.1] and Theorem 2.2 we get [10, Theorem 2.2].

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