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Boundedness in a attraction-repulsion chemotaxis system with nonlinear production

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Abstract. *We study the quasilinear attraction-repulsion chemotaxis system of parabolic-elliptic type* $u_t = \Delta u - \chi\nabla\cdot(u\nabla\upsilon) + \xi\nabla\cdot(u\nabla\omega)$ *with nonlinear production* $0 = \Delta\upsilon - \beta\upsilon + \alpha u^s$, $0 = \Delta\omega - \delta\omega + \gamma u^r$, *subject to the homogeneous Neumann boundary conditions in a bounded domain* $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ *with smooth boundary. It is proved that for every* $\alpha, \beta, \delta, \gamma, \chi, \xi > 0$ *and* $s \geq \frac{2}{N}$, $r > \frac{N-2}{N}s$, *there exists* $\xi^* > 0$ *such that if* $\xi > \xi^*$ *and any sufficiently regular initial datum* $u_0(x) \geq 0$, *then the model has a unique global classical solution* (u, υ, ω), *which is bounded in* $\Omega \times (0, \infty)$.

Keywords. Boundedness, chemotaxis, nonlinear production.

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1 Introduction and preliminaries

We consider the following attraction-repulsion chemotaxis system (parabolic-elliptic system) with nonlinear production

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla\upsilon) + \xi\nabla\cdot(u\nabla\omega) & \text{in } \Omega\times(0,T_{\max}), \\ 0 = \Delta\upsilon - \beta\upsilon + \alpha u^s, & \text{in } \Omega\times(0,T_{\max}), \\ 0 = \Delta\omega - \delta\omega + \gamma u^r, & \text{in } \Omega\times(0,T_{\max}), \\ \frac{\partial u}{\partial\nu} = \frac{\partial\upsilon}{\partial\nu} = \frac{\partial\omega}{\partial\nu} = 0, & \text{on } \partial\Omega\times(0,T_{\max}), \\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 3)$, the paramets α, β, δ, γ, χ, ξ, s, $r > 0$, $\frac{\partial}{\partial\nu}$ denotes the derivative with respect to the outer normal of $\partial\Omega$, the scalar function $u = u(x,t)$ denotes the cell density, $\upsilon = \upsilon(x,t)$ and $\omega = \omega(x,t)$ measure the concentration of an attractive signal and the concentration of a repulsive signal, respectively. Here the positive parameters χ and ξ are the chemotactic coefficients, α, β as well as γ and δ are chemical production and degradation rates. It is mentioned that, instead of the linear production u, the nonlinear production u^ν, $\nu > 0$ was used to model the aggregation patterns

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formed by some bacterial chemotxis (refer to Chapter 5 in [13] and [3], [14]-[16]). Correspondingly, here the productions of signals vand win the model are both nonlinear with the forms of αu^s and γu^r. It will be observed that this would substantially affect the behavior of solutions.

Chemotaxis describes oriented movement of cells along the concentration gradient of a chemical signal produced by the cells. A well-known chemotaxis model was initially proposed by Keller and Segel [11] and has been extensively studied in the past four decades from various perspectives (see [6]-[10], [20]-[21], [23]-[26] and references therein).

A more complete scenario is obtained when, in addition, an external source $D(u, v, \omega)$ influences the kinetics of the cells by providing and dissipating density; the corresponding mathematical formulation reads

$$\begin{cases} u_t = \nabla \cdot (A(u, v, \omega) \nabla u + B(u, v, \omega) \nabla v + C(u, v, \omega) \nabla \omega) + D(u, v, \omega), \\ \tau v_t = \Delta v + E(u, v, \omega), \\ \tau \omega_t = \Delta \omega + F(u, v, \omega). \end{cases}$$

Confining our attention to the linear diffusion case $A(u, v, \omega) \equiv 1$, and fixing $B(u, v, \omega) = -\chi \nabla u$, $C(u, v, \omega) = \xi \nabla u$ with $\chi, \xi > 0$, $D(u, v, \omega) \equiv 0$ and $\tau = 0$, production rates $E(u, v, \omega) = -\beta v + \alpha u^s$, $F(u, v, \omega) = -\delta \omega + \gamma u^r$ with α, β, δ, $\gamma > 0$, we have that the sign of $\xi\gamma - \chi\alpha$ (positive repulsion prevails over attraction, negative attraction prevails over repulsion) establishes whether system (1.1) has unbounded solutions or all solutions are bounded: see the significant contribution [19] and [4], [5], [12] for some details on the issue.

In [19], the authors proved that the system (1.1) (when $r = 1$ and $s = 1$) is globally well-posed in high dimensions if repulsion prevails over attraction in the sense that $\xi\gamma - \chi\alpha > 0$. Also, Tao and Wang proved that for any $N \geq 2$ and $\xi\gamma - \chi\alpha > 0$ with $u_0(x) \in W^{1,\infty}(\Omega)$ is a non-negative function, there exists a unique triple (u, v, ω) of non-negative bounded functions belonging to $C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty))$, which solves (1.1) classically. In [22], when zero-flux boundary conditions were fixed, Viglialoro obtained the results which all excluding chemotactic collapse scenarios under certain correlations between the attraction and repulsive effects describing the model. To be precise, for every α, β, γ, δ, $\chi > 0$, and $r > s \geq 1$ (resp. $s > r \geq 1$), there exists $\xi^* > 0$ (resp. $\xi_* > 0$) such that if $\xi > \xi^*$ (resp. $\xi \geq \xi_*$), any sufficiently regular initial datum $u_0(x) \geq 0$ (resp. $u_0(x) \geq 0$ enjoying some smallness assumptions) produces a unique classical solution (u, v, ω) to problem (1.1) which is global, i.e. $T_{\max} = \infty$, and such that u, v and ω are uniformly bounded. Conversely, the same conclusion holds true for every α, β, γ, δ, χ, $\xi > 0$, $0 < s < 1$, $r = 1$ and any sufficiently regular $u_0(x) \geq 0$.

In this study, we obtain results on the improvement of the r and s exponents in the nonlinear production.

The main result in this paper can be stated as follows.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a bounded domain with smooth boundary. Assume that $\alpha, \beta, \delta, \gamma, \chi, \xi > 0$ and*

$$s \geq \frac{2}{N},\ r > \frac{N-2}{N}s.$$

Let $0 \leq u_0(x) \in C^0\left(\overline{\Omega}\right)$ be any nontrivial initial datum. Then there exists $\xi^* > 0$ such that, if $\xi > \xi^*$, problem (1.1) admits a unique solution (u, v, ω) of nonnegative and bounded functions in the class

$$C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)) \times C^{2,0}(\overline{\Omega} \times (0, \infty)) \times C^{2,0}(\overline{\Omega} \times (0, \infty)).$$

Remark 1.1 In our this paper, the conditions $r > s \geq 1$ (resp. $s > r \geq 1$) in [22] extend as conditions $s \geq \frac{2}{N}$, $r > \frac{N-2}{N}s$. Namely, in Theorem 1.1 we improve the current results in [22].

For simplicity, we denote $\|u\|_{L^p(\Omega)} := \|u\|_p$, $\|u\|_{W^{1,p}(\Omega)} := \|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p$, $\|u\|_{W^{2,p}(\Omega)} := \|u\|_{2,p} = \|u\|_p + \|\Delta u\|_p$ $(1 \le p \le \infty)$.

The local solvability to problem (1.1) for sufficiently smooth initial data can be addressed by methods involving standard parabolic regularity theory in a suitable fixed point framework. In fact, one can thereby also derive a sufficient condition for extensibility of a given local-in-time solution. Details of the proof can be founded in [2], [9].

Lemma 1.1 *Let Ω be a bounded and smooth domain in $\mathbb{R}^N$, $N \ge 3$. Assume $\alpha, \beta, \delta, \gamma, \chi, \xi > 0$ and let $0 \le u_0(x) \in C^0\left(\overline{\Omega}\right)$ be any nontrivial initial datum. Then, problem (1.1) admits a unique classical solution (u, v, ω) of nonnegative functions, precisely in the class*

$$C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})) \times C^{2,0}(\overline{\Omega} \times (0, T_{\max})) \times C^{2,0}(\overline{\Omega} \times (0, T_{\max})).$$

Here $T_{\max} \in (0, \infty]$, denoting the maximal existence time, is such that (dichotomy criterion) either $T_{\max} = \infty$ (global-in-time classical solution) or if $T_{\max} < \infty$ (local-in-time classical solution) then necessarily

$$\lim_{t \to T_{\max}} \|u(\cdot, t)\|_\infty = \infty.$$

Moreover,

$$\int_\Omega u(\cdot, x) = m := \int_\Omega u_0 > 0 \tag{1.2}$$

for all $t \in (0, T_{\max})$.

We need the well-known Gagliardo-Nirenberg interpolation inequality.

Lemma 1.2 *(see [17]). Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$, and $p, q, r, s \ge 1$, $j, m \in \mathbb{N}_0$ and $\delta \in \left[\frac{j}{m}, 1\right]$ satisfying*

$$\frac{1}{p} = \frac{j}{m} + \left(\frac{1}{r} - \frac{m}{N}\right)\delta + \frac{1-\delta}{q}.$$

Then there are positive constants C_1 and C_2 such that for all functions $\varphi \in L^q(\Omega)$ with $\nabla\varphi \in L^r(\Omega)$, $\varphi \in L^s(\Omega)$,

$$\left\|D^j\varphi\right\|_p \le C_1 \|D^m\varphi\|_r^\delta \|\nu\|_q^{1-\delta} + C_2 \|\varphi\|_s.$$

Lemma 1.3 *(see [22]). Let $\gamma_0 > 1$ and $l, L, C > 0$ fulfill the strict inequality*

$$C < \left(\frac{l^{\gamma_0}}{L\gamma_0}\right)^{\frac{1}{\gamma_0 - 1}} \left(\frac{\gamma_0 - 1}{\gamma_0}\right).$$

Then there exists $\phi^* > 0$ such that solutions of the initial problem

$$\begin{cases} \phi'(t) \le -l\phi(t) + L\phi^{\gamma_0}(t) \text{ for all } t > 0, \\ \phi(0) \le \phi^*, \end{cases}$$

satisfy $\phi(t) \le \phi^*$ for all $t \in (0, \infty)$.

Lemma 1.4 *(see [22]). Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded and smooth domain. Then we have these estimates:*

(i) For any $\overline{p} > 1$ and $0 < \theta_0 = \frac{\frac{\overline{p}}{2} - \frac{1}{2}}{\frac{\overline{p}}{2} + \frac{1}{N} - \frac{1}{2}} < 1$, there is a constant $c_* > 0$ such that all functions $0 \leq \psi \in L^1(\Omega)$ with $m := \int_\Omega \psi$ and $\nabla \psi^{\frac{\overline{p}}{2}} \in L^2(\Omega)$, fulfill

$$\int_\Omega \psi^{\overline{p}} \leq \frac{4(\overline{p}-1)}{\overline{p}} \int_\Omega \left|\nabla \psi^{\frac{\overline{p}}{2}}\right|^2 + c_*.$$

(ii) For any arbitrary reals $\varepsilon_1 > 0,\ s \geq 1$ and $\overline{p} > \frac{Ns}{2} \geq 1$, there exist computable and m-independent constants $d_1(\varepsilon_1)$, $c_1 > 0$ such that all functions $0 \leq \psi \in L^{\overline{p}}(\Omega)$ with $m := \int_\Omega \psi$ and $\nabla \psi^{\frac{\overline{p}}{2}} \in L^2(\Omega)$, comply with

$$\int_\Omega \psi^{\overline{p}+s} \leq \varepsilon_1 \int_\Omega \left|\nabla \psi^{\frac{\overline{p}}{2}}\right|^2 + d_1(\varepsilon_1)\left(\int_\Omega \psi^{\overline{p}}\right)^{\frac{2\overline{p}+2s-Ns}{2\overline{p}-Ns}} + c_1 m^{\overline{p}+s}.$$

Lemma 1.5 *(see [18]). Under the assumptions of Lemma 1.1, the solution of (1.1) satisfies*

$$\int_\Omega \upsilon(\cdot,t)^l + \int_\Omega \omega(\cdot,t)^l \leq \overline{C}_0 \text{ for all } t \in (0, T_{\max}),$$

where $\overline{C}_0 > 0$ and $l \in \left[1, \frac{N}{(N-2)_+}\right)$.

2 Proof of Theorem 1.1

In this section, we prove the problem (1.1) possesses a unique global-in-time and bounded classical solution. We should at first establish that for any $p > 1$, there exists $C > 0$ such that

$$\|u(\cdot,t)\|_p \leq C$$

for all $t \in (0, T_{\max})$.

Multiplying the first equation in (1.1) by u^{p-1} for any $p > \max\{1, s-r\}$, integrating by parts can calculate that

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\int_\Omega u^p \leq & -(p-1)\int_\Omega u^{p-2}|\nabla u|^2 + (p-1)\chi\int_\Omega u^{p-1}\nabla u \cdot \nabla \upsilon \\ & -(p-1)\xi \int_\Omega u^{p-1}\nabla u \cdot \nabla\omega \\ := & \ I_1 + I_2 + I_3 \end{aligned} \tag{2.1}$$

for all $t \in (0, T_{\max})$. We estimate the terms $I_1 + I_2 + I_3$. We rewrite the first term I_1 as

$$I_1 = -(p-1)\int_\Omega u^{p-2}|\nabla u|^2 = -\frac{4(p-1)}{p^2}\int_\Omega \left|\nabla u^{\frac{p}{2}}\right|^2. \tag{2.2}$$

We next deal with the second term I_2 and third term I_3. As to the former, integration by parts and the second equation in (1.1) lead to

$$\begin{aligned} I_2 &= (p-1)\chi \int_\Omega u^{p-1}\nabla u\cdot\nabla v \\ &= (p-1)\chi \int_\Omega \nabla\left[\int_0^u s^{p-1}ds\right]\cdot\nabla v \\ &= -(p-1)\chi \int_\Omega \left[\int_0^u s^{p-1}ds\right]\Delta v \\ &= -(p-1)\chi \int_\Omega \left[\int_0^u s^{p-1}ds\right](\beta v-\alpha u^s) \\ &\leq \frac{(p-1)\chi\alpha}{p}\int_\Omega u^{p+s} \end{aligned} \tag{2.3}$$

since $v\geq 0$. Similarly, we have

$$\begin{aligned} I_3 &= -(p-1)\xi \int_\Omega u^{p-1}\nabla u\cdot\nabla\omega \\ &= (p-1)\xi \int_\Omega \left[\int_0^u s^{p-1}ds\right]\Delta\omega \\ &= (p-1)\xi\delta \int_\Omega \left[\int_0^u s^{p-1}ds\right]\omega \\ &\quad -(p-1)\xi\gamma \int_\Omega \left[\int_0^u s^{p-1}ds\right]u^r \\ &= \frac{(p-1)\xi\delta}{p}\int_\Omega u^p\omega - \frac{(p-1)\xi\gamma}{p}\int_\Omega u^{p+r} \\ &\leq \frac{(p-1)\xi\delta}{p}\int_\Omega u^p\omega - \frac{(p-1)\xi\gamma}{p}\int_\Omega u^{p+r} \end{aligned} \tag{2.4}$$

for all $t\in(0,T_{\max})$. Substituting $(2.2)-(2.4)$ into (2.1), we derive

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\int_\Omega u^p \leq & -\frac{4(p-1)}{p^2}\int_\Omega \left|\nabla u^{\frac{p}{2}}\right|^2 + \frac{(p-1)\chi\alpha}{p}\int_\Omega u^{p+s} \\ & +\frac{(p-1)\xi\delta}{p}\int_\Omega u^p\omega - \frac{(p-1)\xi\gamma}{p}\int_\Omega u^{p+r} \end{aligned} \tag{2.5}$$

for all $t\in(0,T_{\max})$. By using Young inequality we get

$$\frac{(p-1)\xi\delta}{p}\int_\Omega u^p\omega \leq \frac{(p-1)\xi\gamma}{2p}\int_\Omega u^{p+r} + C_1\int_\Omega \omega^{\frac{p+r}{r}} \tag{2.6}$$

for some $C_1>0$. We estimate the term $\int_\Omega \omega^{\frac{p+r}{r}}$. Noting that ω solves the following linear elliptic equations

$$\begin{cases} -\Delta\omega+\alpha_1\omega=\alpha_2 u^s, & x\in\Omega, \\ \frac{\partial\omega}{\partial\nu}=0, & x\in\partial\Omega \end{cases}$$

for all $t\in(0,T_{\max})$ and $\alpha_1,\alpha_2>0$. Thus applying the Agmon-Douglis-Nirenberg L^p estimates on linear elliptic equations with the homogeneous Neumann boundary condition, we conclude that there exists $C_0>0$ depending only on p and Ω such that

$$\|\omega(\cdot,t)\|_{2,p} \leq C_0\|u^s(\cdot,t)\|_p,$$

for all $t \in (0, T_{\max})$. For any $p > \max\{1, s-r\}$, we can find $\mu_0 \in \left[1, \frac{N}{(N-2)_+}\right)$. Consequently, an application of the Gagliardo-Nirenberg inequality with Lemma 1.5, there are $C_2, C_3, C_4 > 0$ such that

$$\begin{aligned} C_1 \int_\Omega \omega^{\frac{p+r}{r}} &= C_1 \|\omega\|_{\frac{p+r}{r}}^{\frac{p+r}{r}} \\ &\le C_2 \left(\|\Delta\omega\|_{\frac{p+r}{s}}^{\frac{p+r}{r}\theta_1} \|\omega\|_{\mu_0}^{\frac{p+r}{r}(1-\theta_1)} + \|\omega\|_{\mu_0}^{\frac{p+r}{r}} \right) \\ &\le C_3 \left(\|\Delta\omega\|_{\frac{p+r}{s}}^{\frac{p+r}{r}\theta_1} + 1 \right) \\ &\le C_4 \left(\|u^s\|_{\frac{p+r}{s}}^{\frac{p+r}{r}\theta_1} + 1 \right) \\ &= C_4 \left(\|u\|_{p+r}^{\frac{s(p+r)}{r}\theta_1} + 1 \right), \end{aligned} \tag{2.7}$$

for all $t \in (0, T_{\max})$, where $\theta_1 = \frac{\frac{1}{\mu_0} - \frac{r}{p+r}}{\frac{1}{\mu_0} + \frac{2}{N} - \frac{s}{p+r}} \in (0,1)$. Due to $r > \frac{N-2}{N} s$, implies that

$$\frac{s(p+r)}{r} . \frac{\frac{N-2}{N} - \frac{r}{p+r}}{1 - \frac{s}{p+r}} = \frac{p+r}{\frac{r}{s}} . \frac{\frac{N-2}{N} - \frac{r}{p+r}}{1 - \frac{s}{p+r}} < p + r. \tag{2.8}$$

Therefore, from (2.7) with the inequality (2.8), we get

$$C_1 \int_\Omega \omega^{\frac{p+r}{r}} \le C_4 \int_\Omega u^{p+r} + C_4. \tag{2.9}$$

From (2.5), (2.6) and (2.9), we obtain

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \int_\Omega u^p \le& -\frac{4(p-1)}{p^2} \int_\Omega \left| \nabla u^{\frac{p}{2}} \right|^2 + \frac{(p-1)\chi\alpha}{p} \int_\Omega u^{p+s} \\ &+ \left(C_4 - \frac{(p-1)\xi\gamma}{2p} \right) \int_\Omega u^{p+r} + C_4. \end{aligned} \tag{2.10}$$

On the other hand, there exists a constant $\xi^* > 0$ such that $\xi > \xi^* = \frac{2pC_4}{\gamma(p-1)}$ in the inequality (2.10), we have

$$\frac{d}{dt} \int_\Omega u^p \le -\frac{4(p-1)}{p} \int_\Omega \left| \nabla u^{\frac{p}{2}} \right|^2 + (p-1)\chi\alpha \int_\Omega u^{p+s} + pC_4. \tag{2.11}$$

Hence, from (2.11) and Lemma $1.4\,(i)$, we get

$$\frac{d}{dt} \int_\Omega u^p \le -\frac{2(p-1)}{p} \int_\Omega \left| \nabla u^{\frac{p}{2}} \right|^2 + (p-1)\chi\alpha \int_\Omega u^{p+s} - \frac{1}{2} \int_\Omega u^p + C_5$$

with $C_5 = pC_4 + \frac{c_*}{2}$. We estimate the term $\int_\Omega u^{p+s}$. Let $p > \frac{Ns}{2}$ with $s \geq \frac{2}{N}$. Now using the Gagliardo-Nirenberg inequality with (1.2), we have

$$\begin{aligned}(p-1)\,\chi\alpha \int_\Omega u^{p+s} &= (p-1)\,\chi\alpha \left\| u^{\frac{p}{2}} \right\|_{\frac{2(p+s)}{p}}^{\frac{2(p+s)}{p}} \\ &\leq \widetilde{C} \left(\left\| \nabla u^{\frac{p}{2}} \right\|_2^{\theta_2} \left\| u^{\frac{p}{2}} \right\|_2^{1-\theta_2} + \left\| u^{\frac{p}{2}} \right\|_{\frac{2}{p}} \right)^{\frac{2(p+s)}{p}} \\ &\leq \widetilde{C}_1 \left(\left\| \nabla u^{\frac{p}{2}} \right\|_2^{\frac{2(p+s)}{p}\theta_2} \left\| u^{\frac{p}{2}} \right\|_2^{\frac{2(p+s)}{p}(1-\theta_2)} + m^{p+s} \right) \\ &= \widetilde{C}_1 \left(\left\| \nabla u^{\frac{p}{2}} \right\|_2^{\frac{2(p+s)}{p}\theta_2} \left\| u \right\|_p^{(p+s)(1-\theta_2)} + m^{p+s} \right),\end{aligned}$$

for some $\widetilde{C}, \widetilde{C}_1 > 0$ and since $p > \frac{Ns}{2} \Rightarrow p > (N-2)\,\frac{s}{2}$ with $s \geq \frac{2}{N}$ and $\theta_2 = \frac{Ns}{2(p+s)} \in (0,1)$. Next, an application of the Young inequality we have

$$\begin{aligned}(p-1)\,\chi\alpha \int_\Omega u^{p+s} &\leq \widetilde{C}_1 \left(\left\| \nabla u^{\frac{p}{2}} \right\|_2^{2\frac{p+s}{p}\frac{Ns}{2(p+s)}} \left\| u \right\|_p^{(p+s)\left(1-\frac{Ns}{2(p+s)}\right)} + m^{p+s} \right) \\ &= \widetilde{C}_1 \left(\left\| \nabla u^{\frac{p}{2}} \right\|_2^{2\frac{Ns}{2p}} \left\| u \right\|_p^{\frac{2p+2s-Ns}{2}} + m^{p+s} \right) \\ &\leq \varepsilon \int_\Omega \left| \nabla u^{\frac{p}{2}} \right|^2 + \widetilde{C} \left(\int_\Omega u^p \right)^{\frac{2p+2s-Ns}{2p-Ns}} + \widetilde{C}_1 m^{p+s}, \qquad (2.12)\end{aligned}$$

for some $\varepsilon > 0$, $\widetilde{C} = C(\varepsilon) > 0$. Hence, from (2.11), (2.12) and Lemma $1.4\,(ii)$, we deduce that

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^p \leq{}& \left(\varepsilon - \frac{2\,(p-1)}{p}\right) \int_\Omega \left|\nabla u^{\frac{p}{2}}\right|^2 + \widetilde{C}\left(\int_\Omega u^p\right)^{\frac{2p+2s-Ns}{2p-Ns}} \\ &-\frac{1}{2}\int_\Omega u^p + C_6\end{aligned}$$

with $C_6 = C_5 + \widetilde{C}_1 m^{p+s}$. Taking $\varepsilon = \frac{2(p-1)}{p}$, we obtain

$$\frac{d}{dt}\int_\Omega u^p \leq \widetilde{C}\left(\int_\Omega u^p\right)^{\frac{2p+2s-Ns}{2p-Ns}} - \frac{1}{2}\int_\Omega u^p + C_6$$

for all $t \in (0, T_{\max})$. Finally, by using Lemma 1.3 with $\phi(t) := \int_\Omega u^p$ and $\gamma_0 := \frac{2p+2s-Ns}{2p-Ns} > 1$, we get

$$\phi'(t) \leq \widetilde{C}\phi^{\frac{2p+2s-Ns}{2p-Ns}}(t) - \frac{1}{2}\phi(t) + C_6$$

for all $t \in (0, T_{\max})$. Because of this, there exists $C > 0$ such that $\|u(\cdot,t)\|_p \leq C$ for all $t \in (0, T_{\max})$.

Proof of Theorem 1.1. Let $p > \max\{sN, rN, 1\}$. By the elliptic L^p-estimate to the two elliptic equations in (1.1), we get

$$\|v(\cdot,t)\|_{2,\frac{p}{s}} < C \text{ and } \|\omega(\cdot,t)\|_{2,\frac{p}{r}} < C \text{ for all } t \in (0, T_{\max}),$$

and hence

$$\|v(\cdot,t)\|_{C^1(\overline{\Omega})} < C \text{ and } \|\omega(\cdot,t)\|_{C^1(\overline{\Omega})} < C \text{ for all } t \in (0, T_{\max})$$

by the Sobolev imbedding theorem. Now the Moser iteration technique [1], [20] ensures

$$\|u(\cdot,t)\|_{\infty} \leq C \text{ for all } t \in (0, T_{\max}) .$$

This concludes by Lemma 1.1 that $T_{\max} = \infty$. The proof of Theorem 1.1 is completed.

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