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Constructing the asymptotics of the solution to a boundary value problem with an inner boundary layer for a higher order singularly perturbed elliptic equation

Mahir M. Sabzaliev, I.M. Sabzalieva

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Abstract. *The asymptotics of the solution to a boundary value problem is constructed for a higher order elliptic equation degenerated into an elliptic equation lower order with regard to internal boundary layer arising near some surface located inside the domain under consideration and a remainder term is estimated.*

Keywords. asymptotics · boundary layer type function · remainder term

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1 Introduction and problem statement

When studying numerous real phenomena with noniniform transitions from one characteristics to another ones, we need to study boundary values problems for differential equations containing a small parameter for higher derivatives. Such problems are called singularly perturbed boundary value problems. These problems have attracted the attention of many prominent scientists as A.N. Tikhonov , L.S.Pontryagin, N.N. Bogolybov, Yu. A. Mitropolski, V.Vazov, K. Friedrichs, M.I. Vishik, A.A. Lyusternik, O.A. Oleynik, E.F. Misshenko, N. Kh. Rozov, A.M. Ilyin and others. The issues related to the appearance of the so-called internal boundary layers have been studied less though they arise in many problems of physics and mechanics. The phenomenon of internal boundary layer is typical for problems describing relaxational vibrations ([5]).

The internal boundary layer has been constructed for some problems by M.I. Vishik and L.A. Lyusternik in [6] and E.K. Isakova in [1], [2].

In this paper we consider a boundary value problem for a higher order singularly perturbed elliptic equation degenerated into an elliptic equation of lower order. Let $(n-1)$ dimensional surface C^{∞} of the class S partition the bonded domain $\Omega \subset R^n$ with rather smooth boundary Γ into the domains Ω_1 and Ω_2. In Ω we consider the following boundary

M.M. Sabzaliev
Baku Biznes University, Baku, Azerbaijan;
Azerbaijan State Oil and Industry University;
E-mail: sabzaliyevm@mail.ru

I.M. Sabzalieva
Azerbaijan State Oil and Industry University;
Western Caspian University, Baku, Azerbaijan
E-mail: isabzaliyeva@mail.ru

value problem

$$L_\varepsilon u \equiv \varepsilon^{2(l-k)} L_{2l} + L_{2k} u = f(x), \tag{1.1}$$

$$\left.\frac{\partial^i u}{\partial \nu^i}\right|_\Gamma = 0; \;\; i = 0, 1, ..., 2l-1, \tag{1.2}$$

where $\varepsilon > 0$ is a small parameter,

$$L_{2k} = \sum_{|\alpha| \le 2k} a_\alpha(x) D^\alpha, \;\; L_{2l} = \sum_{|\alpha| \le 2l} b_\alpha(x) D^\alpha,$$

$$\alpha = (\alpha_1, \alpha_2, ..., \alpha_n), \;\; |\alpha| = \sum_{j=1}^{n} \alpha_j, \;\; D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} ... D_n^{\alpha_n},$$

$$D_j = \frac{\partial}{\partial x_j}, \;\; x = (x_1, x_2, ..., x_n), \;\; l > k,$$

ν is a normal to Γ, $f(x)$ is a prescribed rather smooth function for $x \in \Omega_p$,$(p = 1, 2)$, possibly with first order discontinuities on S. It is assume that the following conditions are fulfilled:

1) The coeficient $a_\alpha(x), b_\alpha(x)$ are rather smooth, and the polynomials

$$P_{2l} = \sum_{|\alpha| = 2l} a_\alpha(x) \xi^\alpha, \;\; Q_{2k} = \sum_{|\alpha| = 2k} b_\alpha(x) \xi^\alpha$$

are nonzero for $|\xi| \ne 0$ and have the same signs in $\overline{\Omega}$, where $\xi = (\xi_1, \xi_2 ... \xi_n), \xi_i$ are real numbers, $\xi^\alpha = \xi_1^{\alpha_1}, \xi_2^{\alpha_2} ... \xi_n^{\alpha_n}$.

2) $(L_{2l} u, u) \ge \alpha_1^2 (u, u)$, where α_1^2 is independent of u, while u satisfies the conditions (1.2).

3) $(L_{2k} w, w) \ge \alpha_2^2 \left(\sum\limits_{j=1}^{k-1} \left(D^j w, D^j w \right) + (w, w) \right)$, where α_2^2 is independent of w, while satisfies the first k conditions from (1.2).

It follows from the conditions 1), 2), 3) that problem (1.1)-(1.2) has a unique generalized solution. Following the papers [3], [4] we can affirm that the solution of problem (1.1)-(1.2) has continuous derivatives up to the $(2l-1)$- th order inclusively: $u(x) \in C^{2l-1}(\overline{\Omega})$.

The construction of the asymptotics of the first boundary value problem for elliptic equations degenerated into an elliptic equations of lower order in the case of the absence the special surface S was stated in detail in [6]. Our goal in this paper is to construct the asymptotic expansion of the solution of boundary value problem (1.1)-(1.2) with regard to internal boundary layer arising near the surface S.

2 A degenerated problem and constuction of boundary layer functions

We consider a degenerated problem corresponding to the problem (1.1)-(1.2)

$$L_{2k} w = f, \tag{2.1}$$

$$\left.\frac{\partial^i w}{\partial \nu^j}\right|_\Gamma = 0; \; j = 0, 1, ..., k-1. \tag{2.2}$$

It is obvious that $w \in C^{2k-1}(\overline{\Omega})$. The functions $D^{2k} w, D^{2k+1} w, ..., D^{2l-1} w$ may have discontinuities on the surface S i.e. for the $2k, 2k+1, ..., (2l-1)$- th derivatives of the solution $u(x)$ we observe the internal boundary layer phenomenon. To compensate these discontinuities we should construct a boundary layer type function η near the surface S.

Furthermore, since $\frac{\partial^k w}{\partial \nu^k}, \frac{\partial^{k+1} w}{\partial \nu^{k+}}, ..., \frac{\partial^{2l-1} w}{\partial \nu^{2l-1}}$, do not satisfy generally speaking, boundary conditions on Γ, then in the vicinity of Γ for $u(x)$ we can still observe the external boundary layer prhenomenon.

Therefore, a boundary layer type function v should be constructed near the boundary Γ.

To construct the function η in the vicinity of S we introduce local coordinates (ρ, φ) ,where $\varphi = (\varphi_1, \varphi_2, ..., \varphi_{n-1})$ are local cordinates on S, while ρ is a distance along the normal to S at the approriate point φ with approriate signs i.e. if the point is in Ω_1 then $\rho < 0$, if the point is in Ω_2, then $\rho > 0$.

In new coordinates, the operator L_ε has a split in the form

$$L_{\varepsilon,1} \equiv \varepsilon^{2(l-k)} \Big(p(\rho,\varphi) \frac{\partial^{2l}}{\partial \rho^{2l}} + ... \Big) + q(\rho,\varphi) \frac{\partial^{2k}}{\partial \rho^{2k}} + ...,$$

by the dots in the big bracket denote the terms containing derivatives with north respect to ρ of lower orders than $2l$ and by the dots out of the brackets the terms containing derivatives with respect to ρ of lower order than $2k$.

We make substitution of the variable $\rho = \varepsilon \tau$ and expand all the coefficients in Taylor series:

$$p(\varepsilon\tau, \varphi) = p_0(\varphi) + \sum_{s=1}^{N} \varepsilon^S p_S(\varphi) \tau^S + \varepsilon^{N+1} p_{N+1}(\theta_1 \varepsilon \tau, \varphi) \tau^{N+1},$$

$$q(\varepsilon\tau, \varphi) = q_0(\varphi) + \sum_{s=1}^{N} \varepsilon^S q_S(\varphi) \tau^S + \varepsilon^{N+1} q_{N+1}(\theta_2 \varepsilon \tau, \varphi) \tau^{N+1},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Here $p_0(\varphi) = p(\rho,\varphi)|_{\rho=0}$, $q_0(\varphi) = q(\rho,\varphi)|_{\rho=0}$,

$$p_s(\varphi) = \frac{1}{S!} \frac{\partial^S p(\rho,\varphi)}{\partial \rho^s} \bigg|_{\rho=0}, \quad q_s(\varphi) = \frac{1}{S!} \frac{\partial^S q(\rho,\varphi)}{\partial \rho^s} \bigg|_{\rho=0}, \quad |\theta_i| < 1; i = 1, 2.$$

Combining in expansions the terms at the same degres of ε, we obtain :

$$L_{\varepsilon,1} \equiv \varepsilon^{-2k} \Big(R_0 + \sum_{s=1}^{2l} \varepsilon^S R_S \Big), \tag{2.3}$$

where $R_0 = p_0(\varphi) \frac{\partial^{2l}}{\partial \tau^{2l}} + q_0(\varphi) \frac{\partial^{2k}}{\partial \tau^{2k}}$, $R_s; s = 1, 2, .., 2$ are the known linear operators.

We will look for the solution $L_{\varepsilon,1}\eta = 0$ in the form

$$\eta = \varepsilon^{2k}(\eta_0 + \varepsilon\eta_1 + ...). \tag{2.4}$$

Substituting the expressions for η from (2.4) in equation $L_{\varepsilon,1}\eta = 0$ and taking into account (2.3) in the first approximation we obtain an equation to determine η_0 in the from

$$R_0 \eta_0 \equiv p_0(\varphi) \frac{\partial^{2l} \eta_0}{\partial \tau^{2l}} + q_0(\varphi) \frac{\partial^{2k} \eta_0}{\partial \tau^{2k}} = 0. \tag{2.5}$$

It follows from condition 1) that $p_0(\varphi) \cdot q_0(\varphi) > 0$. If $l - k$ is an even number, then a characteristical equation corresponding to the ordinary differential equation (2.5) in addition to zero roots has exactly $l - k$ roots $\lambda_1, \lambda_2, ..., \lambda_{l-k}$ inside the left and as many $\lambda_{l-k+1}, \lambda_{l-k+2}, ..., \lambda_{2l-2k}$ inside the right half-plane.

We consider the follwing solution of equation (2.5):

$$\eta_0=\begin{cases}\sum\limits_{r=1}^{l-k}C_r(\varphi)\left(1+\frac{\lambda_r}{1!}\tau+...+\frac{\lambda_r^{2k-l}}{(2k-1)!}\cdot\tau^{2k-1}-e^{\lambda_r\tau}\right) & for\ \ \tau>0,\\ \sum\limits_{s=l-k+1}^{2l-2k}C_r(\varphi)\left(1+\frac{\lambda_S}{1!}\tau+...+\frac{\lambda_s^{2k-l}}{(2k-1)!}\cdot\tau^{2k-1}-e^{\lambda_S\tau}\right) & for\ \ \tau<0.\end{cases}$$

$C_1(\varphi),C_2(\varphi),...,C_{2l-2k}(\varphi)$ are determined from the condition of discontinuity of functions $\frac{\partial^j}{\partial\rho^j}(W+\eta);j=2k,2k+1,...,2l-1$ on the surface S, i.e.

$$\Big(\frac{\partial^j}{\partial\rho^j}(W+\varepsilon^{2k}\eta_0)\Big)(\varphi)=0;\ \ j=2k,2k+1,...,2l-1. \tag{2.6}$$

Here $[F](\varphi)$ means the jump of the function $F(\rho,\varphi)$ for $\rho=0$,i.e. $[F](\varphi)=F(+0,\varphi)-$ $-F(-0,\varphi)$.

We can write condition (2.6) as follows:

$$\varepsilon^{2k-j}\Big(\frac{\partial^j\eta_0}{\partial\tau^j}\Big)(\varphi)=-\Big(\frac{\partial^jW}{\partial\rho^j}\Big)(\varphi);\ j=2k,2k+1,...,2l-1. \tag{2.7}$$

Using the explicit expression for η_0 from (2.7) we have the following system of linear equations with respect to $C_1(\varphi),C_2(\varphi),...,C_{2l-2k}(\varphi)$:

$$\begin{cases}-\sum\limits_{i=1}^{l-k}\lambda_i^{2k}C_i+\sum\limits_{j=l-k+1}^{2l-2k}\lambda_j^{2k}C_j=-\Big(\frac{\partial^{2k}W}{\partial\rho^{2k}}\Big)(\varphi),\\ -\sum\limits_{i=1}^{l-k}\lambda_i^{2k+1}C_i+\sum\limits_{j=l-k+1}^{2l-2k}\lambda_j^{2k+1}C_j=-\varepsilon\Big(\frac{\partial^{2k+1}W}{\partial\rho^{2k+1}}\Big)(\varphi),\\ \dots\dots\dots\dots\dots\dots\dots\dots\\ -\sum\limits_{i=1}^{l-k}\lambda_i^{2l-1}C_i+\sum\limits_{j=l-k+1}^{2l-2k}\lambda_j^{2l-1}C_j=-\varepsilon^{2l-2k-1}\Big(\frac{\partial^{2l-1}W}{\partial\rho^{2l-1}}\Big)(\varphi).\end{cases} \tag{2.8}$$

Since all $\lambda_j;j=1,2,...,2l-2k$ are non-zero and different, then the main determinant of the system (2.8) is nonzero. Therefore the functions $C_1(\varphi),C_2(\varphi),...,C_{2l-2k}(\varphi)$ from the system (2.8) are uniquely determined.

We multiply the constructed function η by the smoothing function and for the obtained new function leave the previous denotation.

Let us construct boundary layer functions near the boundary Γ. For that, near Γ we introduce local coordinates $(y_1,y_2,...,y_n)$, where y_1 is a distance along the internal normal to the boundary Γ, while $y'=(y_2,y_3,...,y_n)$ are local coordinates on Γ. Writing the operator L_ε in new coordinates, making a change of the variable $y_1=\varepsilon\xi$, having expanded all the coefficients in Taylor series and groupping the terms with the same degrees with respect to ε, we obtain

$$L_{\varepsilon,2}\equiv\varepsilon^{-2k}\Big(M_0+\sum_{S=1}^{2l}\varepsilon^SM_S\Big),$$

where $M_0\equiv A(y')\frac{\partial^{2l}}{\partial\xi^{2l}}+B(y')\frac{\partial^{2k}}{\partial\xi^{2k}},\ \ M_S;S=1,2,...,2l$ are the known linear differential operators.

We look for the solution of the equation $L_{\varepsilon,2}V=0$ in the form

$$V=\varepsilon^k(V_0+\varepsilon V_1+...+\varepsilon^mV_m). \tag{2.9}$$

Having substituted the expression of V from (2.9) in equation $L_{\varepsilon,2}V = 0$, we obtain an equation to determine the functions $V_0, V_1, ...V_m$:

$$M_0V_0 \equiv A(y')\frac{\partial^{2l}V_0}{\partial\xi^{2l}} + B(y')\frac{\partial^{2lk}V_0}{\partial\xi^{2k}} = 0, \tag{2.10}$$

$$MV_S = -\sum_{r=1}^{S} M_r V_{s-r}; S = 1, 2, ..., m. \tag{2.11}$$

We require

$$\frac{\partial^i}{\partial\nu^j}(w + \eta + V)\bigg|_{\Gamma} = 0;\ j = k, k+1, ..., l-1.$$

These conditions can be writthen as follows

$$\frac{\partial^i}{\partial\xi^j}(V_0 + \varepsilon V_1 + ... + \varepsilon^m V_m)\bigg|_{\xi=0} = -\varepsilon^{j-k}\ \frac{\partial^i}{\partial\nu^j}(w + \eta)\bigg|_{\Gamma};\ j = k, k+1, ..., l-1, \tag{2.12}$$

Note that if $2k \geq l-1$ then the number m in (2.9) can be considered equal to $k, (m = k)$, but if $2k < l-1$, we should take $m = l-k-1$. Thus, the boundary conditions under which will solve the equations (2.10),(2.11) are found from (2.12). For example, the function V_0 we will be a boundary layer type equation (2.10) satisfying the boundary conditions

$$\frac{\partial^k V_0}{\partial\xi^k}\bigg|_{\xi=0} = -\ \frac{\partial^k}{\partial\nu^k}(w + \eta)\bigg|_{\Gamma},\ \frac{\partial^{k+1}V_0}{\partial\xi^{k+1}}\bigg|_{\xi=0} = 0, ...,\ \frac{\partial^{l-1}V_0}{\partial\xi^{l-1}}\bigg|_{\xi=0} = 0. \tag{2.13}$$

Since $A(y')$ and $B(y')$ are nonzero and by the condition 1) have the same signs, the characteristical equation corresponding to the differential equation (2.10) has $l-k$ roots inside the left half-plane They are denoted by $\sigma_1, \sigma_2, ..., \sigma_{l-k}$. The boundary layer type solution of equation (2.10) near Γ, is of the form

$$V_0 = c_1(y')e^{\sigma_1\xi} + c_2(y')e^{\sigma_2\xi} + ... + c_{l-k}(y')e^{\sigma_{l-k}\xi}.$$

The functions $c_1(y'), c_2(y'), ..., c_{l-k}(y')$ are uniquely determined from the conditions (2.13).

Continuing the process, we determine the function $V_1, V_2, ..., V_m$ from equation (2.11) and from the conditions obtained from the equality (2.12). Multiply all the functions V_i by the smoothing function and denote the obtained new functions again by $V_i; i = 0, 1, .., m$.

Thus, for the solution of the problem (1.1),(1.2) in the first approximation we obtain the representation

$$u = w + \eta + V + \varepsilon z, \tag{2.14}$$

where $\eta = \varepsilon^{2k}\eta_0,\ \ V = \sum_{i=1}^{m}\varepsilon^{k+i}V_i$, εz while is a remainder term $m = \max\{k, l-k-1\}$.

Summarizing what are stated in this paper, we can formulate the following statement.

Theorem. *Assume that the $(n-1)$ dimensional surface S of the class C^∞ partitions the bounded domain $\Omega \subset R^n$ with a rather smooth boundary Γ into two parts $f(x_1, x_2, ..., x_n)$ is a smooth function in $\Omega \backslash S$ with first order discontinuity on the surface and S. Let L_{2l} and L_{2k} be elliptic differential operators with variable coefficients of orders $2l$ and $2k$, respectively. Then subject to conditions 1), 2),3) the solution of the boundary value problem (1.1),(1.2) can be represented in the form (2.14), where W is the solution of the degenerated problem, η is is a boundary layer function near the surface S , V is a boundary layer function near the boundary Γ, while εz is a remainder term, and the following estimate is valid for z:*

$$\|z\|_{W_2^{k-1}(\Omega)} \leq C,$$

where $C > 0$ is a constant independent of ε.

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