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On recovering the source term for the heat equation with memory term

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Abstract. In this paper, we first study the inverse source problem for the heat equation with a memory term. This problem is non-well-posed in the sense of Hadamard. We also investigate the regularized solution by the exponential Tikhonov regularization method. The error estimates between the regularized solution and the exact solution are obtained under a priori and posteriori parameter choice rules.

Keywords. Inverse source problem, parabolic equation, memory term, regularization method; error estimate.

Mathematics Subject Classification (2010): 35K99, 47J06, 47H10, 35K05

1 Introduction

In this paper, we consider the parabolic equation with memory as follows

$$\begin{cases} u_t(t,x) = u_{xx}(t,x) - m\displaystyle\int_0^t u(s,x)ds + \rho(x), & (t,x) \in (0,M) \times (0,\pi), \\ u(t,0) = u(t,\pi) = 0, & 0 < t < M, \\ u(0,x) = 0, & 0 < x < \pi, \end{cases} \tag{1.1}$$

where M be a positive constant, and $m > 0$. The function ρ on the right hand side of the first equation is called a source term. If the function ρ is known, we can work out the function u by the initial boundary value problem. However, in this paper, our main problem is determining the source term ρ from the additional information as follows

$$u(M,x) = g(x), \quad 0 < x < \pi. \tag{1.2}$$

Let assume that g is noised by g_ϵ such that

$$\left\|g_\epsilon - g\right\|_{L^2(\mathcal{D})} \le \epsilon. \tag{1.3}$$

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Our mentioned problem (1.1)-(1.2) is called the inverse space-dependent source problem. It is well-known that the inverse source problem is ill-posed in the sense of Hadamard. In practice, the terminal condition g is unknown and it is only available as noisy data g_ε with a noise level ε. When we use the noisy data for our problem (1.1), we will obtain the corresponding source term which has a large deviation from the source function corresponding to g. As we all know, the above inverse source problem is ill-posed and they are required approximately by regularization methods.

- If $m = 0$ the problem (1.1) is called classical parabolic equation. This problem has been studied a lot in [9,11,10,2,18,20,12,22,21,5,17,19,14,13].
- If $m \neq 0$ then problem (1.1) have a memory term $m \int_0^t u_{xx}(s,x)ds$ which is called Volterra integro differential equations, researched extensively in the literature [26,16,7, 15,8,3,6,24].

The parabolic with memory term has many applications in many various fields such as heat conduction in materials with memory, population dynamics, nuclear reactors, [15]. In [26], the authors considered Volterra diffusion equations with nonlinear terms. They investigated the stability properties of solutions in L^p norms. Local existence results of solutions to a system of partial functional differential equations are also investigated. In [16], the authors studied the predator-prey system in the form of a coupled system of reaction-diffusion equations.

Regarding some regularization methods, we see in [27], the generalized and revised generalized Tikhonov regularization methods, see [4] with simplified Tikhonov method. In [25], the fractional Tikhonov method, these studies focus on the error estimates between the exact solution and the regularized solution for both methods were provided under the a-priori and a-posteriori regularization parameter choice rules.

Recently, we proposed a novel regularization method, which we call the exponential Tikhonov regularization method with an exponential parameter γ, see [23]. This method was developed from the Tikhonov regularization method. Until now, the Tikhonov exponential regularization method still has little research results. Our current paper may be one of the first studies to apply this approach. We apply this method to construct the regularized solution, the convergence estimates are established under a-priori, and a-posteriori regularization parameter choice rules continue to be considered.

The layout of the article is shown as follows. Some preliminaries are given in Section 2, and attached are the results on the stability of the source function ρ. In Section 3, we construct the exponential Tikhonov regularization method for solving the inverse source problem of the time fractional diffusion equation and present convergence estimates under the a-priori and a-posteriori regularization parameter choice rules. In Section 4.

2 Preliminaries and inverse source problem

We begin this section by introducing some notations and assumptions that are needed for our analysis in the next sections.

Definition 2.1 *(Hilbert scale space). Let $\mathcal{D} = (0,\pi)$. The Hilbert scale space $\mathbb{H}^{2\tau}(\mathcal{D})$, $(\tau > 0)$ defined by*

$$\mathbb{H}^{2\tau}(\mathcal{D}) := \Big\{ f \in L^2(\mathcal{D}) : \sum_{k=1}^{\infty} k^{4\tau} \langle f, \xi_k \rangle^2_{L^2(\mathcal{D})} < \infty \Big\},$$

is equipped with the norm defined by

$$\|f\|^2_{\mathbb{H}^{2\tau}(\mathcal{D})} = \sum_{k=1}^{\infty} k^{4\tau} f_k^2, \quad f_k = \langle f, \xi_k \rangle_{L^2(\mathcal{D})}, \quad \xi_k(x) = \sqrt{\frac{2}{\pi}} \sin(kx).$$

Next, we introduce an exponent operator of $-\mathcal{A}$ *defined by, see [23]*

$$\exp\Big((-\mathcal{A})^\beta\Big) = I + \sum_{k=1}^{\infty} \frac{1}{k!}\big(-\mathcal{A}\big)^{k\beta},$$

where I *is a unit operator and* $(-\mathcal{A})^{s\beta}\xi_k = k^{2s\beta}\xi_k, k = 0, 1, 2, \cdots$. *For* $\beta \in \mathbb{R}$ *we define*

$$\mathbb{H}\big(\exp\big(\frac{(-\mathcal{A})^\beta}{2}\big)\big) = \Big\{ f \in L^2(\mathcal{D}); \sum_{k=1}^{\infty} \exp\big(k^{2\beta}\big)\big|\langle f, \xi_k \rangle\big|^2 < \infty \Big\},$$

where $\langle \cdot, \cdot \rangle$ *is the inner product in* $L^2(\mathcal{D})$ *with the following norms*

$$\|f\|_{\beta,exp} := \Bigg(\sum_{k=1}^{\infty} \exp\big(k^{2\beta}\big)\Big|\langle f, \xi_k \rangle\Big|^2 \Bigg)^{\frac{1}{2}}.$$

21 **The fomula of source term for problem** (1.1)

The solution to the problem (1.1) can be represented in the form of an expansion in the orthogonal series

$$u(t,x) = \sum_{k=1}^{\infty} u_k(t)\xi_k(x), \quad \text{with } u_k(t) = \langle u(t,\cdot), \xi_k \rangle_{L^2(\mathcal{D})}. \tag{2.1}$$

By considering that the series (2.1) converges and allows a term by term differentiation (the required number of times), we construct a formal solution to the problem. We obtain the problems

$$\begin{cases} \dfrac{d}{dt}u_k(t) + k^2 u_k(t) - m\displaystyle\int_0^t u_k(s)ds = \rho_k, & t \in (0, M), \\ u_k(M) = g_k, u_k(0) = 0, \end{cases} \tag{2.2}$$

where

$$g(x) = \sum_{k=1}^{\infty} g_k \xi_k(x), \quad \text{and } \rho(x) = \sum_{k=1}^{\infty} \rho_k \xi_k(x).$$

Let us set

$$z_k(t) = \int_0^t u_k(s)ds.$$

Then we get

$$z_k'(t) = u_k(t).$$

The problem (2.2) becomes to

$$\begin{cases} \dfrac{d^2}{dt^2}z_k(t) + k^2 z_k'(t) - m z_k(t) = \rho_k, & t \in (0, M), \\ z_k'(M) = g_k, z_k'(0) = 0, \ \ z_k(0) = 0. \end{cases} \tag{2.3}$$

The solution z_k of (2.3) is given by

$$z_k(t) = \mathcal{A}_k^+ e^{\mathcal{B}_k^+ t} + \mathcal{A}_k^- e^{\mathcal{B}_k^- t} + \mathcal{C}_k^+(t) e^{\mathcal{B}_k^+ t} + \mathcal{C}_k^-(t) e^{\mathcal{B}_k^- t}.$$

Here we have

$$\mathcal{B}_k^+ = \frac{\Big(-k^2 + \sqrt{k^4+4m}\Big)}{2}, \quad \mathcal{B}_k^- = \frac{\Big(-k^2 - \sqrt{k^4+4m}\Big)}{2}\cdot$$

and $\mathcal{A}_k^+, \mathcal{A}_k^-$ are two constants. Using the Lagrange constant variation method, we have

$$\frac{d}{dt}\mathcal{C}_k^+(t) = \frac{e^{-\mathcal{B}_k^+ t}\rho_k}{\sqrt{k^4+4m}}, \quad \frac{d}{dt}\mathcal{C}_k^-(t) = \frac{-e^{-\mathcal{B}_k^- t}\rho_k}{\sqrt{k^4+4m}}\cdot$$

Thus, we get that

$$\mathcal{C}_k^+(t) = \int_0^t \frac{e^{-\mathcal{B}_k^+ s}\rho_k}{\sqrt{k^4+4m}} ds + \mathcal{C}_k^+(0) = \frac{1 - e^{-\mathcal{B}_k^+ t}}{\mathcal{B}_k^+ \sqrt{k^4+4m}}\rho_k + \mathcal{C}_k^+(0),$$

and

$$\mathcal{C}_k^-(t) = \int_0^t \frac{-e^{-\mathcal{B}_k^- s}\rho_k}{\sqrt{k^4+4m}} ds + \mathcal{C}_k^-(0) = \frac{-1 + e^{-\mathcal{B}_k^- t}}{\mathcal{B}_k^- \sqrt{k^4+4m}}\rho_k + \mathcal{C}_k^-(0)\cdot \tag{2.4}$$

This implies that

$$\begin{aligned} z_k(t) =& \left(\mathcal{A}_k^+ + \mathcal{C}_k^+(0) + \frac{\rho_k}{\mathcal{B}_k^+\sqrt{k^4+4m}} \right) e^{\mathcal{B}_k^+ t} - \frac{\rho_k}{\mathcal{B}_k^+\sqrt{k^4+4m}} \\ &+ \left(\mathcal{A}_k^- + \mathcal{C}_k^-(0) - \frac{\rho_k}{\mathcal{B}_k^-\sqrt{k^4+4m}} \right) e^{\mathcal{B}_k^- t} + \frac{\rho_k}{\mathcal{B}_k^-\sqrt{k^4+4m}}\cdot \end{aligned} \tag{2.5}$$

Since $z_k(0) = z_k'(0) = 0$, we know that

$$z_k(0) = \mathcal{A}_k^+ + \mathcal{C}_k^+(0) + \mathcal{A}_k^- + \mathcal{C}_k^-(0) = 0. \tag{2.6}$$

and

$$z_k'(0) = \Big(\mathcal{A}_k^+ + \mathcal{C}_k^+(0) \Big) \mathcal{B}_k^+ + \frac{\rho_k}{\sqrt{k^4+4m}} + \Big(\mathcal{A}_k^- + \mathcal{C}_k^-(0) \Big) B_k^- - \frac{\rho_k}{\sqrt{k^4+4m}} = 0. \tag{2.7}$$

Since (2.6) and (2.7), we obtain that

$$\mathcal{A}_k^+ + \mathcal{C}_k^+(0) = \mathcal{A}_k^- + \mathcal{C}_k^-(0) = 0.$$

This follows from (2.5) that

$$z_k(t) = \frac{\rho_k}{\mathcal{B}_k^+\sqrt{k^4+4m}} \Big(e^{\mathcal{B}_k^+ t} - 1 \Big) - \frac{\rho_k}{\mathcal{B}_k^-\sqrt{k^4+4m}} \Big(e^{\mathcal{B}_k^- t} - 1 \Big).$$

Since the fact that $u_k(t) = z_k'(t)$, we know that

$$\begin{aligned} u_k(t) =& \frac{\rho_k e^{\mathcal{B}_k^+ t}}{\sqrt{k^4+4m}} - \frac{\rho_k e^{\mathcal{B}_k^- t}}{\sqrt{k^4+4m}} \\ =& \frac{\rho_k}{\sqrt{k^4+4m}} \Bigg(\exp\Big\{ \frac{\big(-k^2+\sqrt{k^4+4m}\big)}{2} t \Big\} \\ & - \exp\Big\{ \frac{\big(-k^2-\sqrt{k^4+4m}\big)}{2} t \Big\} \Bigg). \end{aligned}$$

Under the condition $u_k(M) = g_k$, we derive that

$$\begin{aligned} g_k =& \frac{\rho_k}{\sqrt{k^4+4m}} \Bigg(\exp\Big\{ \frac{\big(-k^2+\sqrt{k^4+4m}\big)}{2} t \Big\} \\ & - \exp\Big\{ \frac{\big(-k^2-\sqrt{k^4+4m}\big)}{2} t \Big\} \Bigg). \end{aligned}$$

Thus, we derive that

$$\rho(x) = \sum_{k=1}^{\infty} \frac{\sqrt{k^4+4m}}{\exp\Big\{\big(\frac{-k^2+\sqrt{k^4+4m}}{2}\big)M\Big\} - \exp\Big\{\big(\frac{-k^2-\sqrt{k^4+4m}}{2}\big)M\Big\}} \langle g, \xi_k \rangle \xi_k(x). \quad (2.8)$$

22 Ill-posedness

In the following, we provide an example of which shows that the function (2.8) does not depend continuously on the given data g. For $n \in \mathbb{N}^*$, we set

$$\widetilde{g}^n(x) = \frac{1}{\sqrt{n}} \xi_n(x). \quad (2.9)$$

Combining (2.8) and (2.9), we get that

$$\rho^n(x) = \frac{\sqrt{n^4+4m}}{\exp\Big\{\big(\frac{-n^2+\sqrt{n^4+4m}}{2}\big)M\Big\} - \exp\Big\{\big(\frac{-n^2-\sqrt{n^4+4m}}{2}\big)M\Big\}} \frac{1}{\sqrt{n}} \xi_n(x).$$

It is obvious to see that

$$\big\|\widetilde{g}^n\big\|_{L^2(\Omega)} = \frac{1}{\sqrt{n}} \to 0, \quad n \to +\infty, \quad (2.10)$$

and

$$\big\|\rho^n\big\|_{L^2(\Omega)} = \frac{\sqrt{n^4+4m}}{\exp\Big\{\big(\frac{-n^2+\sqrt{n^4+4m}}{2}\big)M\Big\} - \exp\Big\{\big(\frac{-n^2-\sqrt{n^4+4m}}{2}\big)M\Big\}} \frac{1}{\sqrt{n}}.$$

It is easy to check that

$$\frac{-n^2+\sqrt{n^4+4m}}{2} = \frac{2m}{n^2+\sqrt{n^4+4m}} \le 2m,$$

This leads to

$$\exp\Big\{(\frac{-n^2+\sqrt{n^4+4m}}{2})M\Big\}-\exp\Big\{(\frac{-n^2-\sqrt{n^4+4m}}{2})M\Big\}\le\exp\big\{2mM\big\}.$$

Since two latter estimates, we find that

$$\|\rho^n\|_{L^2(\Omega)}\ge\frac{2\sqrt{n^4+4m}}{\exp\big\{2mM\big\}}\to\infty \text{ when } n\to\infty. \tag{2.11}$$

Combining (2.10) and (2.11), the inverse source problem (1.1) is ill-posed in the sense of Hadamard.

23 Conditional stability of source term f

It is easy to see that

$$\frac{\sqrt{k^4+4m}}{\exp\Big\{\frac{\big(-k^2+\sqrt{k^4+4m}\big)}{2}M\Big\}-\exp\Big\{\frac{\big(-k^2-\sqrt{k^4+4m}\big)}{2}M\Big\}}=\frac{\sqrt{k^4+4m}\exp\big\{\frac{k^2}{2}M\big\}}{\exp\big\{\frac{\sqrt{k^4+4m}}{2}M\big\}-1}$$
$$=\exp\Big\{\frac{\big(k^2-\sqrt{k^4+4m}\big)}{2}M\Big\}\frac{\sqrt{k^4+4m}\ \exp\big\{\frac{\sqrt{k^4+4m}}{2}M\big\}}{\exp\big\{\frac{\sqrt{k^4+4m}}{2}M\big\}-1}.$$

We have

$$\frac{k^2-\sqrt{k^4+4m}}{2}\le 0,$$

Therefore

$$\exp\Big\{\frac{\big(k^2-\sqrt{k^4+4m}\big)}{2}M\Big\}\le 1.$$

We derive that

$$\frac{\exp\big\{\frac{\sqrt{k^4+4m}}{2}M\big\}}{\exp\big\{\frac{\sqrt{k^4+4m}}{2}M\big\}-1}=\frac{1}{1-\exp\big\{-\frac{\sqrt{k^4+4m}}{2}M\big\}}\le\frac{1}{1-\exp\big\{-M\frac{\sqrt{1+4m}}{2}\big\}}.$$

From some above observations, we deduce that

$$\frac{\sqrt{k^4+4m}}{\exp\Big\{\frac{\big(-k^2+\sqrt{k^4+4m}\big)}{2}M\Big\}-\exp\Big\{\frac{\big(-k^2-\sqrt{k^4+4m}\big)}{2}M\Big\}}$$
$$\le\frac{\sqrt{k^4+4m}}{1-\exp\big\{-M\frac{\sqrt{1+4m}}{2}\big\}}\le\frac{\sqrt{1+4m}}{1-\exp\big\{-M\frac{\sqrt{1+4m}}{2}\big\}}\,k^2.$$

In this section, we introduce conditional stability by the following theorem.

Theorem 2.1 *Let $\|\rho\|_{\mathbb{H}^{2\tau}(\mathcal{D})} \le \mathcal{E}_1$, for $\mathcal{E}_1 > 0$ then*

$$\big\|\rho\big\|_{L^2(\mathcal{D})} \le \mathcal{C}(m,M)^{\frac{\tau}{\tau+1}} \mathcal{E}_1^{\frac{1}{\tau+1}} \|g\|_{L^2(\mathcal{D})}^{\frac{\tau}{\tau+1}}.$$

Next, we assume that $\big\|\rho\big\|_{\beta,exp} \le \mathcal{E}_2$, $\beta > 0$ then we have

$$\big\|\rho\big\|_{\beta,exp} \le \mathcal{C}(m,M)^{\frac{\beta}{\beta+1}} \mathcal{E}_2^{\frac{1}{\beta+1}} \|g\|_{L^2(\mathcal{D})}^{\frac{\beta}{\beta+1}}.$$

Proof. Combining (2.8) and Hölder inequality, we have:

$$\begin{aligned}
\big\|\rho\big\|_{L^2(\mathcal{D})}^2 &= \sum_{k=1}^{\infty} \frac{\sqrt{k^4+4m}}{\exp\Big\{\big(\frac{-k^2+\sqrt{k^4+4m}}{2}\big)M\Big\} - \exp\Big\{\big(\frac{-k^2-\sqrt{k^4+4m}}{2}\big)M\Big\}} \big|\langle g, \xi_k\rangle\big|^2 \\
&\le \sum_{k=1}^{\infty} \frac{\sqrt{k^4+4m}}{\exp\Big\{\big(\frac{-k^2+\sqrt{k^4+4m}}{2}\big)M\Big\} - \exp\Big\{\big(\frac{-k^2-\sqrt{k^4+4m}}{2}\big)M\Big\}} \big|\langle g, \xi_k\rangle\big|^{\frac{2}{\tau+1}} \big|\langle g, \xi_k\rangle\big|^{\frac{2\tau}{\tau+1}} \\
&\le \Bigg(\sum_{k=1}^{\infty} \frac{\big(\sqrt{k^4+4m}\big)^{2\tau} \big|\langle \rho, \xi_k\rangle\big|^2}{\Big|\Big(\exp\big\{\big(\frac{-k^2+\sqrt{k^4+4m}}{2}\big)M\big\} - \exp\big\{\big(\frac{-k^2-\sqrt{k^4+4m}}{2}\big)M\big\}\Big)\Big|^{2\tau}} \Bigg)^{\frac{1}{\tau+1}} \|g\|_{L^2(\mathcal{D})}^{\frac{2\tau}{\tau+1}}.
\end{aligned}$$

This inequality leads to

$$\begin{aligned}
\big\|\rho\big\|_{L^2(\mathcal{D})}^2 &= \Bigg(\sum_{k=1}^{\infty} \frac{\big(\sqrt{k^4+4m}\big)^{2\tau} \ \big|\langle \rho, \xi_k\rangle\big|^2}{\Big|\Big(\exp\Big\{\big(\frac{-k^2+\sqrt{k^4+4m}}{2}\big)M\Big\} - \exp\big\{\big(\frac{-k^2-\sqrt{k^4+4m}}{2}\big)M\big\}\Big)\Big|^{2\tau}} \Bigg)^{\frac{1}{\tau+1}} \|g\|_{L^2(\mathcal{D})}^{\frac{2\tau}{\tau+1}} \\
&\le \Bigg(\frac{\sqrt{1+4m}}{1-\exp\big\{-M\frac{\sqrt{1+4m}}{2}\big\}} \Bigg)^{\frac{2\tau}{\tau+1}} \|g\|_{L^2(\mathcal{D})}^{\frac{2\tau}{\tau+1}} \Bigg(\sum_{k=1}^{\infty} k^{4\tau} \big|\langle \rho, \xi_k\rangle\big|^2 \Bigg)^{\frac{1}{\tau+1}} \\
&\le \mathcal{C}(m,M)^{\frac{2\tau}{\tau+1}} \|\rho\|_{\mathbb{H}^{2\tau}(\mathcal{D})}^{\frac{2}{\tau+1}} \|g\|_{L^2(\mathcal{D})}^{\frac{2\tau}{\tau+1}}, \text{ where } \mathcal{C}(m,M) \\
&= \frac{\sqrt{1+4m}}{1-\exp\big\{-M\frac{\sqrt{1+4m}}{2}\big\}}.
\end{aligned}$$

3 Exponential Tikhonov Regularization method

From now on, we denote

$$\mathcal{F}_k(m,M) = \frac{\exp\Big\{\big(\dfrac{-k^2+\sqrt{k^4+4m}}{2}\big)M\Big\} - \exp\Big\{\big(\dfrac{-k^2-\sqrt{k^4+4m}}{2}\big)M\Big\}}{\sqrt{k^4+4m}}. \quad (3.1)$$

We know that retrieving the source $\rho(x)$ from formula (2.8) is ill-posed. Next, we use an exponential Tikhonov regularization method to solve 1.1, and present the corresponding convergence estimates under a-priori and a-posterior regularization parameter choice rules. We define a linear forward operator $\mathcal{K} : L^2(\mathcal{D}) \to L^2(\mathcal{D})$ as follows

$$(\mathcal{K}\rho)(x) := \sum_{k=1}^{\infty} g_k \xi_k(x). \quad (3.2)$$

Thus, Eq. (3.2) is rewritten as

$$\mathcal{K}\rho = g. \tag{3.3}$$

Obviously, $\mathcal{K} : L^2(\mathcal{D}) \to L^2(\mathcal{D})$ is a linear self-conjugate compact operator, and operator equation (3.3) is ill-posed, see [1]. In order to stably reconstruct the source $\rho(x)$ from the noisy data $g_\epsilon(x)$ of $g(x)$, with an exponential penalty, we minimize the following Tikhonov regularization functional :

$$J(f) = \left\|\mathcal{K}\rho - g_\epsilon\right\|^2_{L^2(\mathcal{D})} + \gamma(\epsilon)\|\rho\|^2_{\beta,\exp}, \tag{3.4}$$

whereby $\gamma(\epsilon) \in \mathbb{R}^+$ is a regularization parameter. Similarly, the exponential Tikhonov functional (3.4) has a unique minimizer in $L^2(\mathcal{D})$. Let $\rho_\epsilon^{\gamma,\beta}$ be the unique minimizer of exponential Tikhonov functional (3.4), then we get

$$\mathcal{K}^*\mathcal{K}\rho_\epsilon^{\gamma(\epsilon),\beta} + \gamma(\epsilon)\exp(k^{2\beta})\rho_\epsilon^{\gamma(\epsilon),\beta} = \mathcal{K}^* g_\epsilon. \tag{3.5}$$

From (3.5), we obtain

$$\rho_\epsilon^{\gamma(\epsilon),\beta}(x) = \sum_{k=1}^{\infty} \frac{\mathcal{F}_k(m,M)}{\left|\mathcal{F}_k(m,M)\right|^2 + \gamma(\epsilon)\exp(k^{2\beta})}\langle g_\epsilon, \xi_k\rangle \xi_k(x),$$

31 A priori parameter choice rule

Theorem 3.1 *Let $\rho_\epsilon^{\gamma(\epsilon),\beta}(x)$ be the regularization solution with respect to the noise data $g_\epsilon(x)$ and the noise assumption $\left\|g_\epsilon - g\right\|_{L^2(\mathcal{D})} \le \epsilon$ be held.*

- *Under case of $\beta \le 0$, assume that $\|\rho\|_{\mathbb{H}^{2\tau}(\mathcal{D})} \le \mathcal{E}_1$, one has the convergence estimate*
  - *If $\tau > 2$, the choice $\gamma(\epsilon) = \epsilon^{\frac{2}{3}}$, we have*

$$\left\|\rho_\epsilon^{\gamma(\epsilon),\beta} - \rho\right\|_{L^2(\mathcal{D})} \le (\mathcal{C} + \mathcal{C}_3\mathcal{E}_1)\,\epsilon^{\frac{2}{3}}.$$

  - *If $\tau \in (0,2]$, the choice $\gamma(\epsilon) = \epsilon^{\frac{2}{\tau+1}}$, then we get*

$$\left\|\rho_\epsilon^{\gamma(\epsilon),\beta} - \rho\right\|_{L^2(\mathcal{D})} \le (\mathcal{C} + \mathcal{C}_4\mathcal{E}_1)\,\epsilon^{\frac{\tau}{\tau+1}}.$$

  where $\mathcal{C}_3$, and $\mathcal{C}_4$ are defined in the (3.8) *and* (3.9)
- *Under case of $\beta > 0$:suppose that $\|\rho\|_{\beta,exp} \le \mathcal{E}_2$, we choice $\gamma(\epsilon) = \epsilon^{\frac{16+8\beta}{3(4+2\beta)}}$ yields the convergence estimate*

$$\left\|\rho_\epsilon^{\gamma(\epsilon),\beta} - \rho\right\|_{L^2(\mathcal{D})} \le \mathcal{C}_5\epsilon^{\frac{\beta}{\beta+2}} + \mathcal{C}_8\mathcal{E}_2\epsilon^{\frac{1}{3}}.$$

  where $\mathcal{C}_5$ is defined in above the formula (3.10).

Proof. From (3.1), it follows that

$$\frac{\mathcal{C}_1}{k^2} \le \mathcal{F}_k(m,M) \le \frac{\mathcal{C}_2}{k^2},$$

inwhich $\mathcal{C}_1 = \frac{1-\exp\{-M\frac{\sqrt{1+4m}}{2}\}}{\sqrt{1+4m}}, \mathcal{C}_2 = \exp\{2mM\}$. By the triangular inequality, one has

$$\left\|\rho_\epsilon^{\gamma(\epsilon),\beta} - \rho\right\|_{L^2(\mathcal{D})} \le \left\|\rho_\epsilon^{\gamma(\epsilon),\beta} - \rho^{\gamma(\epsilon),\beta}\right\|_{L^2(\mathcal{D})} + \left\|\rho^{\gamma(\epsilon),\beta} - \rho\right\|_{L^2(\mathcal{D})}. \tag{3.6}$$

– For $\beta \leq 0$, the first term on the right-hand side of inequality (3.6) has

$$\begin{aligned}\left\|\rho_\epsilon^{\gamma(\epsilon),\beta} - \rho^{\gamma(\epsilon),\beta}\right\|^2_{L^2((\mathcal{D}))} &= \Big\| \sum_{k=1}^{\infty} \frac{\mathcal{F}_k(m,M)}{\big|\mathcal{F}_k(m,M)\big|^2 + \gamma(\epsilon)\exp(k^{2\beta})} \big\langle g_\epsilon - g, \xi_k \big\rangle \Big\|^2_{L^2((\mathcal{D}))} \\ &\leq \sum_{k=1}^{\infty} \left| \frac{\frac{\mathcal{C}_2}{k^2}}{\frac{\mathcal{C}_1^2}{k^4} + \gamma(\epsilon)\exp(0)} \right|^2 \big|\big\langle g_\epsilon - g, \xi_k \big\rangle\big|^2 \\ &\leq \mathcal{C}_2^2 \sup_{k\in\mathbb{N}} \left| \frac{k^2}{\mathcal{C}_1^2 + \gamma(\epsilon)k^4} \right|^2 \sum_{k=1}^{\infty} \big|\big\langle g_\epsilon - g, \xi_k \big\rangle\big|^2 \\ &\leq \left(\frac{\mathcal{C}_2}{2\mathcal{C}_1}\right)^2 \frac{\epsilon^2}{\gamma(\epsilon)}. \end{aligned} \tag{3.7}$$

It follows that

$$\left\|\rho_\epsilon^{\gamma(\epsilon),\beta} - \rho^{\gamma(\epsilon),\beta}\right\|_{L^2(\mathcal{D})} \leq \mathcal{C} \frac{\epsilon}{\sqrt{\gamma(\epsilon)}}.$$

where $\mathcal{C} = \left(\dfrac{\mathcal{C}_2}{2\mathcal{C}_1}\right)$. The second term on the right-hand side of inequality (3.6) has

$$\begin{aligned} &\left\|\rho^{\gamma(\epsilon),\beta} - \rho\right\|^2_{L^2(\mathcal{D})} \\ &= \left\| \sum_{k=1}^{\infty} \left(\frac{\mathcal{F}_k(m,M)}{\big|\mathcal{F}_k(m,M)\big|^2 + \gamma(\epsilon)\exp(k^{2\beta})} - \frac{1}{\mathcal{F}_k(m,M)} \right) \big|\big\langle g, \xi_k \big\rangle\big| \right\|^2_{L^2(\mathcal{D})} \\ &= \left\| \sum_{k=1}^{\infty} \frac{-\gamma(\epsilon)\exp(k^{2\beta})}{\big|\mathcal{F}_k(m,M)\big|^2 + \gamma(\epsilon)\exp(k^{2\beta})} \big\langle f, \xi_k \big\rangle \right\|^2_{L^2(\mathcal{D})} \\ &\leq \sum_{k=1}^{\infty} \left(\frac{\gamma(\epsilon)\exp(k^{2\beta})}{\big(\frac{\mathcal{C}_1}{k^2}\big)^2 + \gamma(\epsilon)\exp(k^{2\beta})} \right)^2 \big|\big\langle f, \xi_k \big\rangle\big|^2 \\ &= \sum_{k=1}^{\infty} \left(\frac{\gamma(\epsilon)k^4\exp(k^{2\beta})}{\mathcal{C}_1^2 + \gamma(\epsilon)k^4\exp(k^{2\beta})} \right)^2 \big|\big\langle f, \xi_k \big\rangle\big|^2 \\ &\leq \sum_{k=1}^{\infty} \left(\frac{\gamma(\epsilon)k^{4-2\tau}\exp(1)}{\mathcal{C}_1^2 + \gamma(\epsilon)k^4} \right)^2 k^{4\tau} \big|\big\langle f, \xi_k \big\rangle\big|^2 \\ &\leq \left(\sup_{k\in\mathbb{N}} \frac{\gamma(\epsilon)k^{4-2\tau}\exp(1)}{\mathcal{C}_1^2 + \gamma(\epsilon)k^4} \right)^2 \sum_{k=1}^{\infty} k^{4\tau} \big|\big\langle f, \xi_k \big\rangle\big|^2. \end{aligned}$$

We present two cases as follows:

– Under case $\tau \geq 2$, we know that

$$\sup_{k\in\mathbb{N}} \frac{\gamma(\epsilon)k^{4-2\tau}\exp(1)}{\mathcal{C}_1^2 + \gamma(\epsilon)k^4} \leq \mathcal{C}_3\gamma(\epsilon),\ \mathcal{C}_3 = \frac{\exp(1)}{\mathcal{C}_1^2}. \tag{3.8}$$

– Under case $\tau < 2$, we get

$$\sup_{k\in\mathbb{N}} \frac{\gamma(\epsilon)k^{4-2\tau}\exp(1)}{\mathcal{C}_1^2 + \gamma(\epsilon)k^4} \leq \sup_{z>0} \frac{\exp(1)z^{4-2\tau}}{\mathcal{C}_1^2 + \gamma(\epsilon)z^4}\gamma(\epsilon) \leq \mathcal{C}_4[\gamma(\epsilon)]^{\frac{\tau}{2}}. \tag{3.9}$$

where by $\mathcal{C}_4 = \mathcal{C}_1^{-\tau} \exp(1)\Big(\dfrac{2-\tau}{2}\Big)\left(\dfrac{2-\tau}{\tau}\right)^{-\frac{\tau}{2}}$.

In case $0 < \gamma(\epsilon) < 1$ one has

$$\left\|\rho^{\gamma(\epsilon),\beta} - \rho\right\|_{L^2(\mathcal{D})} \leq \begin{cases} \mathcal{C}_3\mathcal{E}_1\gamma(\epsilon), \tau \geq 2, \\ \mathcal{C}_4\mathcal{E}_2[\gamma(\epsilon)]^{\frac{\tau}{2}}, 0 < \tau < 2, \end{cases}$$

where $\mathcal{C}_3$, and $\mathcal{C}_4$ are defined in the (3.8) and (3.9). Placing the above together yields that

- If $\tau > 2$, we choice $\gamma(\epsilon) = \epsilon^{\frac{2}{3}}$, we receive

$$\left\|\rho_\epsilon^{\gamma(\epsilon),\beta} - \rho\right\|_{L^2(\mathcal{D})} \leq \mathcal{C}\frac{\epsilon}{\sqrt{\gamma(\epsilon)}} + \mathcal{C}_3\mathcal{E}_1\gamma(\epsilon).$$

  This leads to

$$\left\|\rho_\epsilon^{\gamma(\epsilon),\beta} - \rho\right\|_{L^2(\mathcal{D})} \leq (\mathcal{C} + \mathcal{C}_3\mathcal{E}_1)\,\epsilon^{\frac{2}{3}}.$$

- If $\tau \in (0.2]$, we choice $\gamma(\epsilon) = \epsilon^{\frac{2}{\tau+1}}$, we receive

$$\left\|\rho_\epsilon^{\gamma(\epsilon),\beta} - \rho\right\|_{L^2(\mathcal{D})} \leq \mathcal{C}\frac{\epsilon}{\sqrt{\gamma(\epsilon)}} + \mathcal{C}_4\mathcal{E}_2[\gamma(\epsilon)]^{\frac{\tau}{2}}.$$

  This leads to

$$\left\|\rho_\epsilon^{\gamma(\epsilon),\beta} - \rho\right\|_{L^2(\mathcal{D})} \leq (\mathcal{C} + \mathcal{C}_4\mathcal{E}_2)\,\epsilon^{\frac{\tau}{\tau+1}}.$$

- For $\beta > 0$, the first term on the right-hand side of inequality (3.6) has

$$\begin{aligned}
\left\|\rho_\epsilon^{\gamma(\epsilon),\beta} - \rho^{\gamma(\epsilon),\beta}\right\|_{L^2(\mathcal{D})}^2 &= \left\|\sum_{k=1}^{\infty} \frac{\mathcal{F}_k(m,M)}{\left|\mathcal{F}_k(m,M)\right|^2 + \gamma(\epsilon)\exp\left(k^{2\beta}\right)}\left\langle g_\epsilon - g, \xi_k\right\rangle\right\|_{L^2(\mathcal{D})}^2 \\
&\leq \sum_{k=1}^{\infty}\left|\frac{\frac{\mathcal{C}_2}{k^2}}{\frac{\mathcal{C}_1^2}{k^4} + \gamma(\epsilon)\exp(k^{2\beta})}\right|^2 \left|\left\langle g_\epsilon - g, \xi_k\right\rangle\right|^2 \\
&\leq \mathcal{C}_2^2 \sup_{k\in\mathbb{N}}\left|\frac{k^3}{\mathcal{C}_1^2 + \gamma(\epsilon)k^{4+2\beta}}\right|^2 \sum_{k=1}^{\infty}\left|\left\langle g_\epsilon - g, \xi_k\right\rangle\right|^2 \\
&\leq \left|\frac{3^{\frac{3}{4+2\beta}}(1+2\beta)^{\frac{1+2\beta}{4+2\beta}}}{\mathcal{C}_1^{\frac{1+2\beta}{2+\beta}}(4+2\beta)}[\gamma(\epsilon)]^{-\frac{3}{4+2\beta}}\right|^2\epsilon^2.
\end{aligned}$$

Thus there exists $\mathcal{C}_5 = \dfrac{3^{\frac{3}{4+2\beta}}(1+2\beta)^{\frac{1+2\beta}{4+2\beta}}}{\mathcal{C}_1^{\frac{1+2\beta}{2+\beta}}(4+2\beta)}$ such that

$$\left\|\rho_\epsilon^{\gamma(\epsilon),\beta} - \rho^{\gamma(\epsilon),\beta}\right\|_{L^2(\mathcal{D})} \leq \mathcal{C}_5\frac{\epsilon}{\gamma(\epsilon)^{\frac{3}{4+2\beta}}}. \tag{3.10}$$

The second term on the right-hand side of inequality (3.6) has

$$
\begin{aligned}
\left\|\rho^{\gamma(\epsilon),\beta}-\rho\right\|^2_{L^2(\mathcal{D})} &= \left\|\sum_{k=1}^{\infty}\Big(\frac{\mathcal{F}_k(m,M)}{\left|\mathcal{F}_k(m,M)\right|+\gamma(\epsilon)\exp(k^{2\beta})}-\frac{1}{\mathcal{F}_k(m,M)}\Big)\langle g,\xi_k\rangle\right\|^2_{L^2(\mathcal{D})}\\
&= \left\|\sum_{k=1}^{\infty}\frac{-\gamma(\epsilon)\exp\left(k^{2\beta}\right)}{\left|\mathcal{F}_k(\alpha,L)\right|^2+\gamma(\epsilon)\exp(k^{2\beta})}\langle f,\xi_k\rangle\right\|^2_{L^2(\mathcal{D})}\\
&= \sum_{k=1}^{\infty}\frac{\gamma(\epsilon)^2\exp(k^{2\beta})}{\left(\left|\mathcal{F}_k(m,M)\right|^2+\gamma(\epsilon)\exp(k^{2\beta})\right)^2}\exp(k^{2\beta})\left|\langle f,\xi_k\rangle\right|^2\\
&\le \left(\sup_{k\in\mathbb{N}}\frac{\gamma(\epsilon)\exp\left(\frac{k^{2\beta}}{2}\right)}{\left|\mathcal{F}_k(\alpha,L)\right|^2+\gamma(\epsilon)\exp(k^{2\beta})}\right)^2\sum_{k=1}^{\infty}\exp(k^{2\beta})\left|\langle f,\xi_k\rangle\right|^2.
\end{aligned}
$$

Obviously, there exists an integer $k_0>0$ such that $k^4\le\exp(k^{2\beta})$ for $k\ge k_0$.

- The case of $k\ge k_0$ we can derive that

$$
\begin{aligned}
\frac{\gamma(\epsilon)\exp\left(\frac{k^{2\beta}}{2}\right)}{\left|\mathcal{F}_k(m,M)\right|^2+\gamma(\epsilon)\exp(k^{2\beta})} &\le \frac{\gamma(\epsilon)\exp\left(\frac{k^{2\beta}}{2}\right)}{\frac{\mathcal{C}_1^2}{k^4}+\gamma(\epsilon)\exp(k^{2\beta})}\le\frac{\gamma(\epsilon)\exp\left(\frac{k^{2\beta}}{2}+k^{2\beta}\right)}{\mathcal{C}_1^2+\gamma(\epsilon)\exp\left(2k^{2\beta}\right)}\\
&\le \sup_{z>0}\frac{\gamma(\epsilon)z^3}{\mathcal{C}_1^2+\gamma(\epsilon)z^4}\le\mathcal{C}_6[\gamma(\epsilon)]^{\frac14}
\end{aligned}
$$

  where $\mathcal{C}_6=\frac{3^{\frac34}}{4\mathcal{C}_1}$.
- The case of $k\le k_0$ we have

$$
\frac{\gamma(\epsilon)\exp\left(\frac{k^{2\beta}}{2}\right)}{\left|\mathcal{F}_k(m,M)\right|^2+\gamma(\epsilon)\exp(k^{2\beta})}\le\frac{\gamma(\epsilon)\exp\left(\frac{k^{2\beta}}{2}\right)}{\frac{\mathcal{C}_1^2}{k^4}+\gamma(\epsilon)\exp(k^{2\beta})}\le\frac{\gamma(\epsilon)\exp\left(\frac{k_0^{2\beta}}{2}\right)}{\frac{\mathcal{C}_1^2}{k_0^4}}\le\mathcal{C}_7\gamma(\epsilon).
$$

  whereby $\mathcal{C}_7=\mathcal{C}_1^{-2}\,k_0^4\exp\left(\frac{k_0^{2\beta}}{2}\right)$.

Combining the above two cases yields that

$$
\left\|\rho^{\gamma(\epsilon),\beta}-\rho\right\|_{L^2(\mathcal{D})}\le\mathcal{C}_8\mathcal{E}_2\gamma(\epsilon).
$$

For $0<\gamma(\epsilon)<1$, where $\mathcal{C}_8=\max\left\{\mathcal{C}_6,\mathcal{C}_7\right\}$. Therefore, we have

$$
\begin{aligned}
\left\|\rho_\epsilon^{\gamma(\epsilon),\beta}-\rho\right\|_{L^2(\mathcal{D})} &\le \left\|\rho_\epsilon^{\gamma(\epsilon),\beta}-\rho^{\gamma(\epsilon),\beta}\right\|_{L^2(\mathcal{D})}+\left\|\rho^{\gamma(\epsilon),\beta}-\rho\right\|_{L^2(\mathcal{D})}\\
&= \mathcal{C}_5\frac{\epsilon}{\gamma(\epsilon)^{\frac{3}{4+2\beta}}}+\mathcal{C}_8\mathcal{E}_2\gamma(\epsilon)^{\frac14}.
\end{aligned}
$$

By choosing $\gamma(\epsilon)=\epsilon^{\frac{16+8\beta}{3(4+2\beta)}}$ yields

$$
\left\|\rho_\epsilon^{\gamma(\epsilon),\beta}-\rho\right\|_{L^2(\mathcal{D})}\le\mathcal{C}_5\epsilon^{\frac{\beta}{\beta+2}}+\mathcal{C}_8\mathcal{E}_2\epsilon^{\frac13}.
$$

The proof is completed.

32 An a posteriori parameter choice rule

Theorem (3.1) offers an a priori approach for choosing regularization parameters, but it necessitates knowing the precise regularity of the exact solution $\rho(x)$ before choosing. However, the regularity of $\rho(x)$ is not well understood in many real-world circumstances. Therefore, it is necessary to research posterior techniques for selecting regularization parameters. Here, we choose the regularization parameters based on the Morozov discrepancy principle [29,28], and we give estimates of the convergence of the regularization solutions. According to the Morozov principle, the regularization parameter $\gamma(\epsilon)$ is selected as the solution to the discrepancy equation.

$$\left\|\mathcal{K}\rho_\epsilon^{\gamma(\epsilon),\beta} - g_\epsilon\right\|_{L^2(\mathcal{D})} = \upsilon\epsilon, \tag{3.11}$$

where $\upsilon > 1$ is a given constant. The solvability of the discrepancy equation (3.11) is guaranteed by the following lemma for $0 < \upsilon\epsilon < \|g_\epsilon\|_{L^2(\mathcal{D})}$.

Lemma 3.1 *Let $g_\epsilon \in L^2(\mathcal{D})$ and $\Xi(\gamma(\epsilon)) = \left\|\mathcal{K}f_\epsilon^{\gamma(\epsilon),\beta} - g_\epsilon\right\|^2_{L^2(\mathcal{D})}$. Then the following results hold:*

- *$\Xi(\gamma(\epsilon))$ is a continuous function;*
- *$\lim_{\gamma(\epsilon)\to 0} \Xi(\gamma(\epsilon)) = 0, \lim_{\gamma(\epsilon)\to\infty} \Xi(\gamma(\epsilon)) = \|g_\epsilon\|^2_{L^2(\mathcal{D})}$;*
- *$\Xi(\gamma(\epsilon))$ is a strictly increasing function for $\gamma(\epsilon) \in (0,\infty)$.*

Proof. We have

$$\Xi(\gamma(\epsilon)) = \sum_{k=1}^{\infty} \left| \frac{\gamma(\epsilon)\exp(k^{2\beta})}{\left|\mathcal{F}_k(m,M)\right|^2 + \gamma(\epsilon)\exp(k^{2\beta})} \right|^2 \left|\left\langle g_\epsilon, \xi_k\right\rangle\right|^2 \le \sum_{k=1}^{\infty} \left|\left\langle g_\epsilon, \xi_k\right\rangle\right|^2 < +\infty, \tag{3.12}$$

which implies that $\Xi(\gamma(\epsilon))$ is continuous on $[0,+\infty)$.

Theorem 3.2 *Suppose that the observed data $g_\epsilon(x)$ satisfies* (1.3)*, and $0 < \upsilon\epsilon < \|g_\epsilon\|$ for $\upsilon > 1$. Let $\rho_\epsilon^{\gamma(\epsilon),\beta}$ be the exponential regularization solution in which the regularization parameter $\gamma(\epsilon)$ is selected by the Morozov's discrepancy principle* (3.11).

1 *For $\beta \le 0$, there exists $\mathcal{E}_1 > 0$ such that $\|\rho\|_{\mathbb{H}^{2\tau}(\mathcal{D})} \le \mathcal{E}_1$, then we get*
   - *In case $\tau \ge 1$, we have*

$$\left\|\rho_\epsilon^{\gamma(\epsilon),\beta} - \rho\right\|_{L^2(\mathcal{D})} \le \left(\mathcal{C}_1 \Big(\frac{\mathcal{C}_3\mathcal{E}_1}{\tau-1}\Big)^{\frac{1}{2}} + (\mathcal{C}(\alpha,L))^{\frac{2\tau}{\tau+1}} \mathcal{E}_1^{\frac{1}{\tau+1}} (1+\upsilon)^{\frac{\tau}{\tau+1}} \right)^{\frac{1}{2}} \epsilon^{\frac{1}{2}}. \tag{3.13}$$

   - *In case $\tau \in (0,1)$, we have*

$$\left\|\rho_\epsilon^{\gamma(\epsilon),\beta} - \rho\right\|_{L^2(\mathcal{D})} \le \left(\mathcal{C}_1 \left(\frac{\mathcal{C}_4\mathcal{E}_1}{\tau-1}\right)^{\frac{1}{\tau+1}} + (\mathcal{C}(\alpha,L))^{\frac{2\tau}{\tau+1}} \mathcal{E}_1^{\frac{1}{\tau+1}} (1+\upsilon)^{\frac{\tau}{\tau+1}} \right) \epsilon^{\frac{\tau}{\tau+1}}. \tag{3.14}$$

2 *For $\beta > 0$, there exists $\mathcal{E}_2 > 0$ such that $\|\rho\|_{\beta,exp} \le \mathcal{E}_2$, then we get*

$$\left\|\rho_\epsilon^{\gamma(\epsilon),\beta} - \rho\right\|_{L^2(\mathcal{D})} \le \big(\mathcal{C}(\alpha,L)\big)^{\frac{2\tau}{\tau+2}} (2\mathcal{E}_2)^{\frac{2}{\gamma+2}} (1+\upsilon)^{\frac{\tau}{\tau+2}} \epsilon^{\frac{\tau}{\tau+2}}. \tag{3.15}$$

Proof. – For $\beta \le 0$. The discrepancy principle (3.11) for choosing the regularization parameter β yields that

$$\begin{aligned} \upsilon\epsilon &= \left\|\mathcal{K}\rho_\epsilon^{\gamma(\epsilon),\beta} - g_\epsilon\right\|_{L^2(\mathcal{D})} \\ &= \left\|\sum_{k=1}^{\infty}\left(\frac{\left|\mathcal{F}_k(m,M)\right|^2}{\left|\mathcal{F}_k(m,M)\right|^2+\gamma(\epsilon)\exp(k^{2\beta})}-1\right)\left\langle g_\epsilon,\xi_k\right\rangle\right\|_{L^2(\mathcal{D})} \\ &\le \epsilon + \left\|\sum_{k=1}^{\infty}\frac{\gamma(\epsilon)\exp(k^{2\beta})}{\left|\mathcal{F}_k(\alpha,L)\right|^2+\gamma(\epsilon)\exp(k^{2\beta})}\left\langle g_\epsilon,\xi_k\right\rangle\right\|_{L^2(\mathcal{D})}. \end{aligned} \tag{3.16}$$

It follows that

$$\begin{aligned} \left[(\upsilon-1)\epsilon\right]^2 &\le \left\|\sum_{k=1}^{\infty}\frac{\mu\exp(k^{2\beta})}{\left|\mathcal{F}_k(\alpha,L)\right|^2+\gamma(\epsilon)\exp(k^{2\beta})}\left|\left\langle g,\xi_k\right\rangle\right|\right\|^2_{L^2(\mathcal{D})} \\ &= \sum_{k=1}^{\infty}\left(\frac{\gamma(\epsilon)\exp(k^{2\beta})k^{-2\tau}\mathcal{F}_k(m,M)}{|\mathcal{F}_k(\alpha,L)|^2+\gamma(\epsilon)\exp(k^{2\beta})}\right)^2 k^{4\tau}\left|\left\langle f,\xi_k\right\rangle\right|^2 \\ &\le \left(\sup_{k\in\mathbb{N}}\frac{\gamma(\epsilon)\exp(1)k^{4-2\tau}}{\mathcal{C}_1^2+\gamma(\epsilon)k^4}\right)^2\sum_{k=1}^{\infty}k^{4\tau}\left|\left\langle f,\xi_k\right\rangle\right|^2. \end{aligned} \tag{3.17}$$

It is easy to see that

$$\sup_{k\in\mathbb{N}}\frac{\gamma(\epsilon)\exp(1)k^{4-2\tau}}{\mathcal{C}_1^2+\gamma(\epsilon)k^4} \le \mathcal{C}_3\gamma(\epsilon), \quad \text{for } \tau \ge 1\ ,$$

and

$$\sup_{k\in\mathbb{N}}\frac{\gamma(\epsilon)\exp(1)k^{4-2\tau}}{\mathcal{C}_1^2+\gamma(\epsilon)k^4} \le \mathcal{C}_4\gamma(\epsilon)^{\frac{\tau}{2}}, \text{ for } 0<\tau<1. \tag{3.18}$$

Combining (3.16) to (3.18), it yields

- If $\tau \ge 1$, by choosing $[\gamma(\epsilon)]^{-\frac{1}{2}} \le \left(\frac{\mathcal{C}_3\mathcal{E}_1}{\tau-1}\right)^{\frac{1}{2}}\epsilon^{-\frac{1}{2}}$, then we have

$$\left\|\rho_\epsilon^{\gamma(\epsilon),\beta}-\rho^{\gamma(\epsilon),\beta}\right\|_{L^2(\mathcal{D})} \le \mathcal{C}_1\left(\frac{\mathcal{C}_3\mathcal{E}_1}{\tau-1}\right)^{\frac{1}{2}}\epsilon^{\frac{1}{2}}.$$

- If $\tau \in (0,1)$, by choosing $[\gamma(\epsilon)]^{-\frac{1}{2}} \le \left(\frac{\mathcal{C}_4\mathcal{E}_1}{\tau-1}\right)^{\frac{1}{p+1}}\epsilon^{-\frac{1}{\tau+1}}$, then we get

$$\left\|\rho_\epsilon^{\gamma(\epsilon),\beta}-\rho^{\gamma(\epsilon),\beta}\right\|_{L^2(\mathcal{D})} \le \mathcal{C}_1\left(\frac{\mathcal{C}_4\mathcal{E}_1}{\tau-1}\right)^{\frac{1}{\tau+1}}\epsilon^{\frac{\tau}{\tau+1}}.$$

On the other hand, we have

$$\begin{aligned} \left\|\rho^{\gamma(\epsilon),\beta}-\rho\right\|^2_{\mathbb{H}^\tau(\mathcal{D})} &= \left\|\sum_{k=1}^{\infty}\left(\frac{\mathcal{F}_k(m,M)}{\left|\mathcal{F}_k(m,M)\right|^2+\gamma(\epsilon)\exp(k^{2\beta})}-\frac{1}{\mathcal{F}_k(m,M)}\right)\left|\left\langle g,\xi_k\right\rangle\right|\right\|^2_{\mathbb{H}^\tau(\mathcal{D})} \\ &= \sum_{k=1}^{\infty}\left(\frac{\mathcal{F}_k(m,M)}{\left|\mathcal{F}_k(m,M)\right|^2+\gamma(\epsilon)\exp(k^{2\beta})}-\frac{1}{\mathcal{F}_k(m,M)}\right)^2 k^{4\tau}\left|\left\langle g,\xi_k\right\rangle\right|^2 \\ &= \sum_{k=1}^{\infty}\left(\frac{\gamma(\epsilon)\exp(k^{2\beta})}{\left|\mathcal{F}_k(m,M)\right|^2+\gamma(\epsilon)\exp(k^{2\beta})}\right)^2 k^{4\tau}\left|\left\langle f,\xi_k\right\rangle\right|^2 \le \sum_{k=1}^{\infty}k^{4\tau}\left|\left\langle f,\xi_k\right\rangle\right|^2 \le \mathcal{E}_1^2. \end{aligned} \tag{3.19}$$

Next, one has

$$\begin{aligned}
\mathcal{K}\big(\rho^{\gamma(\epsilon),\beta}-\rho\big) &= \sum_{k=1}^{\infty}\mathcal{F}_k(m,M)\bigg(\frac{\mathcal{F}_k(m,M)}{\big|\mathcal{F}_k(m,M)\big|^2+\gamma(\epsilon)\exp(k^{2\beta})}-\frac{1}{\mathcal{F}_k(m,M)}\bigg)\langle g,\xi_k\rangle\xi_k(x)\\
&= \sum_{k=1}^{\infty}\frac{-\gamma(\epsilon)\exp(k^{2\beta})}{\big|\mathcal{F}_k(m,M)\big|^2+\gamma(\epsilon)\exp(k^{2\beta})}\langle g,\xi_k\rangle\xi_k(x)\\
&= \sum_{k=1}^{\infty}\frac{-\gamma(\epsilon)\exp(k^{2\beta})}{\big|\mathcal{F}_k(m,M)\big|^2+\gamma(\epsilon)\exp(k^{2\beta})}\langle g_\epsilon-g,\xi_k\rangle\xi_k(x)\\
&\qquad+\sum_{k=1}^{\infty}\frac{-\gamma(\epsilon)\exp(k^{2\beta})}{\big|\mathcal{F}_k(\alpha,L)\big|^2+\gamma(\epsilon)\exp(k^{2\beta})}\langle g_\epsilon,\xi_k\rangle\xi_k(x)\\
&\le \epsilon+\upsilon\epsilon=\epsilon(1+\upsilon).
\end{aligned}$$

The conditional stability estimate in Theorem 2.1, we obtain

$$\big\|\rho^{\gamma(\epsilon),\beta}-\rho\big\|_{L^2(\mathcal{D})}\le(\mathcal{C}(m,M))^{\frac{2\tau}{\tau+1}}\mathcal{E}_1^{\frac{1}{\tau+1}}(1+\upsilon)^{\frac{\tau}{\tau+1}}\epsilon^{\frac{\tau}{\tau+1}}. \tag{3.20}$$

By combining (3.19) with (3.20), the assertion of the theorem is proved for the case of $\beta\le 0$.

– For $\beta>0$. Since $\rho_\epsilon^{\gamma(\epsilon),\beta}$ is the exponential regularization solution, it gives

$$\begin{aligned}
\big\|\mathcal{K}\rho_\epsilon^{\gamma(\epsilon),\beta}-g_\epsilon\big\|^2_{L^2(\mathcal{D})}+\gamma(\epsilon)\big\|\rho_\epsilon^{\gamma(\epsilon),\beta}\big\|^2_{\beta,exp} &\le \big\|\mathcal{K}\rho-g_\epsilon\big\|^2_{L^2(\mathcal{D})}+\gamma(\epsilon)\|\rho\|^2_{\gamma(\epsilon),exp}\\
&= \big\|g-g_\epsilon\big\|^2_{L^2(\mathcal{D})}+\gamma(\epsilon)\|\rho\|^2_{\beta,exp}.
\end{aligned}$$

The discrepancy principle (3.11) for choosing the regularization parameter β directly yields that

$$\big\|\rho_\epsilon^{\gamma(\epsilon),\beta}\big\|^2_{\beta,exp}\le\|\rho\|^2_{\beta,exp}+\beta^{-1}\big(1-\upsilon^2\big)\epsilon^2\le\big\|\rho\big\|^2_{\beta,exp}\le\mathcal{E}_2^2.$$

Thereby, one has

$$\big\|\rho_\epsilon^{\gamma(\epsilon),\beta}-\rho\big\|_{\beta,exp}\le\big\|\rho_\epsilon^{\gamma(\epsilon),\beta}\big\|_{\beta,exp}+\big\|\rho\big\|_{\beta,exp}\le 2\mathcal{E}_2.$$

On the other hand, we have

$$\big\|\mathcal{K}\rho_\epsilon^{\gamma(\epsilon),\beta}-g\big\|_{L^2(\mathcal{D})}\le\big\|\mathcal{K}\rho_\epsilon^{\gamma(\epsilon),\beta}-g_\epsilon\big\|_{L^2(\mathcal{D})}+\big\|g_\epsilon-g\big\|_{L^2(\mathcal{D})}\le\upsilon\epsilon+\epsilon=(1+\upsilon)\epsilon.$$

From the latter conditional stability in Theorem 1, we obtain

$$\begin{aligned}
\big\|\rho_\epsilon^{\gamma(\epsilon),\beta}-\rho\big\|_{L^2(\mathcal{D})} &\le \big(\mathcal{C}(m,M)\big)^{\frac{2\tau}{\tau+2}}\big\|\rho_\epsilon^{\gamma(\epsilon),\beta}-\rho\big\|^{\frac{2}{\tau+2}}_{\beta,exp}\big\|\mathcal{K}\big(\rho_\epsilon^{\gamma(\epsilon),\beta}-\rho\big)\big\|^{\frac{\tau}{\tau+2}}\\
&\le \big(\mathcal{C}(m,M)\big)^{\frac{2\tau}{\tau+2}}\,(2\mathcal{E}_2)^{\frac{2}{\tau+2}}\,(1+\upsilon)^{\frac{\tau}{\tau+2}}\epsilon^{\frac{\tau}{\tau+2}}.
\end{aligned}$$

The proof is completed.

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