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Isoperimetric problem of the calculus of variations with a quadratic functional

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Abstract. *In this paper, we give a complete study of the isoperimetric problem of the calculus of variations with a quadratic functional. The main purpose of this paper is to investigate the case when the Jacobi condition is satisfied, but the strong Jacobi condition isn't satisfied.*

Keywords. Calculus of Variations, isoperimetric problem, quadratic functional, Euler equation, Legendre condition, Jacobi equation, Jacobi condition.

1 Introduction

The extrema is sought in the function space $x(\cdot) \in C^1([t_0; t_1], \mathbb{R})$ with given conditions:

$$J_0(x(\cdot)) \to \min;\ J_i(x(\cdot)) = \alpha_i,\ i = 1, \dots, m;\ \begin{array}{l} x(t_0) = x_0, \\ x(t_1) = x_1, \end{array} \qquad (P)$$

$$\alpha_i \in \mathbb{R},\ J_i(x(\cdot)) := \int_{t_0}^{t_1} f_i(t, x, \dot{x})\, dt,\ i = 0, 1, \dots, m.$$

Definition 1.1 We shall say that functional $J_0(x)$ has a *weak local minimum* for $\widehat{x}(\cdot) \in C^1([t_0;\, t_1])$ $(\widehat{x}(\cdot) \in \mathrm{wlocmin}\, P)$ if there exists $\delta > 0$ such that $J_0(x(\cdot)) \geq J_0(\widehat{x}(\cdot))$ for any feasible function $x(\cdot)$ such that $\|x(\cdot) - \widehat{x}(\cdot)\|_{C^1([t_0;\, t_1])} < \delta$.

Definition 1.2 We shall say that functional $J_0(x)$ has a *strong local minimum* for $\widehat{x}(\cdot) \in PC^1([t_0;\, t_1])$ $(\widehat{x}(\cdot) \in \mathrm{strlocmin}\, P)$ if there exists $\delta > 0$ such that $J_0(x(\cdot)) \geq J_0(\widehat{x}(\cdot))$ for any feasible function $x(\cdot)$ such that $\|x(\cdot) - \widehat{x}(\cdot)\|_{C([t_0;\, t_1])} < \delta$.

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2 Definitions of Legendre condition, Jacobi condition and regularity condition

We shall suppose that for any $i = 0, \dots, m$ functions f_i are twice continuously differentiable in a neighborhood around $\Gamma_{\widehat{x}\dot{\widehat{x}}} = \{(t, x(t), \widehat{x}(t)) : t \in [t_0; t_1]\}$. Let $\widehat{x}(\cdot) \in C^2([t_0;\, t_1], \mathbb{R})$ be feasible extrema of (P) with Lagrange multiplier $\lambda_0 = 1$, i.e. it satisfies the Euler equation for $L = f_0 + \sum\limits_{i=1}^{m} \lambda_i f_i$ with Lagrange multiplier λ_i, $i = 1, \dots, m$,

$$-\frac{d}{dt}\widehat{L}_{\dot{x}} + \widehat{L}_x = 0.$$

As all f_i are twice continuously differentiable then functional $J(x)$ has second variation at the point $x = \widehat{x}$:

$$J''(\widehat{x})[h, h] = \int\limits_{t_0}^{t_1} \big(\widehat{L}_{\dot{x}\dot{x}}\dot{h}^2 + 2\widehat{L}_{x\dot{x}}\dot{h}h + \widehat{L}_{xx}h^2\big)\,dt.$$

Definition 2.1 *We say that the minimum problem on the extrema $\widehat{x}$ satises the Legendre condition if $\widehat{L}_{\dot{x}\dot{x}}(t) \geq 0\ \forall\, t \in [t_0;\, t_1]$ and the strong Legendre condition if $\widehat{L}_{\dot{x}\dot{x}}(t) > 0$ $\forall\, t \in [t_0;\, t_1]$.*

Definition 2.2 *We say that the minimum problem on the extrema $\widehat{x}$ satises regularity condition if functions $g_i(t) := -\frac{d}{dt}\widehat{f}_{i\dot{x}} + \widehat{f}_{ix}$ for all $i = 1, \dots, m$ are linearly independent on any segments $[t_0; \tau]$, $[\tau;\, t_1]$ for any $\tau \in [t_0;\, t_1]$*

Let's suppose that extrema $\widehat{x}$ satisfies the strong Legendre condition and regularity condition. If $\widehat{x} \in \operatorname{wlocmin} P$ then according to the necessary second-order condition in optimization problem with constraints second derivative of J is non-negative on feasible space, which is equivalent that function $\overline{h} \equiv 0 \in \operatorname{absmin} \widetilde{P}$, where

$$J_0''(\widehat{x})[h, h] \to \min;\ J_i'(\widehat{x})[h] = 0,\ i = 1, \dots, m;\ \begin{matrix} h(t_0) = 0, \\ h(t_1) = 0, \end{matrix} \qquad (\widetilde{P})$$

$$J_i'(\widehat{x})[h] = \int\limits_{t_0}^{t_1} \big(\widehat{f}_{i\dot{x}}\dot{h} + \widehat{f}_{ix}h\big)\,dt = \int\limits_{t_0}^{t_1} \Big(-\frac{d}{dt}\widehat{f}_{i\dot{x}} + \widehat{f}_{ix}\Big)h\,dt = \int\limits_{t_0}^{t_1} g_i h\,dt,$$

$g_i(t) := -\frac{d}{dt}\widehat{f}_{i\dot{x}} + \widehat{f}_{ix}$. According to Lagrange multiplier method for isoperimetric problem there exist non-zero Lagrange multipliers $\mu_0, \mu_1, \dots, \mu_m$ such that Lagrangian of problem $(\widetilde{P})$ $\widetilde{L} = \widetilde{L}(t, h, \dot{h}) := \mu_0\big(\widehat{L}_{\dot{x}\dot{x}}\dot{h}^2 + 2\widehat{L}_{x\dot{x}}\dot{h}h + \widehat{L}_{xx}h^2\big) + \sum\limits_{i=1}^{m} \mu_i g_i h$ satisfies the Euler equation.

The Euler equation of $\widetilde{L}$ with $\mu_0 = \frac{1}{2}$ $(\mu_0 \neq 0)$

$$-\frac{d}{dt}\big(\widehat{L}_{\dot{x}\dot{x}}\dot{h} + \widehat{L}_{\dot{x}x}h\big) + \widehat{L}_{x\dot{x}}\dot{h} + \widehat{L}_{xx}h + \sum_{i=1}^{m} \mu_i g_i = 0$$

is *the Jacobi equation* for initial problem (P). As there is a term $\sum\limits_{i=1}^{m} \mu_i g_i$ then the Jacobi equation is a second-order non-homogeneous linear dierential equation.

Definition 2.3 *A point τ is called conjugate to a point t_0 if there exist a nontrivial solution $h(\cdot)$ of the Jacobi equation such that*

$$\int_{t_0}^{\tau} g_i(t)h(t)\,dt = 0,\ i = 1,\dots,m,\ h(t_0) = h(\tau) = 0.$$

Definition 2.4 *The Jacobi condition is satised on the extrema $\widehat{x}$ if there are no points conjugate to t_0 in the interval $(t_0;t_1)$, and the strong Jacobi condition is satised if in the semi open interval $(t_0;t_1]$ there are no points conjugate to t_0.*

We shall consider analytical approach to find conjugate points. Let $h_0(\cdot)$ be the solution of the homogeneous Jacobi equation ($\mu_1 = \dots = \mu_m = 0$) with boundary condition $h_0(t_0) = 0$, $\dot h_0(t_0) = 1$ ($\dot h_0(t_0) \neq 0$); $h_i(\cdot)$ is the solution of the non-homogeneous Jacobi equation with $\mu_i = 1$ ($\mu_i \neq 0$), $\mu_j = 0$, $j \neq i$ and boundary conditions $h_i(t_0) = \dot h_i(t_0) = 0$, $i = 1,\dots,m$. We'll show that τ is conjugate to t_0 if and only if matrix

$$H(t) = \begin{pmatrix} h_0(t) & \dots & h_m(t) \\ \int_{t_0}^{t} g_1h_0ds & \dots & \int_{t_0}^{t} g_1h_mds \\ \dots & \dots & \dots \\ \int_{t_0}^{t} g_mh_0ds & \dots & \int_{t_0}^{t} g_mh_mds \end{pmatrix}$$

has determinant which is equal to zero. The determinant of matrix $H(t)$ is equal to zero in and only if columns of matrix are linearly dependent with non-zero coefficients $\alpha_0, \alpha_1, \dots, \alpha_m$. Then function $h = \sum_{i=0}^{m} \alpha_i h_i$ satisfies boundary conditions $h(\tau) = \sum_{i=0}^{m} \alpha_i h_i(\tau) = 0$; $\sum_{k=0}^{m} \alpha_k \int_{t_0}^{\tau} h_k g_i \iff \int_{t_0}^{\tau} g_i \sum_{k=0}^{m} \alpha_k = \int_{t_0}^{\tau} g_i h dt = 0,\ i = 1,\dots,m$. Herewith $h(t_0) = \sum_{i=0}^{m} \alpha_i h_i(t_0) = 0$. Then h at point τ satisfies definition of conjugate point.

3 Quadratic functional

$$J_0(x(\cdot)) = \int_{t_0}^{t_1} \big(A(t)\dot x^2(t) + B(t)x^2(t)\big)dt \to \min;$$

$$J_i(x(\cdot)) = \int_{t_0}^{t_1} (a_i\dot x + b_i x)dt = \alpha_i, \qquad i = 1,\dots,m,$$

$$x(t_0) = x_0, \qquad x(t_1) = x_1. \tag{P}$$

The main result of this paper is the following theorem about quadratic functional. Parts a) and d) were formulated in [1] with no proof. The essential results are parts b) and c) where necessary condition for extrema is satisfied (the Jacobi condition), but sufficient condition for extrema is not satisfied (the strong Jacobi condition).

Theorem 3.1 *Let the functions $A, a_1, \dots, a_m \in C^1([t_0; t_1])$, $B, b_1, \dots, b_m \in C([t_0; t_1])$, the strong Legendre condition for a minimum and regularity conditions be satisfied. Then*

$a)$ *if the strong Jacobi condition is satisfied, then the feasible extrema exists, is unique and $\hat{x} \in \operatorname{absmin} P$;*

$b)$ *if the Jacobi condition is satisfied but the strong Jacobi condition is not satisfied and feasible extrema exists, $\hat{x} \in \operatorname{absmin} P$;*

$c)$ *if the Jacobi condition is satisfied, but the strong Jacobi condition is not satisfied and the feasible extrema does not exist and*

$$G = A\dot{h}(t_1)x_1 - A\dot{h}(t_0)x_0 - \sum_{i=1}^{m}(x_0 a_i(t_0) - x_1 a_i(t_1) + \alpha_i) \neq 0,$$

then $S_{\text{absmin}} = -\infty$;

$d)$ *if the Jacobi condition is not satisfied, then $S_{\text{absmin}} = -\infty$.*

Proof. $a)$ We suppose that the strong Jacobi condition is satisfied. We shall prove that feasible extrema exists and is unique.

Existence. Let's consider case $m = 1$. We can find the extrema in the form of $\widehat{x} = C_0 h_0(t) + C_1 h_1(t) + C_2 h^1(t)$, here $h_0(t)$ is the solution of the homogeneous Jacobi equation such that $h_0(t_0) = 0$, $\dot{h}_0(t_0) = 1$. Note $\int\limits_{t_0}^{t_1} h_0(-\dot{a}_1 + b_1)dt = \int\limits_{t_0}^{t_1} h_0 g_1 dt := I_0$.

As $h_1(t)$ we take the solution of the non-homogeneous Jacobi equation (with $\mu_1 = 1$) such that $h_1(t_0) = \dot{h}_1(t_0) = 0$ (such solution exists by Existence and Uniqueness Theorem (EUT) for the second-order differential equations). Notice that either $h_1(t_1) \neq 0$ or $\int\limits_{t_0}^{t_1} h_1 g_1 dt := I_1 \neq 0$ or both are satisfied simultaneously as the strong Jacobi condition is satisfied. The strong Jacobi condition implies that

$$\mathrm{detH(t)} = \det\begin{pmatrix} h_0(t) & h_1(t) \\ \int\limits_{t_0}^{t} g_1 h_0 ds & \int\limits_{t_0}^{t} g_1 h_1 ds \end{pmatrix} \neq 0 \qquad \forall\, \mathrm{t} \in (\mathrm{t_0}, \mathrm{t_1}].$$

Then $h_0(t_1) \cdot I_1 \neq h_1(t_1) \cdot I_0$.

Let $h^1(t)$ be the solution of the non-homogeneous Jacobi equation such that $h^1(t_0) = \dot{h}^1(t_0) = 1$ (such solution exists by Existence and Uniqueness Theorem (EUT) for the second-order differential equations). Note $\int\limits_{t_0}^{t_1} h^1 g_1 dt := I^1$. We'll study two following cases (the third case is proven similarly):

1) $h_1(t_1) = 0 \Rightarrow h_0(t_1) \cdot I_1 \neq 0 \Rightarrow I_1 \neq 0,\ h_0(t_1) \neq 0.$

Let's take values of $\widehat{x}$ at points t_0 and t_1:

$\widehat{x}(t_0) = C_0 h_0(t_0) + C_1 h_1(t_0) + C_2 h^1(t_0) = C_2 = x_0$

$\widehat{x}(t_1) = C_0 h_0(t_1) + C_1 h_1(t_1) + C_2 h^1(t_1) = x_1 \Rightarrow$

$C_0 h_0(t_1) = x_1 - C_2 h^1(t_1) \Rightarrow C_0 = \dfrac{x_1 - x_0 h^1(t_1)}{h_0(t_1)},\ h_0(t_1) \neq 0.$

We can find constant C_1:

$$\int_{t_0}^{t_1} (a_1 \dot{\widehat{x}} + b_1 \widehat{x}) dt = C_0 \int_{t_0}^{t_1} (a_1 \dot{h}_0 + b_1 h_0) dt$$

$$+C_1 \int_{t_0}^{t_1} (a_1 \dot{h}_1 + b_1 h_1) dt + C_2 \int_{t_0}^{t_1} (a_1 \dot{h}^1 + b_1 h^1) dt$$

$$= C_0 \left(a_1 h_0 \Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} h_0(-\dot{a}_1 + b_1) dt \right)$$

$$+C_1 \left(a_1 h_1 \Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} h_1(-\dot{a}_1 + b_1) dt \right)$$

$$+C_2 \left(a_1 h^1 \Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} h^1(-\dot{a}_1 + b_1) dt \right)$$

$$= C_0(a_1(t_1) h_0(t_1) + I_0) + C_1 I_1 + C_2(a_1(t_1) h^1(t_1) - a_1(t_0) + I^1) = \alpha_1.$$

Now use that $C_0 = \dfrac{x_1 - x_0 h^1(t_1)}{h_0(t_1)},\ C_2 = x_0 \Rightarrow$

$$(x_1 - x_0 h^1(t_1)) \cdot a_1(t_1) + \frac{I_0}{h_0(t_1)}(x_1 - x_0 h^1(t_1)) + C_1 I_1$$

$$+x_0(a_1(t_1) h^1(t_1) - a_1(t_0) + I^1) = \alpha_1 \Rightarrow$$

$$C_1 = \frac{\alpha_1 - x_1 a_1(t_1) + x_0 a_1(t_0) - x_0 I^1 - \dfrac{I_0}{h_0(t_1)}(x_1 - x_0 h^1(t_1))}{I_1}.$$

As $I_1 \neq 0,\ h_0(t_1) \neq 0$ then can define $\widehat{x}(t)$

2) $I_1 = 0 \Rightarrow I_0 \neq 0,\ h_1(t_1) \neq 0.$

Take values of $\widehat{x}$ at points t_0 and t_1:

$\widehat{x}(t_0) = C_0 h_0(t_0) + C_1 h_1(t_0) + C_2 h^1(t_0) = C_2 = x_0$

$\widehat{x}(t_1) = C_0 h_0(t_1) + C_1 h_1(t_1) + C_2 h^1(t_1) = x_1 \Rightarrow$

$C_1 h_1(t_1) = x_1 - x_0 h^1(t_1) - C_0 h_0(t_1) \Rightarrow$

$C_1 = \dfrac{x_1 - x_0 h^1(t_1) - C_0 h_0(t_1)}{h_1(t_1)},\ h_1(t_1) \neq 0.$

We can find constant C_0:

$$\int_{t_0}^{t_1}(a_1\dot{\widehat{x}}+b_1\widehat{x})dt = C_0\int_{t_0}^{t_1}(a_1\dot{h}_0+b_1h_0)dt$$

$$+C_1\int_{t_0}^{t_1}(a_1\dot{h}_1+b_1h_1)dt + C_2\int_{t_0}^{t_1}(a_1\dot{h}^1+b_1h^1)dt$$

$$= C_0\left(a_1h_0\Big|_{t_0}^{t_1}+I_0\right)+C_1\left(a_1h_1\Big|_{t_0}^{t_1}+I_1\right)+C_2\left(a_1h^1\Big|_{t_0}^{t_1}+I^1\right)$$

$$= C_0(a_1(t_1)h_0(t_1)+I_0)+C_1a_1(t_1)h_1(t_1)$$
$$+x_0(a_1h^1(t_1)-a_1(t_0)+I^1)=\alpha_1$$

$$C_0a_1(t_1)h_0(t_1)+C_0I_0+a_1(t_1)(x_1-x_0h^1(t_1)-C_0h_0(t_1))$$

$$+x_0a_1h^1(t_1)-x_0a_1(t_0)+x_0I^1 = C_0I_0+a_1(t_1)x_1-a_1(t_1)x_0h^1(t_1)$$

$$+x_0a_1(t_1)h^1(t_1)-x_0a_1(t_0)+x_0I^1$$
$$= C_0I_0+a_1(t_1)x_1-x_0a_1(t_0)+x_0I^1=\alpha_1$$

$$\Rightarrow C_0=\frac{\alpha_1+x_0a_1(t_0)-x_1a_1(t_1)-x_0I^1}{I_0},\quad I_0\neq 0$$

so we can define $\widehat{x}(t)$.
We can use the similar idea to find feasible extrema in a case of m isoperimetric constraints: $\widehat{x}(t) = C_0h_0(t)+\ldots+C_mh_m(t)+C^1h^1(t)$, here $h_0(t)$ is the solution of the homogeneous Jacobi equation such that $h_0(t_0)=0, \dot{h}_0(t_0)=1;$ $h_i(t)$ is the solution of the non-homogeneous Jacobi equation with $\mu_i=1$ ($\mu_j=0,\ i\neq j$) such that $h_i(t_0)=\dot{h}_i(t_0)=0$, $i=1,\ldots,m$.

Uniqueness. Let $\overline{x}$ be different feasible extrema. Then $h=\widehat{x}-\overline{x}$ is non-trivial solution of the Jacobi equation such that $h(t_0)=h(t_1)=0$,

$$\int_{t_0}^{t_1} g_ih\,dt = \int_{t_0}^{t_1}(-\dot{a}_i+b_i)h\,dt$$

$$= \int_{t_0}^{t_1}(a_i\dot{h}+b_ih)dt = \int_{t_0}^{t_1}\big(a_i(\dot{\widehat{x}}-\dot{\overline{x}})+b_i(\widehat{x}-\overline{x})\big)dt$$

$$= \int_{t_0}^{t_1}(a_i\dot{\widehat{x}}+b_i\widehat{x})dt - \int_{t_0}^{t_1}(a_i\dot{\overline{x}}+b_i\overline{x})dt = \alpha_i-\alpha_i=0,$$

here $i = 1, \ldots, m$. Then $\tau = t_1$ is conjugate point to t_0 and it contradicts the strong Jacobi condition.

By strong Legendre and Jacobi condition feasible extrema can be imbedded in a central field of extremals. Let $x \in C^1([t_0;\, t_1])$ be arbitrary feasible function. Then by the formula for the Weierstrass E-Function in quadratic case

$$J(x) - J(\widehat{x}) = \int\limits_{t_0}^{t_1} A(\dot{x} - u)^2\, dt \geq 0,$$

here $A(t) > 0\, \forall\, t \in [t_0;\, t_1]$ as the strong Legendre condition is satisfied. Thus, $\widehat{x} \in \ \mathrm{absmin}\, P$.

b) Let the Jacobi condition be satisfied, but the strong Jacobi condition isn't satisfied and a feasible extrema $\widehat{x}$ exists. The extrema $\hat{x}(t)$ can be imbedded in a central field of extremals that covers $t_0 \leq t \leq t_1 - \varepsilon$. Let $x_\varepsilon \in C^1([t_0, t_1 - \varepsilon])$ be arbitrary function such that $x_\varepsilon(t_0) = x_0,\ x_\varepsilon(t_1 - \varepsilon) = \hat{x}(t_1 - \varepsilon)$ and $\int\limits_{t_0}^{t_1-\varepsilon} (a_i \dot{x}_\varepsilon + b_i x_\varepsilon) dt = \int\limits_{t_0}^{t_1-\varepsilon} (a_i \dot{\hat{x}} + b_i \hat{x}) dt$, $i = 1, \ldots, m$. Then the function $h = x_\varepsilon - \hat{x}$ satisfies the following conditions:

$$h(t_0) = x_\varepsilon(t_0) - \hat{x}(t_0) = x_0 - x_0 = 0,$$
$$h(t_1 - \varepsilon) = x_\varepsilon(t_1 - \varepsilon) - \hat{x}(t_1 - \varepsilon) = 0,$$

$$\int\limits_{t_0}^{t_1-\varepsilon} (a_i \dot{h} + b_i h) dt = \int\limits_{t_0}^{t_1-\varepsilon} (a_i \dot{x}_\varepsilon + b_i x_\varepsilon) dt - \int\limits_{t_0}^{t_1-\varepsilon} (a_i \dot{\hat{x}} + b_i \hat{x}) dt = 0, \qquad i = 1, \ldots, m.$$

Hence $h \in \mathit{\Pi}_\varepsilon$ — feasible space. Here

$$\mathit{\Pi}_\varepsilon = \{h \in C_0^1[t_0, t_1 - \varepsilon] : \int\limits_{t_0}^{t_1-\varepsilon} (a_i \dot{h} + b_i h) dt = 0\}.$$

According to the formula for the Weierstrass E-Function in quadratic case

$$J_0(x_\varepsilon) - J_0(\hat{x}) = \int\limits_{t_0}^{t_1-\varepsilon} A(\dot{x}_\varepsilon - u)^2 dt \geq 0.$$

Taking limit $\varepsilon \to 0$, we obtain that $J(x) - J(\hat{x}) \geq 0$. Thus, $\hat{x} \in \mathrm{absmin}\, P$.

c) Assume that the Jacobi condition is satisfied, but the strong Jacobi condition is not satisfied and feasible extrema doesn't exist. Let's suppose that the feasible space is non-empty. Then we can take any feasible function $\tilde{x}$ such that $\tilde{x}(t_0) = x_0$, $\tilde{x}(t_1) = x_1$ and

$$\int\limits_{t_0}^{t_1} (a_i \dot{\tilde{x}} + b_i \tilde{x}) dt = \alpha_i, \qquad i = 1, \ldots, m.$$

Using Taylor's theorem we obtain that

$$J(\tilde{x} + h) = J(\tilde{x}) + J'(\tilde{x})[h] + \frac{1}{2} J''[h, h]. \tag{3.1}$$

Here h is the solution of the Jacobi equation such that $h(t_0) = h(t_1) = 0$ and

$$\int_{t_0}^{t_1} (a_i \dot{h} + b_i h) dt = \int_{t_0}^{t_1} a_i dh + \int_{t_0}^{t_1} b_i h dt$$

$$= a_i h \Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} h(-\dot{a}_i + b_i) dt = \int_{t_0}^{t_1} h(-\dot{a}_i + b_i) dt = 0, \tag{3.2}$$

$i = 1, \ldots, m$. We can find solution that complies with these boundary conditions and isoperimetric constraints as the Jacobi condition is satisfied, but the strong Jacobi condition is not satisfied. Now we shall calculate the second variation of the functional $J(\cdot)$ at the point h.

$$\frac{1}{2} J''[h,h] = \int_{t_0}^{t_1} (A\dot{h}^2 + Bh^2) dt = \int_{t_0}^{t_1} A\dot{h} dh + \int_{t_0}^{t_1} Bh^2 dt$$

$$= A\dot{h}h \Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} \left(-h \frac{d}{dt}(A\dot{h}) + Bh^2 \right) dt$$

$$= \int_{t_0}^{t_1} h \left(-\frac{d}{dt}(A\dot{h}) + Bh \right) dt,$$

here the first term after integration by parts is equal to zero as $h(t_0) = h(t_1) = 0$. Using the fact that h is the solution of the Jacobi equation, we obtain that

$$-\frac{d}{dt}(A\dot{h}) + Bh = -\sum_{i=1}^{m} (-\dot{a}_i + b_i). \tag{3.3}$$

Thus,

$$\frac{1}{2} J''[h,h] = -\sum_{i=1}^{m} \int_{t_0}^{t_1} (-\dot{a}_i + b_i) h dt = -\sum_{i=1}^{m} \int_{t_0}^{t_1} (a_i \dot{h} + b_i h) dt = 0,$$

by (3.2). Now we can rewrite (3.1) in the form of $J(\tilde{x} + h) - J(\tilde{x}) = J'(\tilde{x})[h]$. We'll use integration by parts to obtain formula for $J'(\tilde{x})[h]$:

$$J'(\tilde{x})[h] = 2 \int_{t_0}^{t_1} (A\dot{\tilde{x}}\dot{h} + B\tilde{x}h) dt$$

$$= 2 \int_{t_0}^{t_1} A\dot{h} d\tilde{x} + 2 \int_{t_0}^{t_1} B\tilde{x}h dt = 2(A\dot{h}\tilde{x}) \Big|_{t_0}^{t_1}$$

$$+2 \int_{t_0}^{t_1} \left(-\frac{d}{dt}(A\dot{h}) + Bh \right) \tilde{x} dt = 2(A\dot{h}(t_1)x_1 - A\dot{h}(t_0)x_0) - 2\sum_{i=1}^{m} \int_{t_0}^{t_1} (-\dot{a}_i + b_i)\tilde{x} dt.$$

The last formula is obtained by (3.3). Integrate by parts the following expression:

$$\sum_{i=1}^{m}\int_{t_0}^{t_1}(-\dot{a}_i\tilde{x}+b_i\tilde{x})dt=\sum_{i=1}^{m}\left(\int_{t_0}^{t_1}-\tilde{x}da_i+\int_{t_0}^{t_1}b_i\tilde{x}dt\right)$$

$$=\sum_{i=1}^{m}\left(x_0a_i(t_0)-x_1a_i(t_1)+\int_{t_0}^{t_1}(a_i\dot{\tilde{x}}+b_i\tilde{x})dt\right)$$

$$=\sum_{i=1}^{m}(x_0a_i(t_0)-x_1a_i(t_1)+\alpha_i)\equiv C.$$

Then $J(\tilde{x}+h)-J(\tilde{x})=2(G-C)$. Consequently, $J(\tilde{x}+\lambda h)=J(\tilde{x})+2\lambda(G-C)$. According to the condition of theorem 3.1 $G\neq C$. Thus, $J(\tilde{x}+\lambda h)\to-\infty$ as $\lambda\to+\infty$ or $\lambda\to-\infty$.

d) Assume that the Jacobi condition isn't satisfied. Then according to the necessary conditions for weak local minimum function $\overline{h}\equiv 0\notin \operatorname{absmin} P''$ in the following problem:

$$\int_{t_0}^{t_1}(A\dot{h}^2+Bh^2)dt\to\min;\quad \int_{t_0}^{t_1}(a_i\dot{h}+b_ih)dt=0,\ i=1,\dots,m,$$

$$h(t_0)=h(t_1)=0.(P'')$$

Hence, $S_{\operatorname{absmin} P''}<0$. Consequently, there exists function $h\in\mathit{\Pi}$ such that $J(h)<0$ and we obtain that

$$J(\widehat{x}+\lambda h)=J(\widehat{x})+J(\lambda h)=J(\widehat{x})+\lambda^2J(h)\to-\infty$$

as $\lambda\to+\infty$, i.e. $S_{\operatorname{absmin} P}=-\infty$.

Theorem 3.1 is proved.

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