

An analog of Titchmarsh's and Younis's theorems for the q -Bessel Fourier transform in the space $\mathcal{L}_{q,\alpha}^p(\mathbb{R}^+)$

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Abstract. In this paper by using a q -translation operator, we give an analog of Titchmarsh's theorem and Younis's theorem for q -Bessel Fourier transform satisfying q -Bessel-Lipschitz and q -Bessel-Dini-Lipschitz conditions in the space $\mathcal{L}_{q,\alpha}^p(\mathbb{R}^+)$, where $1 < p \leq 2$.

Keywords. q -Bessel operator, q -Bessel Fourier transform, q -translation operator, q -Bessel-Lipschitz class, q -Bessel-Dini-Lipschitz class

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1 Introduction

In the recent mathematical literature one finds many articles which deal with the theory of q -Fourier analysis associated with the q -Hankel transform. This theory was elaborated first by Koornwinder and R.F.Swarttouw [13] and then by Fitouhi and Al [7]. They were interested in q -analogue of different integral transformations. In connection with q -difference Bessel operator and with the basic Bessel functions, they introduced several generalized q -Fourier transform. So, it is natural to look for the q -analogue of some well-known classical theorems.

Titchmarsh ([15], Theorem 84) characterized the set of functions in $L^p(\mathbb{R})$, $1 < p \leq 2$, satisfying the q -Bessel-Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transform, namely we have

Theorem 1.1 [15] Let f belong to $L^p(\mathbb{R})$, $1 < p \leq 2$, such that

$$\int_{-\infty}^{+\infty} |f(x+h) - f(x-h)|^p dx = O(h^{\alpha p}), \quad 0 < \alpha \leq 1 \quad \text{as } h \rightarrow 0.$$

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Then, its Fourier transform $\mathcal{F}(f)$ belong to $L^\beta(\mathbb{R})$ for

$$\frac{p}{p + \alpha p - 1} < \beta \leq \frac{p}{p - 1}.$$

On the other hand, Younis in ([16], Theorem 3.3) studied the same phenomena for the wider Dini-Lipschitz class as well as for some other allied classes of functions. More precisely

Theorem 1.2 [16] Let $f \in L^p(\mathbb{R})$ with $1 < p \leq 2$, such that

$$\left(\int_{-\infty}^{+\infty} |f(x+h) - f(x)|^p dx \right)^{\frac{1}{p}} = O \left(\frac{h^\alpha}{(\log \frac{1}{h})^\gamma} \right), \quad h \rightarrow 0, \quad 0 < \alpha \leq 1, \quad \gamma > 0.$$

Then $\mathcal{F}(f) \in L^\beta(\mathbb{R})$ for

$$\frac{p}{p + \alpha p - 1} \leq \beta < p' = \frac{p}{p - 1}$$

and

$$\frac{1}{\beta} < \gamma,$$

where $\mathcal{F}(f)$ stands for the Fourier transform of f .

The main aim of this paper is to generalize these theorems for the q -Bessel Fourier transform setting by means of the q -translation operator.

In recent years, these two results have been generalized in several different versions and for several different types of transform (for exemple, see [2, 3, 8, 14]).

2 Preliminaries and auxiliary results

In the first we collect some definitions, notations and properties of the q -shifted factorials, the q -hypergeometric functions, the Jackson's q -derivative and the Jackson's q -integrals (see [10, 12]). Throughout this paper, we assume that $0 < q < 1$ and $\alpha > -\frac{1}{2}$. we denote by

$$\mathbb{R}_q^+ = \{q^n, n \in \mathbb{Z}\}.$$

Let $x \in \mathbb{C}$, the q -shifted factorials are defined by

$$(x; q)_0 = 1, \quad (x; q)_n = \prod_{k=0}^{n-1} (1 - xq^k), \quad n = 1, 2, \dots, \quad (x; q)_\infty = \prod_{k=0}^{\infty} (1 - xq^k)$$

and for $a \in \mathbb{C}$ and $n \in \mathbb{N}$ we also denote

$$[a]_q = \frac{1 - q^a}{1 - q}, \quad [n]_q! = \frac{(q; q)_n}{(1 - q)^n}.$$

The q -derivative of a function f is here defined by

$$\mathcal{D}_q f(x) = \frac{f(x) - f(qx)}{(1 - q)x} \quad \text{if } x \neq 0$$

$\mathcal{D}_q f(0) = f'(0)$ provided $f'(0)$ exists.

We also consider

$$\mathcal{D}_q^+ f(x) = q^{-1} \mathcal{D}_q f(q^{-1}x).$$

For all $s \in \mathbb{R}$ and $x \in \mathbb{R}_q^+$, we have

$$\mathcal{D}_q x^s = [s]_q x^{s-1} \quad (2.1)$$

and

$$\mathcal{D}_q^+ x^s = q^{-1} [s]_q x^{s-1} = -[-s]_q x^{s-1}. \quad (2.2)$$

In [11], the q-Jackson integrals from 0 to a , from a to b , from 0 to $+\infty$ and from $-\infty$ to $+\infty$ are defined by

$$\begin{aligned} \int_0^a f(x) d_q x &= (1-q)a \sum_{n=0}^{+\infty} q^n f(aq^n), \\ \int_a^b f(x) d_q x &= \int_0^b f(x) d_q x - \int_0^a f(x) d_q x, \\ \int_0^{+\infty} f(x) d_q x &= (1-q) \sum_{n=-\infty}^{+\infty} q^n f(q^n), \\ \int_{-\infty}^{+\infty} f(x) d_q x &= (1-q) \sum_{n=-\infty}^{+\infty} q^n [f(q^n) + f(-q^n)]. \end{aligned}$$

The q-analogue of the integration theorem by a change of variable can be stated as follows

$$\int_a^b g\left(\frac{\lambda}{r}\right) \lambda^{2\alpha+1} d_q \lambda = r^{2\alpha+2} \int_{\frac{a}{r}}^{\frac{b}{r}} g(t) t^{2\alpha+1} d_q \lambda \quad \forall r \in \mathbb{R}_q^+. \quad (2.3)$$

The q-integration by part formulas associated with \mathcal{D}_q and \mathcal{D}_q^+ are given by

$$\int_a^b g(x) \mathcal{D}_q f(x) d_q x = [f(b)g(b) - f(a)g(a)] - \int_a^b f(qx) \mathcal{D}_q g(x) d_q x. \quad (2.4)$$

$$\int_a^b g(q^{-1}x) \mathcal{D}_q^+ f(x) d_q x = [f(q^{-1}b)g(q^{-1}b) - f(q^{-1}a)g(q^{-1}a)] - \int_a^b f(x) \mathcal{D}_q^+ g(x) d_q x. \quad (2.5)$$

Note that for any function f we can write

$$f = f_e + f_o$$

where f_e and f_o are respectively, the even and the odd parts of f defined by

$$f_e = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f_o = \frac{f(x) - f(-x)}{2}$$

Now, we briefly collect the pertinent definitions and facts relevant for q-Bessel Fourier transform (see [5, 7, 9, 13]).

In [1] the normalized third Jackson q-Bessel function of order α is defined by

$$j_\alpha(x, q^2) = \sum_{n=0}^{+\infty} (-1)^n \frac{\Gamma_{q^2}(\alpha+1) q^{n(n+1)}}{\Gamma_{q^2}(\alpha+n+1) \Gamma_{q^2}(n+1)} \left(\frac{x}{1+q}\right)^{2n}, \quad (2.6)$$

where Γ_q is the q-gamma function defined for $x \in \mathbb{R}_q^+$ by

$$\Gamma_q = \frac{(q, q)_\infty}{(q^x, q)_\infty} (1-q)^{1-x}.$$

The formula (6) with a simple calculation implies that

$$\lim_{x \rightarrow 0} \frac{1 - j_\alpha(x; q^2)}{x^2} = \frac{1}{[\alpha + 1]_{q^2}} \left(\frac{q}{q + 1} \right)^2 \neq 0,$$

hence, there exists $C > 0$ and $\eta > 0$ satisfying

$$|x| \leq \eta \implies |1 - j_\alpha(x, q^2)| \geq Cx^2. \quad (2.7)$$

The function $x \mapsto j_\alpha(\lambda x, q^2)$ is a solution of the following q-differential equation

$$\Lambda_{q,\alpha} f(x) = -\lambda^2 f(x)$$

where $\Lambda_{q,\alpha}$ is the q-Bessel operator

$$\Lambda_{q,\alpha} f(x) = \frac{1}{x^2} [f(q^{-1}x) - (1 + q^{2\alpha})f(x) + q^{2\alpha}f(qx)].$$

For $1 \leq p < \infty$ we denote by $\mathcal{L}_{q,\alpha}^p$ the space of functions defined on \mathbb{R}_q^+ such that

$$\|f\|_{q,p,\alpha} = \left(\int_0^{+\infty} |f(x)|^p x^{2\alpha+1} d_q x \right)^{\frac{1}{p}}.$$

exist.

We denote by $C_{q,0}(\mathbb{R}_q^+)$ the space of functions defined on \mathbb{R}_q^+ tending to 0 as $x \rightarrow \infty$ and continuous at 0 equipped with the topology of uniform convergence. The space $C_{q,0}(\mathbb{R}_q^+)$ is complete with respect to the norm

$$\|f\|_{q,\infty} = \sup_{x \in \mathbb{R}_q^+} |f(x)|.$$

The q-Bessel Fourier transform $\mathcal{F}_{q,\alpha}$ is defined by [5, 7, 13]

$$\mathcal{F}_{q,\alpha} f(x) = C_{q,\alpha} \int_0^{+\infty} f(t) j_\alpha(xt, q^2) t^{2\alpha+1} d_q t \quad \forall x \in \mathbb{R}_q^+$$

where

$$C_{q,\alpha} = \frac{1}{1-q} \frac{(q^{2\alpha+2}; q^2)_\infty}{(q^2; q^2)_\infty}.$$

From [5] we have the following result

Proposition 2.1 *The q-Bessel Fourier transform satisfies*

i) *If $f \in \mathcal{L}_{q,\alpha}^1$, then $\mathcal{F}_{q,\alpha} f \in C_{q,0}$ and we have*

$$\|\mathcal{F}_{q,\alpha} f\|_{q,\infty} \leq B_{q,\alpha} \|f\|_{q,1,\alpha} \quad (2.8)$$

where

$$B_{q,\alpha} = \frac{1}{1-q} \frac{(-q^2; q^2)_\infty (-q^{2\alpha+2}; q^2)_\infty}{(q^2; q^2)_\infty}.$$

ii) *For all function $f \in \mathcal{L}_{q,\alpha}^p$*

$$\mathcal{F}_{q,\alpha}^2 f = f. \quad (2.9)$$

iii) *For all function $f \in \mathcal{L}_{q,\alpha}^2$*

$$\|\mathcal{F}_{q,\alpha} f\|_{q,2,\alpha} = \|f\|_{q,2,\alpha}. \quad (2.10)$$

Proposition 2.2 Let $f \in \mathcal{L}_{q,\alpha}^p$ where $p \geq 1$, then $\mathcal{F}_{q,\alpha}f \in \mathcal{L}_{q,\alpha}^{p'}$. Also if $1 \leq p \leq 2$, then

$$\|\mathcal{F}_{q,\alpha}f\|_{q,p',\alpha} \leq B_{q,\alpha}^{\frac{2}{p}-1} \|f\|_{q,p,\alpha} \quad (2.11)$$

where $\frac{1}{p} + \frac{1}{p'} = 1$

Proof. This is an immediate consequence of formulas (2.8), (2.10), the Riesz-Thorin theorem and the inversion formula (2.9).

The q -translation operator is given as follow

$$T_{q,x}^\alpha f(y) = C_{q,\alpha} \int_0^{+\infty} \mathcal{F}_{q,\alpha}(f)(t) j_\alpha(yt, q^2) j_\alpha(xt, q^2) t^{2\alpha+1} d_q t.$$

Let us now introduce

$$Q_\alpha = \{q \in]0, 1[, T_{q,x}^\alpha \text{ is positive for all } x \in \mathbb{R}_q^+\}$$

the set of the positivity of $T_{q,x}^\alpha$. We recall that $T_{q,x}^\alpha$ is called positive if $T_{q,x}^\alpha f \geq 0$ for $f \geq 0$. In a recent paper [6] it was proved that if $-1 < \alpha < \alpha'$ then $Q_\alpha \subset Q_{\alpha'}$. As a consequence:

- if $0 \leq \alpha$ then $Q_\alpha =]0, 1[$.
- if $-\frac{1}{2} < \alpha < 0$ then $]0, q_0[\subset Q_{-\frac{1}{2}} \subset Q_\alpha \subset]0, 1[$, $q \simeq 0.43$.
- if $-1 < \alpha < -\frac{1}{2}$ then $Q_\alpha \subset Q_{-\frac{1}{2}}$. (we don't have the information if this subset is empty or not).

Proposition 2.3 For any function $f \in \mathcal{L}_{q,\alpha}^2$ we have

$$\mathcal{F}_{q,\alpha}(T_{q,x}^\alpha f)(\lambda) = j_\alpha(\lambda x, q^2) \mathcal{F}_{q,\alpha}f(\lambda) \quad \text{for all } \lambda, x \in \mathbb{R}_q^+. \quad (2.12)$$

For $f \in \mathcal{L}_{q,\alpha}^p$, $1 < p \leq 2$, we define the finite differences of the first order and step $h > 0$, $h \in \mathbb{R}_q^+$ by

$$\Delta_{q,h}f(x) = T_{q,h}^\alpha f(x) - f(x) = (T_{q,h}^\alpha - I)f(x)$$

where I is the unit operator in $\mathcal{L}_{q,\alpha}^p$.

Lemma 2.1 For any function $f \in \mathcal{L}_{q,\alpha}^p$, $1 < p \leq 2$, we have

$$\int_0^{+\infty} |1 - j_\alpha(\lambda h, q^2)|^{p'} |\mathcal{F}_{q,\alpha}f(\lambda)|^{p'} |\lambda|^{2\alpha+1} d_q \lambda \leq C_1 \|\Delta_{q,h}f\|_{q,p,\alpha}^{p'}.$$

Proof. By formula (2.12) we have

$$\begin{aligned} \mathcal{F}_{q,\alpha}(\Delta_{q,h}f)(\lambda) &= \mathcal{F}_{q,\alpha}(T_{q,h}^\alpha f - f)(\lambda) \\ &= \mathcal{F}_{q,\alpha}(T_{q,h}^\alpha f)(\lambda) - \mathcal{F}_{q,\alpha}(f)(\lambda) \\ &= j_\alpha(\lambda h, q^2) \mathcal{F}_{q,\alpha}(f)(\lambda) - \mathcal{F}_{q,\alpha}(f)(\lambda) \\ &= (j_\alpha(\lambda h, q^2) - 1) \mathcal{F}_{q,\alpha}(f)(\lambda). \end{aligned}$$

Using formula (2.11) we obtain our result.

Lemma 2.2 For any function f defined on \mathbb{R}_q^+ , we have

$$\mathcal{D}_q \left[\int_a^x f(t) d_q t \right]_o = f_e(x) \quad (2.13)$$

and

$$\mathcal{D}_q^+ \left[\int_a^x f(t) d_q t \right]_e = q^{-1} f_o(q^{-1}x) \quad (2.14)$$

where $x \mapsto \left[\int_a^x f(t) d_q t \right]_o$ and $x \mapsto \left[\int_a^x f(t) d_q t \right]_e$ are respectively, the odd and the even part of $x \mapsto \left[\int_a^x f(t) d_q t \right]$.

Proof. See Lemma 3.2 in [4].

3 Main results

Before giving our main result, we define, first, the q -Bessel-Lipschitz class.

Definition 3.1 Let $0 < \delta < 1$. A function $f \in \mathcal{L}_{q,\alpha}^p$, $1 < p \leq 2$ is said to be in the q -Bessel-Lipschitz class, denoted $q\text{-BLip}(\delta, p, \alpha)$ if

$$\|\Delta_{q,h} f\|_{q,p,\alpha} = O(h^\delta) \quad \text{as } h \rightarrow 0.$$

Theorem 3.1 For $f \in \mathcal{L}_{q,\alpha}^p$ where $1 < p \leq 2$. If f in $q\text{-BLip}(\delta, p, \alpha)$, then $\mathcal{F}_{q,\alpha}(f) \in \mathcal{L}_{q,\alpha}^\beta(\mathbb{R}_q^+)$ where

$$\frac{2p\alpha + 2p}{2p + 2\alpha(p-1) + \delta p - 2} < \beta \leq p' = \frac{p}{p-1}.$$

Proof. If $\beta = p'$ we have by the formula (2.11) that $\mathcal{F}_{q,\alpha}(f) \in \mathcal{L}_{q,\alpha}^{p'}$. And for $\alpha > -\frac{1}{2}$, $0 < \delta < 1$ we get

$$\begin{aligned} 1 + \frac{\delta p'}{2\alpha + 2} &> 1 \\ \iff \frac{p-1}{p} \left(\frac{2\alpha + 2 + \delta p'}{2\alpha + 2} \right) &> \frac{p-1}{p} \\ \iff \frac{(2\alpha + 2)(p-1) + \delta p}{p(2\alpha + 2)} &> \frac{1}{p'} \\ \iff \frac{p(2\alpha + 2)}{(2\alpha + 2)(p-1) + \delta p} &< \beta = p' = \frac{p}{p-1} \\ \iff \frac{2p\alpha + 2p}{2p + 2\alpha(p-1) + \delta p - 2} &< \beta = p' = \frac{p}{p-1}, \end{aligned}$$

then the theorem is proved in the case where $\beta = p'$.

In what follows we assume that $\beta < p'$ and $f \in q\text{-BLip}(\delta, p, \alpha)$, then we have

$$\|\Delta_{q,h} f\|_{q,p,\alpha} = O(h^\delta) \quad \text{as } h \rightarrow 0.$$

The Lemma 2.1 yields

$$\begin{aligned} \int_0^{+\infty} |1 - j_\alpha(\lambda h, q^2)|^{p'} |\mathcal{F}_{q,\alpha}(f)(\lambda)|^{p'} |\lambda|^{2\alpha+1} d_q \lambda &\leq C_1 \|\Delta_{q,h} f\|_{q,p,\alpha}^{p'} \\ &\leq C_2 h^{\delta p'}. \end{aligned}$$

If $0 < \lambda < \frac{\eta}{h}$, then $0 < \lambda h < \eta$ and inequality (2.7) implies that

$$|1 - j_\alpha(\lambda h, q^2)| \geq C \lambda^2 h^2.$$

Therefore

$$\begin{aligned} &\int_0^{\frac{\eta}{h}} h^{2p'} \lambda^{2p'} |\mathcal{F}_{q,\alpha}(f)(\lambda)|^{p'} |\lambda|^{2\alpha+1} d_q \lambda \\ &\leq \frac{1}{C^{p'}} \int_0^{\frac{\eta}{h}} |1 - j_\alpha(\lambda h, q^2)|^{p'} |\mathcal{F}_{q,\alpha}(f)(\lambda)|^{p'} |\lambda|^{2\alpha+1} d_q \lambda \\ &\leq \frac{1}{C^{p'}} \int_0^{+\infty} |1 - j_\alpha(\lambda h, q^2)|^{p'} |\mathcal{F}_{q,\alpha}(f)(\lambda)|^{p'} |\lambda|^{2\alpha+1} d_q \lambda \\ &= O(h^{\delta p'}). \end{aligned}$$

Then

$$\int_0^{\frac{\eta}{h}} \lambda^{2p'} |\mathcal{F}_{q,\alpha}(f)(\lambda)|^{p'} |\lambda|^{2\alpha+1} d_q \lambda = O\left(h^{(\delta-2)p'}\right) \quad \text{as } h \rightarrow 0.$$

Thus

$$\int_0^X \lambda^{2p'} |\mathcal{F}_{q,\alpha}(f)(\lambda)|^{p'} |\lambda|^{2\alpha+1} d_q \lambda = O\left(X^{(2-\delta)p'}\right) \quad \text{as } X \rightarrow +\infty.$$

Let

$$\varphi(X) = \int_1^X |\lambda^2 \mathcal{F}_{q,\alpha}(f)(\lambda)|^\beta |\lambda|^{(2\alpha+1)\beta/p'} d_q \lambda. \quad (3.1)$$

Taking into account the Hölder inequality yields

$$\begin{aligned} \varphi(X) &\leq \left(\int_1^X |\lambda^2 \mathcal{F}_{q,\alpha}(f)(\lambda)|^{p'} |\lambda|^{(2\alpha+1)} d_q \lambda \right)^{\beta/p'} \left(\int_1^X d_q \lambda \right)^{(p'-\beta)/p'} \\ &= O\left(X^{(2-\delta)p' \times \frac{\beta}{p'}} X^{\frac{p'-\beta}{p'}} \right) \\ &= O\left(X^{2\beta - \delta\beta + 1 - \frac{\beta}{p'}} \right). \end{aligned}$$

Let us now estimate the next integral

$$\int_1^X |\mathcal{F}_{q,\alpha}(f)(\lambda)|^\beta |\lambda|^{2\alpha+1} d_q \lambda.$$

This integral is split into two

$$\int_1^X |\mathcal{F}_{q,\alpha}(f)(\lambda)|^\beta |\lambda|^{2\alpha+1} d_q \lambda = I_1 + I_2,$$

where

$$I_1 = \int_1^X \left[|\mathcal{F}_{q,\alpha}(f)(\lambda)|^\beta \right]_e |\lambda|^{2\alpha+1} d_q \lambda$$

and

$$I_2 = \int_1^X \left[|\mathcal{F}_{q,\alpha}(f)(\lambda)|^\beta \right]_o |\lambda|^{2\alpha+1} d_q \lambda.$$

Estimate the summands I_1 and I_2 from above. It follows from formula (3.1) and Lemma 2.2 that

$$\mathcal{D}_q \varphi_o(\lambda) = |\lambda|^{2\beta+(2\alpha+1)\frac{\beta}{p'}} \left[|\mathcal{F}_{q,\alpha}(f)(\lambda)|^\beta \right]_e, \quad (3.2)$$

and

$$\mathcal{D}_q^+ \varphi_e(\lambda) = q^{-1} |q^{-1} \lambda|^{2\beta+(2\alpha+1)\frac{\beta}{p'}} \left[|\mathcal{F}_{q,\alpha}(f)(q^{-1} \lambda)|^\beta \right]_o. \quad (3.3)$$

Using the formula (2.1), the q-integration by parts formula (2.4) and (3.2), we get

$$\begin{aligned} I_1 &= \int_1^X \left[|\mathcal{F}_{q,\alpha}(f)(\lambda)|^\beta \right]_e |\lambda|^{2\alpha+1} d_q \lambda \\ &= \int_1^X \lambda^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+1} \mathcal{D}_q \varphi_o(\lambda) d_q \lambda \\ &= X^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+1} \varphi_o(X) - \varphi_o(1) - \int_1^X \varphi_o(q\lambda) \mathcal{D}_q \left(\lambda^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+1} \right) d_q \lambda \\ &= X^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+1} \varphi_o(X) - \varphi_o(1) - [(2\alpha+1)(1-\beta/p') - 2\beta]_q \\ &\quad \times \int_1^X \varphi_o(q\lambda) \lambda^{2\alpha-2\beta-(2\alpha+1)\frac{\beta}{p'}} d_q \lambda. \end{aligned} \quad (3.4)$$

Furthermore, it follows from (2.2), the q-integration by parts formula (2.5), (2.3) and (3.3) that

$$\begin{aligned} I_2 &= \int_1^X \left[|\mathcal{F}_{q,\alpha}(f)(\lambda)|^\beta \right]_o |\lambda|^{2\alpha+1} d_q \lambda \\ &= \int_q^{qX} q^{-1} \left[|\mathcal{F}_{q,\alpha}(f)(q^{-1} \lambda)|^\beta \right]_o |q^{-1} \lambda|^{2\alpha+1} d_q \lambda \\ &= \int_q^{qX} (q^{-1} \lambda)^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+1} \mathcal{D}_q^+ \varphi_e(\lambda) d_q \lambda \\ &= X^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+1} \varphi_e(X) - \varphi_e(1) - \int_q^{qX} \varphi_e(\lambda) \mathcal{D}_q^+ \left(\lambda^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+1} \right) d_q \lambda \\ &= X^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+1} \varphi_e(X) - \varphi_e(1) + [2\beta - (2\alpha+1)(1-\beta/p')]_q \\ &\quad \times \int_q^{qX} \varphi_e(\lambda) \lambda^{2\alpha-2\beta-(2\alpha+1)\frac{\beta}{p'}} d_q \lambda \end{aligned}$$

$$\begin{aligned}
&= X^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+1} \varphi_e(X) - \varphi_e(1) + [2\beta - (2\alpha + 1)(1 - \beta/p')]_q \\
&\quad q^{-(2\beta-(2\alpha+1)(1-\beta/p'))} \times \int_1^X \varphi_e(q\lambda) \lambda^{2\alpha-2\beta-(2\alpha+1)\frac{\beta}{p'}} d_q \lambda \\
&= X^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+1} \varphi_e(X) - \varphi_e(1) - [(2\alpha + 1)(1 - \beta/p') - 2\beta]_q \\
&\quad \times \int_1^X \varphi_e(q\lambda) \lambda^{2\alpha-2\beta-(2\alpha+1)\frac{\beta}{p'}} d_q \lambda.
\end{aligned} \tag{3.5}$$

Hence, combining the formula (3.4) and (3.5), we conclude that

$$\begin{aligned}
&\int_1^X |\mathcal{F}_{q,\alpha}(f)(\lambda)|^\beta |\lambda|^{2\alpha+1} d_q \lambda \\
&= X^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+1} \varphi(X) - [(2\alpha + 1)(1 - \beta/p') - 2\beta]_q \\
&\quad \times \int_1^X \varphi(q\lambda) \lambda^{2\alpha-2\beta-(2\alpha+1)\frac{\beta}{p'}} d_q \lambda \\
&= O\left(X^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+2-\delta\beta+\beta\left(\frac{p+1}{p}\right)}\right) - [(2\alpha + 1)(1 - \beta/p') - 2\beta]_q \\
&\quad \times \int_1^X \varphi(q\lambda) \lambda^{2\alpha-2\beta-(2\alpha+1)\frac{\beta}{p'}} d_q \lambda \\
&= O\left(X^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+2-\delta\beta+\beta\left(\frac{p+1}{p}\right)}\right) \\
&\quad + O\left(\int_1^X \lambda^{1-\delta\beta+\beta\left(\frac{p+1}{p}\right)+2\alpha-2\beta-(2\alpha+1)\frac{\beta}{p'}} d_q \lambda\right) \\
&= O\left(X^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+2-\delta\beta+\beta\left(\frac{p+1}{p}\right)}\right)
\end{aligned}$$

and this is bounded as $X \rightarrow \infty$ if

$$-2\beta - (2\alpha + 1)\frac{\beta}{p'} + 2\alpha + 2 - \delta\beta + \beta\left(\frac{p+1}{p}\right) < 0,$$

that is

$$\beta > \frac{2p\alpha + 2p}{2p + 2\alpha(p-1) + \delta p - 2}$$

and this ends the proof.

In the rest of this paper, we give our second main result which is a generalization of Younis's theorem 1.2.

For this objective, we need to define the q -Bessel-Dini-Lipschitz class.

Definition 3.2 Let $0 < \delta < 1$, $\gamma > 0$. A function $f \in \mathcal{L}_{q,\alpha}^p$, $1 < p \leq 2$ is said to be in the q -Bessel-Dini-Lipschitz class, denoted D - q - $BLip(\delta, \gamma, p, \alpha)$ if

$$\|\Delta_{q,h} f\|_{q,p,\alpha} = O\left(\frac{h^\delta}{(\log \frac{1}{h})^\gamma}\right) \text{ as } h \rightarrow 0.$$

Theorem 3.2 Let $f \in \mathcal{L}_{q,\alpha}^p$ whis $1 < p \leq 2$. If f belong to D - q - $BLip(\delta, \gamma, p, \alpha)$, then $\mathcal{F}_{q,\alpha}(f)$ belong to $\mathcal{L}_{q,\alpha}^\beta(\mathbb{R}_q^+)$, where

$$\frac{2p\alpha + 2p}{2p + 2\alpha(p-1) + \delta p - 2} < \beta \leq p' = \frac{p}{p-1} \quad \text{and} \quad \beta > \frac{1}{\gamma}.$$

Proof. If $\beta = p'$. From formula (2.11) we have that $\mathcal{F}_{q,\alpha}(f) \in \mathcal{L}_{q,\alpha}^{p'}$. And for $\alpha > -\frac{1}{2}$, $0 < \delta < 1$ we get

$$\begin{aligned} 1 + \frac{\delta p'}{2\alpha + 2} &> 1 \\ \Leftrightarrow \frac{p-1}{p} \left(\frac{2\alpha + 2 + \delta p'}{2\alpha + 2} \right) &> \frac{p-1}{p} \\ \Leftrightarrow \frac{(2\alpha + 2)(p-1) + \delta p}{p(2\alpha + 2)} &> \frac{1}{p'} \\ \Leftrightarrow \frac{p(2\alpha + 2)}{(2\alpha + 2)(p-1) + \delta p} &< \beta = p' = \frac{p}{p-1} \\ \Leftrightarrow \frac{2p\alpha + 2p}{2p + 2\alpha(p-1) + \delta p - 2} &< \beta = p' = \frac{p}{p-1}. \end{aligned}$$

So, we assume that $\beta < p'$ and $f \in \text{D-q-BLip}(\delta, \gamma, p, \alpha)$.

By analogy with the proof of Theorem 3.1, we can establish the following result

$$\int_0^{\eta/h} \lambda^{2p'} |\mathcal{F}_{q,\alpha}(f)(\lambda)|^{p'} |\lambda|^{2\alpha+1} d_q \lambda = O \left(\frac{h^{(\delta-2)p'}}{(\log \frac{1}{h})^{\gamma p'}} \right) \quad \text{as } h \rightarrow 0.$$

Thus

$$\int_0^X \lambda^{2p'} |\mathcal{F}_{q,\alpha}(f)(\lambda)|^{p'} |\lambda|^{2\alpha+1} d_q \lambda = O \left(\frac{X^{(2-\delta)p'}}{(\log X)^{\gamma p'}} \right) \quad \text{as } h \rightarrow 0.$$

Set

$$\varphi(x) = \int_1^X |\lambda^2 \mathcal{F}_{q,\alpha}(f)(\lambda)|^\beta |\lambda|^{(2\alpha+1)\beta/p'} d_q \lambda.$$

We use the Hölder inequality we obtain

$$\varphi(X) = O \left(\frac{X^{2\beta-\delta\beta+1-\frac{\beta}{p'}}}{(\log X)^{\gamma\beta}} \right) \quad \text{as } X \rightarrow \infty.$$

Let us estimate the next integral

$$\int_1^X |\mathcal{F}_{q,\alpha}(f)(\lambda)|^\beta |\lambda|^{2\alpha+1} d_q \lambda.$$

We write

$$\int_1^X |\mathcal{F}_{q,\alpha}(f)(\lambda)|^\beta |\lambda|^{2\alpha+1} d_q \lambda = I_1 + I_2,$$

where

$$I_1 = \int_1^X \left[|\mathcal{F}_{q,\alpha}(f)(\lambda)|^\beta \right]_e |\lambda|^{2\alpha+1} d_q \lambda$$

and

$$I_2 = \int_1^X \left[|\mathcal{F}_{q,\alpha}(f)(\lambda)|^\beta \right]_o |\lambda|^{2\alpha+1} d_q \lambda.$$

Similarly as in the proof of Theorem 3.1 we have

$$\begin{aligned} I_1 &= X^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+1} \varphi_o(X) - \varphi_o(1) - [(2\alpha+1)(1-\beta/p') - 2\beta]_q \\ &\quad \times \int_1^X \varphi_o(q\lambda) \lambda^{2\alpha-2\beta-(2\alpha+1)\frac{\beta}{p'}} d_q \lambda \end{aligned} \quad (3.6)$$

and

$$\begin{aligned} I_2 &= X^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+1} \varphi_e(X) - \varphi_e(1) - [(2\alpha+1)(1-\beta/p') - 2\beta]_q \\ &\quad \times \int_1^X \varphi_e(q\lambda) \lambda^{2\alpha-2\beta-(2\alpha+1)\frac{\beta}{p'}} d_q \lambda. \end{aligned} \quad (3.7)$$

Combining (3.6) and (3.7) we conclude that

$$\int_1^X |\mathcal{F}_{q,\alpha}(f)(\lambda)|^\beta |\lambda|^{2\alpha+1} d_q \lambda = O \left(\frac{X^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+2-\delta\beta+\beta\left(\frac{p+1}{p}\right)}}{(\log X)^{\gamma\beta}} \right).$$

and this is bounded as $X \rightarrow \infty$ if

$$-2\beta - (2\alpha+1)\frac{\beta}{p'} + 2\alpha + 2 - \delta\beta + \beta \left(\frac{p+1}{p} \right) < 0 \quad \text{and} \quad -\gamma\beta < -1.$$

Hence

$$\frac{2p\alpha + 2p}{2p + 2\alpha(p-1) + \delta p - 2} < \beta \leq p' = \frac{p}{p-1}.$$

Then, the theorem is proved.

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