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Stochastic additive functionals of multitype age-dependent branching processes

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Abstract. In this paper we consider stochastic additive functionals of critical multitype age-dependent branching processes with and without immigration. We prove that under mild conditions, those processes have same limit distributions as total progeny.

Keywords. branching process · stochastic additive functional · critical branching process · immigration

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1 Introduction and description of the processes

We will begin with a brief overview of the multi-type age-dependent process, with a more comprehensive description available in [3].

The system under investigation involves n types of particles, each denoted as $T_1, T_2, \ldots, T_n$. Each type of particle, T_i, exhibits a random lifespan represented by τ_i, with a distribution function described as:

$$P(\tau_i \leq t) = G^i(t), G^i(0+) = 0.$$

We will assume that $G^i(t)$ are absolutely continuous.

Upon completing their lifespan, particles of any type can transform into an arbitrary number of particles of any type. The conditional probability of such a transformation, given that the age attained by the original particle was u, is denoted as $p^i_\alpha(u)$, where α is an n-dimensional vector representing the number of particles of each type T_i in the set.

The evolution of particles in this process is defined by the joint distribution of the random variable τ_i and the random vector $v^i = (v^i_1, \ldots, v^i_n)$, which characterizes the progeny of each particle:

$$P(\tau_i \in B, v^i = \alpha) = \int_B p^i_\alpha(u) dG^i(u).$$

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We use the vector $\mu^i(t) = (\mu_1^i(t), \dots, \mu_n^i(t))$ to represent the number of particles of types $T_1, T_2, \dots, T_n$ at time t, assuming that initially, there existed one T_i-type particle.

While we won't delve into the description of the probability space in this introduction, we will note that it can be constructed analogically to [1, Chapter 6].

We also define generating functions as follows:

$$h^i(u,s) = E\left(s^{\mu^i(\tau_i)}|\tau_i = u\right) = \sum_\alpha p_\alpha^i(u)s^\alpha \text{ and } F^i(t,s) = \sum_\alpha P(\mu^i(t) = \alpha)s^\alpha,$$

$i = \overline{1,n}$, where $s = (s_1, ..., s_n), s^\alpha = s_1^{\alpha_1} \cdots s_n^{\alpha_n}$.

Additionally, we define vectors $F(t,s)$ and $h(t,s)$ as:

$$F(t,s) = (F^1(t,s), ..., F^n(t,s)), \quad h(t,s) = (h^1(t,s), ..., h^n(t,s)).$$

2 Stochastic additive functionals

Suppose that each individual during its lifetime, denoted as τ, generates some product. Let $\xi(t), t \leq \tau$, represent the amount of product produced by an individual over time t.

A stochastic additive functional from the branching process, denoted as $\eta(t)$, can be defined as the sum of the product generated by all individuals that existed before time t. Additionally, the processes $\xi(t), t \leq \tau$, corresponding to different individuals, are independent, and for the same type, they are also identically distributed.

The strict definition is given as follows. We can enumerate all individuals that ever existed in chronological order. Denote by i_k, τ_k, κ_k^j, and $\rho_k(t)$ the type of particle, duration of the particle's life, number of j-th type descendants, and the age of the particle before time t, respectively.

Let

$$\xi_n(i, \theta, \kappa, t), t \in [0;\theta], k = 1, 2, ...$$

with fixed $i, \theta, \kappa = (\kappa^1, ..., \kappa^n), \kappa^j = 0, 1, ...$ be a sequence of measurable, jointly independent stochastic processes. These processes are also independent from the branching process, and they satisfy the initial condition

$$\xi_n(i, \theta, \kappa, 0) = 0.$$

Then

$$\eta(t) = \sum_{k=1}^\infty \xi_k(i_k, \tau_k, \kappa_k, \rho_k(t)).$$

A more detailed description of $\eta(t)$ is provided in [4].

Sometimes, it is more convinient to split $\eta(t)$ into vector $(\eta_1(t), ..., \eta_n(t))$, where each component of the vector counts only product generated by specific type of individuals, $\eta_j(t) = \sum\limits_{k_j=1}^\infty \xi_k(j, \tau_{k_j}, \kappa_{k_j}, \rho_{k_j}(t))$.

3 Preliminary results

During the paper we will use following notations:

$$a_j^i(u, s_0) = \frac{\partial h^i(u,s)}{\partial s_j}|_{s=s_0}, \quad a_j^i(u) = a_j^i(u, \mathbf{1}), \quad A_j^i = \int_0^\infty a_j^i(u)dG^i(u),$$

$$b^i_{jk}(u,s_0) = \frac{\partial h^i(u,s)}{\partial s_j \partial s_k}|_{s=s_0}, \quad b^i_{jk}(u) = b^i_{jk}(u,\mathbf{1}), \quad B^i_{jk} = \int_0^\infty b^i_{jk}(u) dG^i(u).$$

Consider matrix $A = \left\| A^i_j \right\|_{i,j=\overline{1,n}}$. We assume that this matrix is irreducible with positive entries. In this case, the matrix A has the largest eigenvalue in magnitude, denoted as ρ, which equals to 1 in critical case we consider. By $u = (u_1, u_2, ..., u_n)$ and $v = (v_1, v_2, ..., v_n)$ denote right and left eigenvectors of matrix A, corresponding to ρ. For these vectors, normalization conditions hold $(u,\mathbf{1}) = \sum\limits_{k=1}^n u_k = 1, (u,v) = \sum\limits_{k=1}^n u_k v_k = 1$.

Let

$$M^k = \int_0^\infty u dG^k(u), \quad M^{lk}_a = \int_0^\infty u a^k_l(u) dG^l(u),$$

$$B = \sum_{l,k,m=1}^n B^l_{mk} v_l u_k u_m, \quad M_a = \sum_{l,k=1}^n M^{lk}_a v_l u_k.$$

We also define

$$M^i_\eta = \int_0^\infty E\left(\eta_i(u) | \tau_i = u\right) dG^i(u), \quad \xi = \sum_{m=1}^n \sup_{t\in[0;\tau]} |\eta_m(t)|,$$

where τ is a moment of death of first individual.

Theorem 3.1 *([5]). Let $M(dy) = \|m_{ij}(dy)\|_{i,j=\overline{1,r}}$ be an r*r square matrix, components of which are finite non-negative measures on $[0,+\infty)$. Let vector function $g(x) = (g_1(x), ..., g_r(x))$ be such that for some $\gamma \geq 0$ it holds $\sup\limits_{x\in\mathbb{R}} \dfrac{g(x)}{max\{1, x^\gamma\}} < \infty$ and $\frac{g(x)}{x^\gamma} \xrightarrow{x\to+\infty} c = (c_1, ..., c_r)$. If the Perron root of $M[0,+\infty)$ equals to 1 and $\int_0^\infty u\, m_{ij}(du) < \infty$, then*

$$\frac{1}{x^{1+\gamma}} \int_0^x g(x-y) dH(y) \xrightarrow{x\to+\infty} \frac{c}{(1+\gamma)a} \left\| u^i v^j \right\|_{i,j=\overline{1,r}},$$

where $H(y)$ is the renewal matrix, which corresponds to matrix $M(dy)$, u and v are right and left eigenvectors of $M[0,+\infty)$, $a = \left(v, \int_0^\infty y M(dy) u\right)$.

Lemma 3.1 *([2]). If for random variables $X(t)$ and $Y(t)$ following conditions are satisfied:*

a) $X(t) \xrightarrow{t\to+\infty} X$ in distribution,
b) $\lim_{t\to\infty} E\left(X(t) - Y(t)\right)^2 = 0$,
then $Y(t) \xrightarrow{t\to+\infty} X$ in distribution.

4 Main results

In order to prove next theorem we will compare processes $\eta(t)$ with processes $N(t)$ - total number of particles born by the moment of time t.

Let $N^i_j(t)$ denote number of particles of type T_j, born by t, assuming that initially, there existed one T_i-type particle. Analogously, $\eta^i_j(t) = (\eta^i_1(t), ..., \eta^i_n(t))$ denote stochastic additive functional, under the same condition. It is known [2], that if $\lim_{t\to\infty} t^2 \int_t^\infty a^k_l(u) dG^l(u) = 0$, $\lim_{t\to\infty} t^2 \left(1 - G^l(t)\right) = 0$, M^j, M^{jk}_a, B^i_{jk} are finite, then

$$E\left(exp\left\{i\sum_{j=1}^{n}\beta^j N_j^i(t)/v^j t^2\right\}|\sum_{j=1}^{n}\mu_j^k(t)>0\right)\xrightarrow{t\to+\infty}$$

$$\left(2B\sum_{j=1}^{n}\beta^j\right)^{1/2}/M_a\left(sh\left(\left(2B\sum_{j=1}^{n}\beta^j\right)^{1/2}/M_a\right)\right). \qquad (4.1)$$

Furthermore, as indicated in [2] (see also ([8, p. 464-465]), we can establish asymptotic behavior of the moments $E(N_j^i(t))$ and $E(N_j^i(t)N_k^i(t))$:

$$E(N_j^i(t))\sim\frac{u_i v_j t}{M_a},\quad E(N_j^i(t)N_k^i(t))\sim B\frac{u_i v_j v_k t^3}{3(M_a)^3}. \qquad (4.2)$$

Theorem 4.1 *If the following conditions are satisfied:*
i) integrals M^j, M_a^{jk}, B_{jk}^i *are finite;*
ii) $M_\eta^j\equiv M_\eta$, $0<|M_\eta|<\infty, E^j(\xi^2)<+\infty$;
iii) $\lim_{t\to\infty}t^2\int_t^\infty a_l^k(u)dG^l(u)=\lim_{t\to\infty}t^2\left(1-G^l(t)\right)=0$, $l,k,j=\overline{1,n}$, *then*

$$E\left(exp\left\{i\sum_{j=1}^{n}\beta^j\eta_j^k(t)/M_\eta v^j t^2\right\}|\sum_{j=1}^{n}\mu_j^k(t)>0\right)\xrightarrow{t\to+\infty}$$

$$\left(\left(2B\sum_{j=1}^{n}\beta^j\right)^{1/2}/M_a\right)\left(sh\left(\left(2B\sum_{j=1}^{n}\beta^j\right)^{1/2}/M_a\right)\right) \qquad (4.3)$$

for all $k=\overline{1,n}$.

Proof. Let $N^i(t)=\sum_{k=1}^n N_k^i(t)$. Processes $N^i(t)$ we will consider as stochastic additive functionals from b.p., where the amount of product produced by one particle - $\xi(t)$ equals to 1 on $(0;\tau],\xi(0)=0$.

Let

$$F^i(t,z,x,s)=E\left(e^{zN^i(t)}e^{x\eta^i(t)}s^{\mu^i(t)}\right),$$

$$h^i(u,x,s)=E\left(e^{x\eta^i(\tau_i)}s^{\mu^i(\tau_i)}|\tau_i=u\right),$$

$$F(t,z,s)=(F^1(t,z,s),...,F^i(t,z,s))$$

where $e^{zN^i(t)}=e^{\sum_{k=1}^{n}z_k N_k^i(t)}, e^{x\eta(t)}=e^{\sum_{k=1}^{n}z_k\eta_k^i(t)}, x_l,z_l\le 0, l=\overline{1,n}$.

Additionally, define the following derivatives

$$D_j^i(t)=\frac{\partial F^i(t,z,x,s)}{\partial x_j}|_{x=\mathbf{0},z=\mathbf{0},s=\mathbf{1}}, D_{lj}^i(t)=\frac{\partial^2 F^i(t,z,x,s)}{\partial x_l\partial x_j}|_{x=\mathbf{0},z=\mathbf{0},s=\mathbf{1}},$$

$$D_{lj}^{i0}(t)=\frac{\partial^2 F^i(t,z,x,s)}{\partial z_l\partial x_j}|_{x=\mathbf{0},z=\mathbf{0},s=\mathbf{1}},$$

where $\mathbf{1}=(\underbrace{1,...,1}_{n}),\mathbf{0}=(\underbrace{0,...,0}_{n})$. By e_k we denote vector $(\underbrace{0,...,0}_{k-1},1,\underbrace{0,...,0}_{n-k})$.

Note that $h^i(u, \mathbf{1}, s) = h^i(u, s)$.

Similarly to [4], using law of total probability, we can derive formula

$$F^i(t,z,x,s) = e^{z_i}\left(s_i E\left(e^{x\eta^i(t)}, t<\tau_i\right) + \int_0^t h^i\left(u,x,F(t-u,z,x,s)\right) dG^i(u)\right). \tag{4.4}$$

Let θ denote the point $(x, z, s) = (\mathbf{0}, \mathbf{0}, \mathbf{1})$.

Differentiating (4.4) with respect to x_j at the point θ we get

$$D^i_j(t) = \delta^i_j\left(E\left(\eta^i_j(t), t<\tau_i\right) + \int_0^t E\left(\eta^i_j(u)|\tau_i = u\right) dG^i(u)\right)$$
$$+\sum_{k=1}^n \int_0^t D^k_j(t-u) a^i_k(u) dG^i(u).$$

Differentiating (4.4) first with respect to x_j, then with respect to x_l at the point θ we get

$$D^i_{lj}(t) = \delta^i_j\delta^i_l E\left(\eta^i_j(t)(\eta^i_l(t) - \delta^j_l), t<\tau_i\right)$$
$$+\delta^i_j\delta^i_l \int_0^t E\left(\eta^i_j(u)(\eta^i_l(u) - \delta^j_l)|\tau_i = u\right) dG^i(u)$$
$$+\delta^i_j \sum_{k=1}^n \int_0^t D^k_l(t-u) E\left(\eta^i_j(u)\mu^i_k(u))|\tau_i = u\right) dG^i(u)$$
$$+\sum_{k,m=1}^n \int_0^t D^k_j(t-u) D^m_l(t-u) b^i_{mk}(u) dG^i(u) + \sum_{k=1}^n \int_0^t D^k_{lj}(t-u) a^i_k(u) dG^i(u). \tag{4.5}$$

And calculating successive derivatives of $F^i(t,z,x,s)$ first with respect to x_j, then with respect to z_l at the point θ we get

$$D^{i0}_{lj}(t) = \delta^i_j\delta^i_l\left(D^i_l(t) + \sum_{k=1}^n \int_0^t E\left(N^k_l(t-u)\right) E\left(\eta^i_j(u)\mu^i_k(u))|\tau_i = u\right) dG^i(u)\right)$$
$$+\sum_{k,m=1}^n \int_0^t D^k_j(t-u) E\left(N^m_l(t-u)\right) b^i_{mk}(u) dG^i(u) + \sum_{k=1}^n \int_0^t D^{k0}_{lj}(t-u) a^i_k(u) dG^i(u). \tag{4.6}$$

Condition ii) implies that

$$E\left(\eta^i_i(t), t<\tau_i\right) + \int_0^t E\left(\eta^i_i(u)|\tau_i = u\right) dG^i(u) \xrightarrow{t\to+\infty} M^i_\eta,$$

and by Theorem 3.1 we can establish that

$$D^i_j(t) \sim \frac{u_i v_j M_\eta t}{M_a}. \tag{4.7}$$

By utilizing (4.2), (4.7) and condition ii) of the theorem, we see that first three summands of (4.5) and first two summands of (4.6) are at most $O(t)$. Expressions (4.2) and (4.7) also imply that

$$\sum_{m,k=1}^n \int_0^t D^k_j(t-u) D^m_l(t-u) b^i_{mk}(u) dG^i(u) \sim v_j v_l \sum_{m,k=1}^n B^i_{mk} u_m u_k \frac{M^2_\eta t^2}{(M_a)^2},$$

$$\sum_{k,m=1}^{n}\int_0^t D_j^k(t-u)E\left(N_l^m(t-u)\right)b_{mk}^i(u)dG^i(u)\sim v_jv_l\sum_{m,k=1}^{n}B_{mk}^iu_mu_k\frac{M_\eta t^2}{(M_a)^2}.$$

Using Theorem 3.1 again, we see that

$$D_{jl}^i(t)\sim B\frac{u_iv_jv_lM_\eta^2t^3}{3(M_a)^3},D_{lj}^{i0}(t)\sim B\frac{u_iv_jv_lM_\eta t^3}{3(M_a)^3}. \tag{4.8}$$

Define

$$\Phi^i(t,z,x)=E\left(e^{zN^i(t)}e^{x\eta^i(t)},\sum_{k=1}^{n}\mu_k^i(t)=0\right),$$

$$K_j^i(t)=\frac{\partial\Phi^i(t,z,x)}{\partial x_j}|_{x=\mathbf{0},z=\mathbf{0}},K_{lj}^i(t)=\frac{\partial^2\Phi^i(t,z,x)}{\partial x_l\partial x_j}|_{x=\mathbf{0},z=\mathbf{0}},K_{lj}^{i0}(t)=\frac{\partial^2\Phi^i(t,z,x)}{\partial z_l\partial x_j}|_{x=\mathbf{0},z=\mathbf{0}},$$

$$\hat{N}_j^i(t)=\frac{\partial\Phi^i(t,z,x)}{\partial z_j}|_{x=\mathbf{0},z=\mathbf{0}},\hat{N}_{jl}^i(t)=\frac{\partial\Phi^i(t,z,x)}{\partial z_j\partial z_l}|_{x=\mathbf{0},z=\mathbf{0}}.$$

Plugging $s=\mathbf{0}$ to (4.4), we get (see Weiner [7] for derivation of this formula and formulas for conditional moments in one-dimensional case)

$$\Phi^i(t,z,x)=e^{z_i}\left(\int_0^t h^i\left(u,x,\Phi(t-u,z,x)\right)dG^i(u)\right), \tag{4.9}$$

$$\Phi(t,z,x)=\left(\Phi^1(t,z,x),\ldots,\Phi^n(t,z,x)\right).$$

Denote $Q_i(t)=P(\sum\limits_{k=1}^{n}\mu_k^i(t)>0)=1-P(\sum\limits_{k=1}^{n}\mu_k^i(t)=0)$, $Q(t)=(Q_1(t),\ldots,Q_n(t))$.

Differentiating (4.9) with respect to x_j at the point $(\mathbf{0},\mathbf{0})$ yield

$$K_j^i(t)=\delta_j^i\left(\int_0^t E\left(\eta_j^i(u)(\mathbf{1}-Q(t-u))^{\mu^i(u)}|\tau_i=u\right)dG^i(u)\right)$$

$$+\sum_{k=1}^{n}\int_0^t K_j^k(t-u)a_k^i(u,\mathbf{1}-Q(t-u))dG^i(u).$$

Expanding $a_k^i(u,\mathbf{1}-Q(t-u))$ into Taylor series at point $\mathbf{1}$ yields

$$K_j^i(t)=\delta_j^i\left(\int_0^t E\left(\eta_j^i(u)(\mathbf{1}-Q(t-u))^{\mu^i(u)}|\tau_i=u\right)dG^i(u)\right)$$

$$+\sum_{k=1}^{n}\int_0^t K_j^k(t-u)a_k^i(u)dG^i(u)-\sum_{k,l=1}^{n}\int_0^t K_j^k(t-u)b_{kl}^i(u)Q_l(t-u)dG^i(u)$$

$$+\sum_{k,l=1}^{n}\int_0^t K_j^k(t-u)e_{kl}^i(u,Q(t))Q_l(t-u)dG^i(u), \tag{4.10}$$

where $e_{kl}^i(u,Q(t))\to 0, Q(t)\to 0$.

Condition iii) allows us to claim [6], that

$$P\left(\sum_{k=1}^{n}\mu_k^i(t)>0\right)\sim\frac{2M_au^i}{Bt}. \tag{4.11}$$

Obviously, $K^i_j(t) \le D^i_j(t)$, and therefore $K^i_j(t)/t$ is bounded sequence. Suppose that $K^i_j(t) = o(t)$. Then (4.11) gives

$$K^i_j(t) = \delta^i_j M_\eta + o(1) + \sum_{k=1}^{n} \int_0^t K^k_j(t-u) a^i_k(u) dG^i(u) + o(1).$$

Thus, Theorem 3.1 yield $K^i_j(t) = u_i v_j M_\eta t / M_a$, which contradicts our assumption.

Let K^i_j be a limit (maybe partial) of $K^i_j(t)/t$. Then $K^i_j(t) \sim K^i_j t$ and from relations (4.11) and (4.10) we get

$$K^i_j t \sim \delta^i_j M_\eta + \sum_{k=1}^{n} \int_0^{t/2} K^k_j \cdot (t-u) a^i_k(u) dG^i(u) - \frac{2M_a}{B} \sum_{k,l=1}^{n} u_l K^k_j B^i_{k,l} + o(1). \quad (4.12)$$

Comparing coefficients near t, we get $K^i_j = \sum\limits_{k=1}^{n} K^k_j A^i_k$, and by definiton of matrix A, it must be the case, that $K^i_j = u^i K_j$. Then, premultiplying (4.12) by v_i and summing over i (recall $\sum_{k=1}^n u_k v_k = 1$), yield

$$0 \sim v_j M_\eta - 3 M_a K_j + o(1),$$

which, in return gives $K^i_j = u_i v_j M_\eta / 3M_a$ and

$$K^i_j(t) \sim \frac{u_i v_j M_\eta}{3M_a} t. \quad (4.13)$$

Analogical arguments show that

$$\hat{N}^i_j(t) \sim \frac{u_i v_j}{3M_a} t. \quad (4.14)$$

Differentiating (4.4) first with respect to x_j then with respect to z_l at the point $(\mathbf{0}, \mathbf{0})$ we get

$$K^{i0}_{lj}(t) = \delta^i_j \delta^i_l K^i_l(t) + \sum_{k,m=1}^{n} \int_0^t K^k_j(t-u) \hat{N}^m_l(t-u) b^i_{mk}(u, \mathbf{1} - Q(t-u)) dG^i(u)$$

$$+ \delta^i_j \delta^i_l \sum_{k=1}^{n} \int_0^t \hat{N}^k_l(t-u) E\left(\eta^i_j(u) \mu^i_k(u) (\mathbf{1} - Q(t-u))^{\mu^i(u) - e_k} | \tau_i = u\right) dG^i(u)$$

$$+ \sum_{k=1}^{n} \int_0^t K^{k0}_{lj}(t-u) a^i_k(u, \mathbf{1} - Q(t-u)) dG^i(u). \quad (4.15)$$

Similar reasoning to (4.10)-(4.12), relations (4.14) and (4.13), along with the binomial formula, show that $K^{i0}_{lj}(t) \sim K^{i0}_{lj} t^3$ and

$$K^{i0}_{lj}(t) \sim K^{i0}_{lj} t^3 \sim \sum_{k,m=1}^{n} \frac{v_l v_j u_k u_m B^i_{km} M_\eta}{9M_a^2} t^2 - \frac{2M_a}{B} \sum_{k,m=1}^{n} B^i_{km} u_m K^{k0}_{lj} t^2$$

$$+ \sum_{k=1}^{n} A^i_k K^{k0}_{lj} t^3 - 3 \sum_{k=1}^{n} K^{k0}_{lj} M^{ik}_a t^2 + o(t^2).$$

Comparing coefficients near t^3 gives $K_{lj}^{i0} = u_i K_{lj}^0$. Then, after multiplying both sides by v_i, summing ovet i and comparing coefficients near t^2, we get $K_{lj}^{i0} = u_i v_l v_j B M_\eta / 45 M_a^3$ and

$$K_{lj}^{i0}(t) \sim B \frac{u_i v_l v_j M_\eta}{45 M_a^3} t^3. \tag{4.16}$$

Same reasoning as in (4.10)-(4.16) gives

$$K_{lj}^{i}(t) \sim B \frac{u_i v_l v_j M_\eta^2}{45 M_a^3} t^3, \hat{N}_{lj}^{i}(t) \sim B \frac{u_i v_l v_j}{45 M_a^3} t^3. \tag{4.17}$$

Now relations (4.2), (4.8), (4.11), (4.16) and (4.17) yield

$$E\left(\left(\frac{(M_a)^2 N_l^i(t)}{v_l t^2} - \frac{(M_a)^2 \eta_j^i(t)}{v_j M_\eta t^2}\right)^2 | \sum_{k=1}^n \mu_k^i(t) > 0\right)$$

$$= E\left(\frac{(M_a)^2 N_l^i(t)}{v_l t^2} - \frac{(M_a)^2 \eta_j^i(t)}{v_j M_\eta t^2}\right)^2 / Q_i(t)$$

$$-E\left(\left(\frac{(M_a)^2 N_l^i(t)}{v_l t^2} - \frac{(M_a)^2 \eta_j^i(t)}{v_j M_\eta t^2}\right)^2, \sum_{k=1}^n \mu_k^i(t) = 0\right) / Q_i(t)$$

$$\sim B M_a u_i \left(\frac{1/3t - 2/3t + 1/3t + 1/45t - 2/45t + 1/45t}{2 M_a u_i / Bt}\right) = 0.$$

From here, (4.1), using Lemma 3.1, we get the result.

Corollary 4.1 *If conditions of theorem* (4.1) *are satisfied, then distributions* $P\left(\frac{v_l \eta_j^i(t)}{M_\eta v_j N_l^i(t)} \leq x | \sum_{k=1}^n \mu_k^i(t) > 0\right), j, l = \overline{1, n}$ *converge to the degenerate distribution, localized at the point* 1.

Now consider a branching process with immigration. We assume that the process starts with no individuals, and $E^0(*)$ denotes the conditional expectation, given that no particles existed at the beginning ($P^0(*)$ is respective conditional probability). It is convenient to consider a branching process with immigration as a decomposable branching process with $n+1$ types, with one extra type T_0. An individual of type T_0 reproduces itself and some number of individuals of other types. Let $p_\alpha^0(u)$, where $\alpha = (\alpha_1, ..., \alpha_n)$, be the probability that α_1 cells of the first type, α_2 cells of the second type, ..., α_n cells of the n-th type, all of age 0, arrive at time u, according to the renewal process with distribution $G^0(u)$. Let's introduce generating functions

$$h^0(u, s) = \sum_\alpha p_\alpha^0(u) s^\alpha,$$

$$F^0(t, z, x, s) = E^0\left(e^{zN(t)} e^{x\eta(t)} s^{\mu(t)}\right),$$

where $N(t), \eta(t), \mu(t)$ denote the same processes as above.

It is easy to check that similarly to (4.4) next integral equation holds

$$F^0(t, z, x, s) = 1 - G^0(t) + \int_0^t h^0(u, F(t-u, z, x, s)) F^0(t-u, z, x, s) dG^0(u). \tag{4.18}$$

Also define

$$M_0 = \int_0^{+\infty} udG^0(u), A_j^0 = \int_0^{\infty} \frac{\partial h^0(u,s)}{\partial s_j}|_{s=1} dG^0(u), A = \sum_{k=1}^{n} A_j^0 u^j.$$

From [2] we know that if conditions i), ii) of Theorem 4.1 holds, A, M_0 are finite, then next convergance takes place

$$E^0\left(exp\left\{i\sum_{j=1}^{n}\beta^j N_j(t)/v_j t^2\right\}\right) \xrightarrow{t\to+\infty}$$

$$\left(ch\left(B\sum_{j=1}^{n}\beta^j/2M_a\right)^{1/2}\right)^{-\frac{2AM_a}{BM_0}}. \tag{4.19}$$

Equivalent to branching processes without immigration, we can obtain similar results for $\eta(t)$ in processes with immigration.

Theorem 4.2 *If conditions i), ii) of Theorem 4.1 holds, A, M_0 are finite, then*

$$E^0\left(exp\left\{i\sum_{j=1}^{n}\beta^j \eta_j(t)/M_\eta v_j t^2\right\}\right) \xrightarrow{t\to+\infty}$$

$$\left(ch\left(B\sum_{j=1}^{n}\beta^j/2M_a\right)^{1/2}\right)^{-\frac{2AM_a}{BM_0}}.$$

Proof. By differentiating (4.18) in a similar manner to what we did in Theorem 4.1, we can establish the second and mixed moments of $\eta_j(t)$ and $N_l(t)$. Comparing these moments using a lemma and (4.19) completes the proof.

This theorem also implies next corollary.

Corollary 4.2 *If conditions of Theorem* (4.2) *are satisfied, then distributions* $P^0\left(\frac{v_l\eta_j(t)}{M_\eta v_j N_l(t)} \le x\right), j, l = \overline{1,n}$ *converge to the degenerate distribution, localized at the point* 1.

5 Conclusion

In this paper, we derived the limiting distributions for stochastic additive functionals, which have finite, non-zero mathematical expectations for the product generated by a single particle, for both processes with and without immigration, under standard conditions of second moment finiteness.

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