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Inverse problem for a system of Dirac-type equations with discontinuous coefficients

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Abstract. *The work considers the inverse problem of scattering on a semi-axis. The uniqueness of the solution to the inverse problem is proved and an algorithm is given for restoring the coefficient of the equation from given scattering functions.*

Keywords. Jost solution · scattering function · inverse scattering problem

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1 Introduction and problem statement

Consider the system of differential equations

$$\begin{cases} \frac{1}{\rho(x)}\left(\rho(x)y_2\right)' + \rho(x)y_1 + q(x)y_2 = \lambda y_1, \\ -y_1' + q(x)y_1 - \rho(x)y_2 = \lambda y_2, \quad 0 < x < \infty \end{cases} \tag{1.1}$$

with boundary condition

$$y_1(0) = 0, \tag{1.2}$$

where $\rho(x) = \alpha$ for $x > c$ and $\rho(x) = 1$ for $x < c,$ α is a positive number, c is a fixed point in $(0, \infty)$, $\rho(x)$ and $q(x)$ are real-valued functions satisfying the condition

$$\int_0^\infty \{|p(x)| + |q(x)|\}\, dx < \infty \tag{1.3}$$

In this work, we study the inverse problem for the boundary value problem (1.1)-(1.2). In case $\alpha = 1$ (i.e.when $\rho(x) \equiv 1$) the inverse scattering problem has been solved in [2] (see also [3]). Similar problem for the Sturm-Liouville operator on the whole axis has been solved in [5] where the references to other works are available which have considered various kinds of inverse problems.

Note that the inverse problem of scattering theory has been first completely solved in [7] for the Sturm-Liouville operator on the half-axis. That works played an important role in

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the further development of the theory of direct and inverse scattering problems for different operators (see, e. g., [1-3]-[5-6]).

1. Let's introduce some special solutions of the equation (1.1). For this, let's first note that the equation (1.1) can be reduced to the system of Dirac equations

$$By' + \Omega(x)y = \lambda y \tag{1.4}$$

with the following conditions at the point $x = c$:

$$y_1(c-0) = y_1(c+0), \qquad y_2(c-0) = \alpha y_2(c+0), \tag{1.5}$$

where $B = \begin{pmatrix} 01 \\ -10 \end{pmatrix}$, $\Omega(x) = \begin{pmatrix} \rho(x) \; q(x) \\ q(x) - \rho(x) \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

Denote by $e(x,\lambda)$ the Jost solution of the equation (1.1) (i.e. of the problem (1.4)-(1.5)) satisfying the condition

$$\lim_{x\to+\infty} e(x,\lambda)\cdot e^{-i\lambda x} = \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

It is not difficult to show that the function

$$e_0(x,\lambda) = \begin{cases} \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{i\lambda x}, & x > c \\ \frac{1+\alpha}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{i\lambda x} + \frac{1-\alpha}{2}\begin{pmatrix} 1 \\ i \end{pmatrix} e^{i\lambda(2c-x)}, & x < c \end{cases}$$

is a Jost solution of the equation (1.1) in case $p(x) = q(x) = 0$.

Theorem 1.1 *Let the conditions* (1.3) *hold. Then the equation* (1.1) *has a Jost solution for all* λ *with Im* $\lambda \geq 0$. *This solution is unique and can be represented as*

$$e(x,\lambda) = e_0(x,\lambda) + \int_x^{+\infty} K(x,t)\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{i\lambda t}dt, \tag{1.6}$$

where $K(x,t)$ *is a matrix function of second order with the elements from* $L_1(x,+\infty)$ *which is related to the potentials* $\Omega(x)$ *as follows (see, e. g., [6]):*

$$\begin{gathered} \lim_{t\to+\infty} \int_c^{+\infty} \|BK(x,x+1) - K(x,x+1)B - \Omega(x)\|\, dx = 0, \\ \lim_{t\to+\infty} \int_0^{c} \left\|BK(x,x+1) - K(x,x+1)B - \tfrac{1+\alpha}{2}\Omega(x)\right\| dx = 0, \end{gathered} \tag{1.7}$$

where $\|\cdot\|$ *is an operator norm in Euclidean space.*

2 The direct and inverse problem of scattering

Denote by $\varphi(x,\lambda)$ the solution of the equation (1.1) satisfying the initial conditions

$$\varphi(0,\lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

2.1. The problem (1.1)-(1.2) generates the self-adjoint operator in the space $L_{2,\rho}(0,+\infty; C_2)$ of vector functions with scalar product

$$(y,z) = \int_0^{+\infty} \rho(x)\, \left\{ y_1(x)\,\overline{z_1(x)} + y_2(x)\,\overline{z_2(x)} \right\} dx.$$

The spectrum of this operator is purely continuous and fills the entire real axis. The eigenfunction of the continuous spectrum has the following form:

$$U(x,\lambda)=\frac{2i\varphi(x,\lambda)}{e_1(0,\lambda)}=\overline{e(x,\lambda)}-S(\lambda)\,e(x,\lambda)\,,\qquad \lambda\in(-\infty,+\infty)\,,$$

where $S(\lambda)=\frac{\overline{e_1(x,\lambda)}}{e_1(x,\lambda)}$ is a scattering function of the problem (1.1)-(1.2).

Lemma 2.1 *Scattering function is continuous on the entire axis and has the following properties*

1 $S^{-1}(\lambda)=\overline{S(\lambda)}=S(\lambda)\,,\qquad \lambda\in(-\infty,+\infty)\,;$
2 *the elements of the matrix function*

$$F_s(x)=Re\frac{1}{2\pi}\int_{-\infty}^{\infty}[S(\lambda)-S_0(\lambda)]\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}e^{i\lambda x}d\lambda$$

belong to the space $L_1(0,\infty)$ *, where*

$$S_0(\lambda)=\frac{1+\frac{1-\alpha}{1+\alpha}e^{-2i\lambda c}}{1+\frac{1-\alpha}{1+\alpha}e^{2i\lambda c}}$$

is a scattering function of the problem (1.1)-(1.2) *in case* $p(x)=q(x)=0$.

2.2. Inverse scattering problem for (1.1)-(1.2) is: provided scattering function $S(\lambda)$, find thecoefficients $p(x)$ and $q(x)$ (I, e. the potential $\Omega(x)$) of the equation (1.1). The relations (1.7) show that to solve the inverse problem it suffices to find the relationship between the kernel $K(x,t)$ defined by (1.6) and the scattering function $S(\lambda)$. To do so, we derive the Marchenko equation, the main equation of the inverse problem:

$$F(x,y)+K(x,y)+\frac{\alpha-1}{\alpha+1}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\ K(x,2c-y)$$

$$+\int_{x}^{+\infty}K(x,t)\,F_s(t+y)\ dt=0,\ \ y>x, \tag{2.1}$$

where

$$F(x,y)=\begin{cases}F_s(x+y)\,, & x>c,\\ \frac{1+\alpha}{2}F_s(x+y)+\frac{1-\alpha}{2}F_s(2e-x+y)\,, & 0<x<c,\end{cases}$$

and $F_s(x)$ is defined in Lemma.

Theorem 2.1 *For every* $x>0$ *, the main equation* (2.1) *has a unique solution* $K(x,\cdot)$ *with the elements from* $L_1(x,\infty)$.

The relations (1.7), the main equation (2.1) and Theorem 2.1 provide the algorithm for finding the potential function $\Omega(x)$, i.e. for finding the coefficients $p(x)$ and $q(x)$ of the equation (1.1) for the given scattering function $S(\lambda)$.

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