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Blow-up and exponential decay of the Dirichlet problem for a nonlinear Kirchhoff-Love equation

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Abstract. *In this paper, the nonlinear Kirchhoff-Love equation*

$$u_{tt} - \frac{\partial}{\partial x}\left[\mathcal{H}(u,u_x) + B_1\left(\|u_x\|^2\right)u_x + B_2\left(\|u_x\|^2\right)u_{tx} + B_3\left(x,t\right)u_{ttx}\right] + \lambda u_t = F(u,u_x) + f(x,t)$$

associated with initial and Dirichlet boundary conditions is considered. Under suitable assumptions on the functions f, B_1, B_2, B_3, F, $\mathcal{H}$ and the initial data, we prove the local existence and uniqueness of a weak solution. We also establish a new blow-up result with a negative initial energy. On the other hand, a sufficient condition is proved to obtain the exponential decay of weak solutions.

Keywords. Nonlinear Kirchhoff-Carrier-Love equation; Blow-up; Exponential decay.

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1 Introduction

In this paper, we consider the following initial boundary value problem

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$$u_{tt}-\frac{\partial}{\partial x}\left[\mathcal{H}(u,u_x)+B_1\left(\|u_x\|^2\right)u_x+B_2\left(\|u_x\|^2\right)u_{tx}+B_3\left(x,t\right)u_{ttx}\right]+\lambda u_t$$
$$=F(u,u_x)+f(x,t),\ x\in\Omega=(0,1),\ 0<t<T, \tag{1.1}$$

$$u(0,t)=u(1,t)=0, \tag{1.2}$$

$$u(x,0)=\tilde{u}_0(x),\ u_t(x,0)=\tilde{u}_1(x), \tag{1.3}$$

where $\lambda>0$ are constants and $\tilde{u}_0,\,\tilde{u}_1\in H_0^1\cap H^2$; $f,\,B_1,\,B_2,\,B_3,\,F,\,\mathcal{H}$ are given functions under suitable assumptions. In Eq. (1.1), the nonlinear terms $B_1\left(\|u_x\|^2\right),\ B_2\left(\|u_x\|^2\right)$ depend on the integral $\|u_x\|^2=\int_0^1 u_x^2(x,t)dx$.

This problem can be regarded as a Kirchhoff-Love type because it connects Kirchhoff and Love equations. Eq. (1.1) has its origin in the model of Kirchhoff [6] which describes small vibrations of an elastic string

$$\rho h u_{tt}=\left(P_0+\frac{Eh}{2L}\int_0^L\left|\frac{\partial u}{\partial y}(y,t)\right|^2dy\right)u_{xx}, \tag{1.4}$$

here u is the lateral deflection, L is the length of the string, h is the cross-sectional area, E is Young's modulus, ρ is the mass density, and P_0 is the initial tension. Eq. (1.1) also arises from the Love equation

$$u_{tt}-\frac{E}{\rho}u_{xx}-2\mu^2\omega^2u_{xxtt}=0, \tag{1.5}$$

see V. Radochová [15]. This equation describes the vertical oscillations of a rod, which was established from Euler's variational equation of an energy functional

$$\int_0^T dt\int_0^L\left[\frac{1}{2}F\rho\left(u_t^2+\mu^2\omega^2u_{tx}^2\right)-\frac{1}{2}F\left(Eu_x^2+\rho\mu^2\omega^2u_xu_{xtt}\right)\right]dx, \tag{1.6}$$

where u is the displacement, L is the length of the rod, F is the area of cross-section, ω is the cross-section radius, E is the Young modulus of the material and ρ is the mass density.

To the best of our knowledge, many works related to those kinds of problems under different hypotheses have been extensively studied by many authors, for example, we refer to [1] - [5], [8] - [20], and references therein.

In [2], M.M. Cavalcanti et al. studied the existence of global solutions and exponential decay for the following Kirchhoff-Carrier model with viscosity

$$\begin{cases}\dfrac{\partial^2y}{\partial t^2}-M\left(\int_\Omega|\nabla y|^2dx\right)\Delta y-\dfrac{\partial}{\partial t}\Delta y=f\text{ in }Q=\Omega\times(0,+\infty),\\ y=0\text{ on }\Sigma_1=\Gamma_1\times(0,+\infty),\\ M\left(\int_\Omega|\nabla y|^2dx\right)\dfrac{\partial y}{\partial\nu}+\dfrac{\partial}{\partial t}\left(\frac{\partial y}{\partial\nu}\right)=g\text{ on }\Sigma_0=\Gamma_0\times(0,+\infty),\\ y(0)=y^0,\ \dfrac{\partial y}{\partial t}(0)=y^1\text{ in }\Omega,\end{cases} \tag{1.7}$$

where M is a C^1 function, $M(\lambda)\geq\lambda_0>0,\forall\lambda\geq0$.

In [14], Kosuke Ono investigated the global existence, decay properties, and blow-up of solutions to the initial boundary value problem for the following nonlinear integrodifferential equations of hyperbolic type with nonlinear dissipative terms

$$\begin{cases} u'' + M\left(\left\|A^{\frac{1}{2}}u\right\|^2\right)Au + |u'|^\beta u' = f(u) \text{ in } Q = \Omega \times [0, +\infty), \\ u(x,0) = u_0(x),\, u'(x,0) = u_1(x), \text{ and } u(x,t)|_{\partial\Omega} = 0, \end{cases} \tag{1.8}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $A = -\Delta = \sum_{j=1}^N \dfrac{\partial^2}{\partial x_j^2}$ is the Laplace operator with the domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $\|\cdot\|$ is the norm of $H = L^2(\Omega)$, $\beta > 0$, $M(r)$ is a nonnegative C^1- function for $r \geq 0$ satisfying $M\left(\left\|A^{\frac{1}{2}}u\right\|^2\right) = a + b\left\|A^{\frac{1}{2}}u\right\|^{2\gamma}$, with $a,\, b \geq 0$, $a + b > 0$, and $\gamma \geq 1$, and $f(u)$ is a nonlinear C^1- function satisfying $|f(u)| \leq k_1 |u|^{\alpha+1}$, $|f'(u)| \leq k_2 |u|^\alpha$, with some constants k_1, $k_1 > 0$ and $\alpha > 0$.

In [19], Z. Yang, Z. Gong considered the viscoelastic equation

$$u_{tt}(x,t) - M\left(\|\nabla u\|_2^2\right)\Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)ds + u_t = |u|^{p-1}u \tag{1.9}$$

with suitable initial data and boundary conditions, where $M(s) = 1 + bs^\gamma$ is a positive C^1- function ($b \geq 0$, $\gamma > 0$, $s \geq 0$), and $\|\cdot\|_2$ is the usual norm of $L^2(\Omega)$. Under certain assumptions on the kernel g and the initial data, the authors established a new blow-up result for arbitrary positive initial energy, by using simple analysis techniques.

Recently, in [4], Z. Far et al. considered the problem of blow-up of solutions for a coupled system of nonlinear Love-equations in 1-dimensional bounded domain with homogeneous Dirichlet boundary conditions and an internal infinite memory. Here, the nonexistence of weak solutions with positive initial energy was proved by using a classical arguments.

In [17], Prob. (1.1) - (1.3) with $B_i = B\left(x, t, u, \|u\|^2, \|u_x\|^2, \|u_t\|^2, \|u_{xt}\|^2\right)$, $i = 1, 2, 3$, was considered, where results related to the existence, blow-up and exponential decay estimates were proved. In case $B = B(x,t)$ and $F = F(u, u_x)$, $\mathcal{H} = \mathcal{H}(u, u_x)$ such that $(F, -\mathcal{H}) = \left(\dfrac{\partial \mathcal{F}}{\partial u}, \dfrac{\partial \mathcal{F}}{\partial u_x}\right)$, the authors proved that the solution blows up in finite time when $f(x,t) \equiv 0$ and the initial energy is negative. On the other hand, a sufficient condition was established, under the assumptions that the initial energy is positive and small, to guarantee the global existence and exponential decay of weak solutions.

This paper is inspired by the results of [17], we shall establish a linear recurrent sequence to prove that Prob. (1.1)-(1.3) has a solution. Furthermore, we shall consider blow-up and decay properties of Prob. (1.1)-(1.3) with $B_i = B_i\left(\|u_x(t)\|^2\right) \neq B_i(x,t)$, $i = 1, 2$, as in [17]. It consists of four sections.

In the Section 2, both existence and uniqueness of weak solutions for Prob. (1.1)-(1.3) are stated in Theorem 2.2, in case $F, \mathcal{H} \in C^1(\mathbb{R}^2)$; $B_1, B_2 \in C^1(\mathbb{R}_+)$, $B_3 \in C^1([0,1] \times [0,T])$, with $B_i(y) \geq b_{i*} > 0$, $\forall y \in \mathbb{R}_+$ $(i = 1,2)$, $B_3(x,t) \geq b_{3*} > 0$, $\forall (x,t) \in [0,1] \times [0,T]$.

In Sections 3, 4, Prob. (1.1)-(1.3) is considered with $(F, -\mathcal{H}) = (\frac{\partial \mathcal{F}}{\partial u}, \frac{\partial \mathcal{F}}{\partial u_x})$. Here, Theorem 3.1, Theorem 4.1 are proved to have a blow up result and the exponential decay of weak solutions via using the Lyapunov functional. More precisely, in Section 3, with $f(x,t) \equiv 0$ and a negative initial energy, the solution of (1.1)-(1.3) blows up in finite time. In Section

4, we give a sufficient condition, where the initial energy is positive and small, any global weak solution is exponentially decaying. By modifying the methods used in [17], the results obtained here are more general than the results established in [17].

2 Existence of a weak solution

First, let $\langle\cdot,\cdot\rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space, $\|\cdot\|$ be the norm in L^2 and $\|\cdot\|_X$ be the norm in the Banach space X. Let $L^p(0,T;X)$, $1\le p\le\infty$, be the Banach space of the real functions $u:(0,T)\to X$ measurable, with

$$\|u\|_{L^p(0,T;X)}=\left(\int_0^T\|u(t)\|_X^p\,dt\right)^{1/p}<\infty \text{ for } 1\le p<\infty,$$

and

$$\|u\|_{L^\infty(0,T;X)}=ess\sup_{0<t<T}\|u(t)\|_X \ \text{ for } p=\infty.$$

Denote $u(t)=u(x,t)$, $u'(t)=u_t(t)=\frac{\partial u}{\partial t}(x,t)$, $u''(t)=u_{tt}(t)=\frac{\partial^2 u}{\partial t^2}(x,t)$, $u_x(t)=\frac{\partial u}{\partial x}(x,t)$, $u_{xx}(t)=\frac{\partial^2 u}{\partial x^2}(x,t)$. With $F\in C^1(\mathbb{R}^2)$, $F=F(u,v)$, we put $D_1F=\frac{\partial F}{\partial u}$, $D_2F=\frac{\partial F}{\partial v}$.

Now, we recall the following properties related to the usual spaces $C([0,1])$, H^1, and

$$H_0^1=\{v\in H^1 : v(1)=v(0)=0\}.$$

Lemma 2.1.
(i) *The imbedding* $H^1\hookrightarrow C([0,1])$ *is compact and*

$$\|v\|_{C[0,1]}\le\sqrt{2}\left(\|v\|^2+\|v_x\|^2\right)^{1/2} \ \textit{for all } v\in H^1. \tag{2.1}$$

(ii) *On* H_0^1, $v\longmapsto\|v_x\|$ *and* $v\longmapsto\|v\|_{H^1}=\left(\|v\|^2+\|v_x\|^2\right)^{1/2}$ *are equivalent norms. Furthermore*

$$\|v\|_{C([0,1])}\le\|v_x\| \quad \textit{for all } v\in H_0^1. \tag{2.2}$$

A weak solution u of Prob. (1.1)-(1.3) is defined in the following manner: Find $u\in W_T=\{u\in L^\infty\left(0,T;H_0^1\cap H^2\right) : u',\, u''\in L^\infty\left(0,T;H_0^1\cap H^2\right)\}$, such that u satisfies the following variational equation

$$\begin{aligned}&\langle u''(t),w\rangle+\langle\mathcal{H}\left(u(t),u_x(t)\right),w_x\rangle+B_1\left(\|u_x(t)\|^2\right)\langle u_x(t),w_x\rangle\\&+B_2\left(\|u_x(t)\|^2\right)\langle u_x'(t),w_x\rangle+\langle B_3(t)u_x''(t),w_x\rangle+\lambda\langle u'(t),w\rangle\\&=\langle F\left(u(t),u_x(t)\right),w\rangle+\langle f(t),w\rangle,\end{aligned} \tag{2.3}$$

for all $w\in H_0^1$, a.e., $t\in(0,T)$, together with the initial conditions

$$u(0)=\tilde{u}_0,\ u'(0)=\tilde{u}_1. \tag{2.4}$$

Next, we make the following assumptions:

(H_1) $\tilde{u}_0, \tilde{u}_1 \in H_0^1 \cap H^2$;
(H_2) $B_i \in C^1(\mathbb{R}_+)$ and there exist the constants $b_{i*} > 0,\ i = 1, 2$ such that $B_i(y) \geq b_{i*},\ \forall y \geq 0,$
(H_3) $B_3 \in C^1([0,1] \times \mathbb{R}_+)$ and there exists a constant $b_{3*} > 0$ such that $B_3(x,t) \geq b_{3*},\ \forall (x,t) \in [0,1] \times \mathbb{R}_+$;
(H_4) $F, \mathcal{H} \in C^1(\mathbb{R}^2)$;
(H_5) $f \in C^1([0,1] \times \mathbb{R}_+)$.

Using the standard Faedo-Galerkin method, which is introduced by Lions in [7], we can prove the following theorem, it implies that the problem (1.1)-(1.3) has a unique weak solution.

Theorem 2.2. *Let $(H_1) - (H_5)$ hold. Then Prob.* (1.1)-(1.3) *has a unique local solution*

$$u \in L^\infty\left(0,T; H_0^1 \cap H^2\right),\ u' \in L^\infty\left(0,T; H_0^1 \cap H^2\right),\ u'' \in L^\infty\left(0,T; H_0^1 \cap H^2\right), \tag{2.5}$$

for $T > 0$ small enough.

Remark 2.1. In the base of the regularity obtained by (2.5), Prob. (1.1)-(1.3) has a unique strong solution

$$u \in C^1\left([0,T]; H_0^1 \cap H^2\right),\ u'' \in L^\infty\left(0,T; H_0^1 \cap H^2\right). \tag{2.6}$$

3 Blow-up

In this section, we will consider problems (1.1)-(1.3) with $f = 0$. Under appropriate assumptions, we show that the solution of this problem blows up in finite time.

First, we add the following assumption.

$(\hat{H}_2)$ $B_1, B_2 \in C^1(\mathbb{R}_+)$ and there exist the positive constants $b_{1*}, \chi_1, \bar{b}_2, r$ such that
(i) $B_1(y) \geq b_{1*} > 0,\ \forall y \geq 0,$
(ii) $yB_1(y) \leq \chi_1 \int_0^y B_1(z)dz,\ \forall y \geq 0,$
(iii) $0 \leq B_2(y) \leq \bar{b}_2\left(1 + y^r\right),\ \forall y \geq 0$;

$(\hat{H}_3)$ $B_3 \in C^1([0,1] \times \mathbb{R}_+)$ and there exist the positive constants b_{3*}, b_3^*, σ_3 such that
(i) $b_{3*} \leq B_3(x,t) \leq b_3^*,\ \forall (x,t) \in [0,1] \times \mathbb{R}_+,$
(ii) $-\sigma_3 \leq B_3'(x,t) \leq 0,\ \forall (x,t) \in [0,1] \times \mathbb{R}_+$;

$(\hat{H}_4)$ There exist $\mathcal{F} \in C^2(\mathbb{R}^2; \mathbb{R})$ and the constants $p, q > 2;\ d_1, \bar{d}_1 > 0$, such that
(i) $\frac{\partial \mathcal{F}}{\partial u}(u,v) = F(u,v),\ \frac{\partial \mathcal{F}}{\partial v}(u,v) = -\mathcal{H}(u,v),$
(ii) $uF(u,v) - v\mathcal{H}(u,v) \geq d_1 \mathcal{F}(u,v)$, for all $(u,v) \in \mathbb{R}^2$,
(iii) $\mathcal{F}(u,v) \geq \bar{d}_1\left(|u|^q + |v|^p\right)$, for all $(u,v) \in \mathbb{R}^2$;

$(\hat{H}_5)$ $0 < r < \dfrac{p-2}{2},\ d_1 > 2\chi_1$ with χ_1, r, d_1, p as in $\left(\hat{H}_2, (ii), (iii)\right)$, $\left(\hat{H}_4, (ii), (iii)\right)$ and $\sigma_3 > 0$ (in $\left(\hat{H}_3, (ii)\right)$) is small enough.

We give the examples of the functions $F, \mathcal{H}$ satisfying $(\hat{H}_4)$ as below.

Example 3.1. $F(u,v) = \left(|u|^{q-2} + \dfrac{\alpha}{\beta}|u|^{\alpha-2}|v|^\beta\right)u,\ \mathcal{H}(u,v) = -\left(|v|^{p-2} + |u|^\alpha |v|^{\beta-2}\right)v,$

where $\alpha, \beta, p, q > 2$ are the constants, with $\min\{p, q, \alpha+\beta\} > 2\chi_1$, and χ_1 as in $(\hat{H}_2, (ii))$.

It is obvious that $(\hat{H}_4)$ holds, because there exists a $\mathcal{F} \in C^2(\mathbb{R}^2; \mathbb{R})$ defined by $\mathcal{F}(u,v) = \dfrac{1}{p}|v|^p + \dfrac{1}{\beta}|u|^\alpha |v|^\beta + \dfrac{1}{q}|u|^q$, such that

$$\begin{aligned}\frac{\partial \mathcal{F}}{\partial u}(u,v) =& F(u,v),\ \frac{\partial \mathcal{F}}{\partial v}(u,v) = -\mathcal{H}(u,v),\\ uF(u,v) - v\mathcal{H}(u,v) =& |v|^p + \frac{\alpha+\beta}{\beta}|u|^\alpha |v|^\beta + |u|^q\\ \geq& d_1\mathcal{F}(u,v) \text{ for all } (u,v)\in\mathbb{R}^2,\end{aligned}$$

in which $d_1 = \min\{p, q, \alpha+\beta\} > 2\chi_1$;

$$\begin{aligned}\mathcal{F}(u,v) =& \frac{1}{p}|v|^p + \frac{1}{\beta}|u|^\alpha |v|^\beta + \frac{1}{q}|u|^q\\ \geq& \bar{d}_1\left(|v|^p + |u|^q\right) \text{ for all } (u,v)\in\mathbb{R}^2,\end{aligned}$$

with $\bar{d}_1 = \min\{1/p, 1/q\}$.

Example 3.2.

$$\begin{aligned}F(u,v) &= |u|^{q-2}u\Phi^{k_2}(u,v) + \frac{2k_1}{p}\frac{|v|^p\Phi^{k_1-1}(u,v)u}{e+u^2+v^2} + \frac{2k_2}{q}\frac{|u|^q u\Phi^{k_2-1}(u,v)}{e+u^2+v^2},\\ \mathcal{H}(u,v) &= -|v|^{p-2}v\Phi^{k_1}(u,v) - \frac{2k_1}{p}\frac{|v|^p v\Phi^{k_1-1}(u,v)}{e+u^2+v^2} - \frac{2k_2}{q}\frac{|u|^q\Phi^{k_2-1}(u,v)v}{e+u^2+v^2},\end{aligned}$$

and $\Phi(u,v) = \ln\left(e+u^2+v^2\right)$, where $p,\ q > 2$; $k_1,\ k_2 > 1$ are the constants, with $\min\{p,q\} > 2\chi_1$.

The assumption $(\hat{H}_4)$ holds, because there exists the function $\mathcal{F}\in C^2(\mathbb{R}^2;\mathbb{R})$ defined by

$$\mathcal{F}(u,v) = \frac{1}{p}|v|^p\Phi^{k_1}(u,v) + \frac{1}{q}|u|^q\Phi^{k_2}(u,v),$$

such that

$$\frac{\partial \mathcal{F}}{\partial u}(u,v) = F(u,v),\ \frac{\partial \mathcal{F}}{\partial v}(u,v) = -\mathcal{H}(u,v);$$

$$\begin{aligned}&uF(u,v) - v\mathcal{H}(u,v)\\ &= |u|^q\Phi^{k_2}(u,v) + |v|^p\Phi^{k_1}(u,v)\\ &\quad + \frac{2k_1}{p}\frac{|v|^p\left(u^2+v^2\right)}{e+u^2+v^2}\Phi^{k_1-1}(u,v) + \frac{2k_2}{q}\frac{|u|^q\left(u^2+v^2\right)}{e+u^2+v^2}\Phi^{k_2-1}(u,v)\\ &\geq |u|^q\Phi^{k_2}(u,v) + |v|^p\Phi^{k_1}(u,v)\\ &\geq d_1\mathcal{F}(u,v) \text{ for all } (u,v)\in\mathbb{R}^2,\end{aligned}$$

in which $d_1 = \min\{p,q\} > 2\chi_1$;

$$\begin{aligned}\mathcal{F}(u,v) =& \frac{1}{p}|v|^p\Phi^{k_1}(u,v) + \frac{1}{q}|u|^q\Phi^{k_2}(u,v)\\ \geq& \frac{1}{p}|v|^p + \frac{1}{q}|u|^q \geq \bar{d}_1\left(|v|^p + |u|^q\right) \text{ for all } (u,v)\in\mathbb{R}^2,\end{aligned}$$

with $\bar{d}_1 = \min\{1/p, 1/q\}$.

Put

$$H(0) = -\frac{1}{2}\|\tilde{u}_1\|^2 - \frac{1}{2}\int_0^{\|\tilde{u}_{0x}\|^2} B_1(y)dy - \frac{1}{2}\left\|\sqrt{B_3(0)}\tilde{u}_{1x}\right\|^2 \tag{3.1}$$

$$+\int_0^1 \mathcal{F}\left(\tilde{u}_0(x), \tilde{u}_{0x}(x)\right) dx.$$

Theorem 3.1. *Let $(\hat{H}_2)-(\hat{H}_5)$ hold. Then, for any $\tilde{u}_0, \tilde{u}_1 \in H_0^1 \cap H^2$ such that $H(0) > 0$, the weak solution $u = u(x,t)$ of Prob.* (1.1)-(1.3) *blows up in finite time.*

Proof of Theorem 3.1. It consists of two steps.

Step 1. We prove that the Problem (1.1)-(1.3) has not a global weak solution.

Indeed, by contradiction, we assume that

$$u \in C^1(\mathbb{R}_+; H^2 \cap H_0^1),\ u'' \in L^\infty(0,T; H^2 \cap H_0^1),\ \forall T > 0, \tag{3.2}$$

is a global weak solution of Prob. (1.1)-(1.3). We define the energy associated with (1.1)-(1.3) by

$$\begin{aligned} E(t) =& \frac{1}{2}\left\|u'(t)\right\|^2 + \frac{1}{2}\int_0^{\|u_x(t)\|^2} B_1(z)dz \\ &+ \frac{1}{2}\left\|\sqrt{B_3(t)}u_x'(t)\right\|^2 - \int_0^1 \mathcal{F}\left(u(x,t), u_x(x,t)\right) dx, \end{aligned} \tag{3.3}$$

and we put $H(t) = -E(t)$, $\forall t \geq 0$. Multiplying (1.1) by $u'(x,t)$ and integrating the resulting equation over $[0,1]$, we have

$$\begin{aligned} H'(t) =& \lambda\left\|u'(t)\right\|^2 + \|u_x(t)\|^2 B_2\left(\|u_x(t)\|^2\right) \\ &- \frac{1}{2}\int_0^1 B_3'(x,t)\left|u_x'(x,t)\right|^2 dx \ \geq 0. \end{aligned} \tag{3.4}$$

It implies that

$$H(t) \geq H(0) > 0,\ \forall t \geq 0, \tag{3.5}$$

so

$$\begin{cases} 0 < H(0) \leq H(t) \leq \int_0^1 \mathcal{F}\left(u(x,t), u_x(x,t)\right) dx; \\ \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\int_0^{\|u_x(t)\|^2} B_1(z)dz + \frac{1}{2}\left\|\sqrt{B_3(t)}u_x'(t)\right\|^2 \\ \qquad \leq \int_0^1 \mathcal{F}\left(u(x,t), u_x(x,t)\right) dx,\ \forall t \geq 0. \end{cases} \tag{3.6}$$

Now, we define the functional

$$L(t) = H^{1-\eta}(t) + \varepsilon\Psi(t), \tag{3.7}$$

where

$$\begin{aligned} \Psi(t) =& \langle u'(t), u(t)\rangle + \langle B_3(t)u_x'(t), u_x(t)\rangle + \frac{\lambda}{2}\|u(t)\|^2 \\ &+ \frac{1}{2}\int_0^{\|u_x(t)\|^2} B_2(z)dz, \end{aligned} \tag{3.8}$$

for ε small enough and

$$0 < \eta \leq \min\left\{\frac{1}{2} - \frac{1}{p},\ \frac{1}{2} - \frac{1}{q},\ 1 - \frac{2r+2}{p}\right\}. \tag{3.9}$$

In what follows, we show that, there exists a constant $\gamma > 0$ such that

$$L'(t) \geq \gamma\left[H(t) + \|u(t)\|_{L^q}^q + \|u_x(t)\|_{L^p}^p + \left\|u'(t)\right\|^2 + \left\|u_x'(t)\right\|^2 + \|u_x(t)\|^2\right]. \tag{3.10}$$

Multiplying (1.1) by $u(x,t)$ and integrating over $[0,1]$, it leads to

$$\begin{aligned}\Psi'(t) =& \left\|u'(t)\right\|^2 - \|u_x(t)\|^2 B_1\left(\|u_x(t)\|^2\right) + \left\|\sqrt{B_3(t)}u_x'(t)\right\|^2 \\ &+ \langle B_3'(t)u_x'(t), u_x(t)\rangle + \langle F\left(u(t),u_x(t)\right), u(t)\rangle \\ &- \langle \mathcal{H}\left(u(t),u_x(t)\right), u_x(t)\rangle.\end{aligned} \tag{3.11}$$

Therefore

$$L'(t) = (1-\eta)H^{-\eta}(t)H'(t) + \varepsilon\Psi'(t) \geq \varepsilon\Psi'(t). \tag{3.12}$$

By $(\hat{H}_4)$, we obtain

$$\begin{cases} \langle F\left(u(t),u_x(t)\right), u(t)\rangle - \langle \mathcal{H}\left(u(t),u_x(t)\right), u_x(t)\rangle \geq d_1 \int_0^1 \mathcal{F}\left(u(x,t),u_x(x,t)\right)dx; \\ \int_0^1 \mathcal{F}\left(u(x,t),u_x(x,t)\right)dx \geq \bar{d}_1\left(\|u(t)\|_{L^q}^q + \|u_x(t)\|_{L^p}^p\right); \\ \int_0^1 \mathcal{F}\left(u(x,t),u_x(x,t)\right)dx = H(t) + \frac{1}{2}\left\|u'(t)\right\|^2 + \frac{1}{2}\int_0^{\|u_x(t)\|^2} B_1(z)dz \\ \qquad + \frac{1}{2}\left\|\sqrt{B_3(t)}u_x'(t)\right\|^2. \end{cases} \tag{3.13}$$

On the other hand, by $(\hat{H}_2, ii)$, $(\hat{H}_3, ii)$, we get

$$\begin{aligned} -B_1\left(\|u_x(t)\|^2\right)\|u_x(t)\|^2 &\geq -\chi_1\int_0^{\|u_x(t)\|^2} B_1(z)dz, \\ \left|\langle B_3'(t)u_x'(t), u_x(t)\rangle\right| &\leq \left\|B_3'(t)u_x'(t)\right\|\|u_x(t)\| \\ &\leq \frac{\delta_1}{2}\|u_x(t)\|^2 + \frac{\sigma_3^2}{2\delta_1}\left\|u_x'(t)\right\|^2, \ \forall\delta_1 > 0. \end{aligned} \tag{3.14}$$

It implies from (3.11), (3.13), (3.14) that

$$\begin{aligned}\Psi'(t) =& \left\|u'(t)\right\|^2 - \|u_x(t)\|^2 B_1\left(\|u_x(t)\|^2\right) + \left\|\sqrt{B_3(t)}u_x'(t)\right\|^2 \\ &+ \langle B_3'(t)u_x'(t), u_x(t)\rangle + \langle F\left(u(t),u_x(t)\right), u(t)\rangle - \langle \mathcal{H}\left(u(t),u_x(t)\right), u_x(t)\rangle \\ \geq& \left\|u'(t)\right\|^2 - \chi_1\int_0^{\|u_x(t)\|^2} B_1(z)dz + b_{3*}\left\|u_x'(t)\right\|^2 \\ &- \frac{1}{2}\left(\delta_1\|u_x(t)\|^2 + \frac{\sigma_3^2}{\delta_1}\left\|u_x'(t)\right\|^2\right) + d_1\int_0^1 \mathcal{F}\left(u(x,t),u_x(x,t)\right)dx \\ =& \left\|u'(t)\right\|^2 - \chi_1\int_0^{\|u_x(t)\|^2} B_1(z)dz + b_{3*}\left\|u_x'(t)\right\|^2 \\ &- \left(\frac{\delta_1}{2}\|u_x(t)\|^2 + \frac{\sigma_3^2}{2\delta_1}\left\|u_x'(t)\right\|^2\right) + d_1\delta\int_0^1 \mathcal{F}\left(u(x,t),u_x(x,t)\right)dx \\ &+ d_1(1-\delta)\left[H(t) + \frac{1}{2}\left\|u'(t)\right\|^2 + \frac{1}{2}\int_0^{\|u_x(t)\|^2} B_1(z)dz + \frac{1}{2}\left\|\sqrt{B_3(t)}u_x'(t)\right\|^2\right] \\ \geq& \left(1 + \frac{1}{2}d_1(1-\delta)\right)\left\|u'(t)\right\|^2 + d_1(1-\delta)H(t) \\ &+ d_1\delta\bar{d}_1\left(\|u(t)\|_{L^q}^q + \|u_x(t)\|_{L^p}^p\right) + \left[\frac{1}{2}d_1(1-\delta) - \chi_1\right]\int_0^{\|u_x(t)\|^2} B_1(z)dz \end{aligned} \tag{3.15}$$

$$-\frac{\delta_1}{2}\|u_x(t)\|^2+\left[\left(1+\frac{d_1}{2}(1-\delta)\right)b_{3*}-\frac{\sigma_3^2}{2\delta_1}\right]\left\|u_x'(t)\right\|^2,$$

for all $\delta\in(0,1)$, $\delta_1>0$.

By $d_1>2\chi_1$, we have

$$\lim_{\delta\to0_+,\ \delta_1\to0_+}\left[\left(\frac{1}{2}d_1(1-\delta)-\chi_1\right)b_{1*}-\frac{\delta_1}{2}\right]=\left(\frac{1}{2}d_1-\chi_1\right)b_{1*}>0,$$

then, we can choose $\delta,\,\delta_1\in(0,1)$ with $\delta,\,\delta_1$ are small enough such that

$$\left(\frac{1}{2}d_1(1-\delta)-\chi_1\right)b_{1*}-\frac{\delta_1}{2}>0. \tag{3.16}$$

Hence, we deduce from (3.16) that

$$\begin{aligned}&\left[\frac{1}{2}d_1(1-\delta)-\chi_1\right]\int_0^{\|u_x(t)\|^2}B_1(z)dz-\frac{\delta_1}{2}\|u_x(t)\|^2 \\ &\geq\left[\left(\frac{1}{2}d_1(1-\delta)-\chi_1\right)b_{1*}-\frac{\delta_1}{2}\right]\|u_x(t)\|^2.\end{aligned} \tag{3.17}$$

Then, if $\sigma_3>0$ satisfies

$$\left(1+\frac{d_1}{2}(1-\delta)\right)b_{3*}-\frac{\sigma_3^2}{2\delta_1}>0, \tag{3.18}$$

then we deduce from (3.12), (3.15)-(3.18) that there exists a constant $\gamma>0$ such that (3.10) holds.

From the formula of $L(t)$ and (3.10), we can choose $\varepsilon>0$ small enough such that

$$L(t)\geq L(0)>0,\ \forall t\geq0.$$

Using the inequality

$$\left(\sum\nolimits_{i=1}^5 x_i\right)^\sigma\leq5^{\sigma-1}\sum\nolimits_{i=1}^5 x_i^\sigma,\ \text{for all }\sigma>1,\ \text{and }x_1,\cdots,x_5\geq0, \tag{3.19}$$

it implies from (3.7)-(3.9) that

$$\begin{aligned}L^{1/(1-\eta)}(t)\leq&\,\mathit{Const}\Bigg[H(t)+\left|\langle u(t),u'(t)\rangle\right|^{1/(1-\eta)}+\left|\langle B_3(t)u_x'(t),u_x(t)\rangle\right|^{1/(1-\eta)} \\ &+\|u(t)\|^{2/(1-\eta)}+\left(\int_0^{\|u_x(t)\|^2}B_2(z)dz\right)^{1/(1-\eta)}\Bigg].\end{aligned} \tag{3.20}$$

Using Young's inequality, we have

$$\begin{aligned}\left|\langle u(t),u'(t)\rangle\right|^{1/(1-\eta)}&\leq\|u(t)\|^{1/(1-\eta)}\left\|u'(t)\right\|^{1/(1-\eta)} \\ &\leq\tfrac{1-2\eta}{2(1-\eta)}\|u(t)\|^\theta+\tfrac{1}{2(1-\eta)}\left\|u'(t)\right\|^2 \\ &\leq\mathit{Const}\left(\|u_x(t)\|^\theta+\left\|u'(t)\right\|^2\right),\end{aligned} \tag{3.21}$$

where $\theta = 2/(1-2\eta)$. Similarly, we also get

$$\begin{aligned}\left|\langle B_3(t)u_x'(t), u_x(t)\rangle\right|^{1/(1-\eta)} &\leq (b_3^*)^{1/(1-\eta)} \|u_x(t)\|^{1/(1-\eta)} \left\|u_x'(t)\right\|^{1/(1-\eta)} \\ &\leq \mathit{Const}\left(\|u_x(t)\|^{\theta} + \left\|u_x'(t)\right\|^2\right), \\ \left(\int_0^{\|u_x(t)\|^2} B_2(z)dz\right)^{1/(1-\eta)} &\leq \left(\bar{b}_2\int_0^{\|u_x(t)\|^2}(1+z^r)\,dz\right)^{1/(1-\eta)} \\ &\leq \left(\bar{b}_2\right)^{1/(1-\eta)}\left[\|u_x(t)\|^2 + \tfrac{1}{r+1}\|u_x(t)\|^{2r+2}\right]^{1/(1-\eta)} \\ &\leq \mathit{Const}\left(\|u_x(t)\|^{2/(1-\eta)} + \|u_x(t)\|^{(2r+2)/(1-\eta)}\right).\end{aligned} \tag{3.22}$$

Combining (3.20) - (3.22), it leads to

$$\begin{aligned}L^{1/(1-\eta)}(t) \leq &\mathit{Const}\Big[H(t) + \left\|u'(t)\right\|^2 + \left\|u_x'(t)\right\|^2 + \|u(t)\|^{2/(1-\eta)} \\ &+ \|u_x(t)\|^{2/(1-\eta)} + \|u_x(t)\|^{\theta} + \|u_x(t)\|^{(2r+2)/(1-\eta)}\Big].\end{aligned} \tag{3.23}$$

We note more a useful property as follows.

Lemma 3.2. *Let $2 \leq r_1 \leq q$, $2 \leq r_2, r_3 \leq \min\{p,q\}$, $2 \leq r_4 \leq p$. Then, for any $v \in H_0^1$, we have*

$$\|v\|^{r_1} + \|v_x\|^{r_2} + \|v_x\|^{r_3} + \|v_x\|^{r_4} \leq 4\left(\|v\|_{L^q}^q + \|v_x\|_{L^p}^p + \|v_x\|^2\right). \tag{3.24}$$

Proof of Lemma 3.2 is not difficult, so we omit the details. □

We note that the condition $0 < \eta \leq \min\left\{\frac{1}{2} - \frac{1}{p}, \frac{1}{2} - \frac{1}{q}, 1 - \frac{2r+2}{p}\right\}$ (as in 3.9) is equivalent to

$$2/(1-2\eta) = \theta \leq \min\{p,q\}\ , (2r+2)/(1-\eta) \leq p.$$

Using (3.23) and Lemma 3.2 with $2 < r_1 = 2/(1-\eta) \leq q$, $2 < r_2 = \theta$, $r_3 = 2/(1-\eta) \leq \min\{p,q\}$, $2 < r_4 = (2r+2)/(1-\eta) \leq p$, we obtain

$$L^{1/(1-\eta)}(t) \leq Const\left[H(t) + \left\|u'(t)\right\|^2 + \left\|u_x'(t)\right\|^2 + \|u_x(t)\|^2 + \|u(t)\|_{L^q}^q + \|u_x(t)\|_{L^p}^p\right] \tag{3.25}$$

for all $t \geq 0$.

It follows from (3.10) and (3.25) that

$$L'(t) \geq \bar{\lambda}L^{1/(1-\eta)}(t),\ \forall t \geq 0, \tag{3.26}$$

where $\bar{\lambda}$ is a positive constant. Integrating (3.26) over $(0,t)$, it leads to

$$L^{\eta/(1-\eta)}(t) \geq \frac{1-\eta}{\bar{\lambda}\eta}\frac{1}{T_* - t},\ 0 \leq t < T_* = \frac{1-\eta}{\bar{\lambda}\eta}L^{-\eta/(1-\eta)}(0). \tag{3.27}$$

Therefore $\lim_{t\to T_*^-} L(t) = +\infty$. This is a contradiction with (3.25) and (3.2). Thus, the Problem (1.1)-(1.3) has not a global weak solution. It implies that $T_\infty < +\infty$, where

$T_\infty = \sup\Big\{T > 0$: Prob. (1.1)-(1.3) has a unique solution

$$u \in C^1([0,T]; H^2 \cap H_0^1),\ u'' \in L^\infty(0,T; H^2 \cap H_0^1)\Big\}.$$

Step 2. Next, we now prove that

$$\lim_{t \to T_\infty^-} \left(\|u(t)\|_{H^2 \cap H_0^1} + \left\|u'(t)\right\|_{H^2 \cap H_0^1} \right) = +\infty. \tag{3.28}$$

Indeed, assume that (3.28) is not true, there exists a constant $M > 0$ and there exists a sequence $\{T_m\}$ with $\{T_m\} \subset (0, T_\infty)$, $T_m \to T_\infty$ such that

$$\|u(T_m)\|_{H^2 \cap H_0^1} + \left\|u'(T_m)\right\|_{H^2 \cap H_0^1} \le M,\ \forall m \in \mathbb{N}.$$

Following the argument as above, for each $m \in \mathbb{N}$, there exists a unique weak solution

$$\bar{u} \in C^1([T_m, T_m + \eta]; H^2 \cap H_0^1),\ u'' \in L^\infty(T_m, T_m + \eta; H^2 \cap H_0^1)$$

of Prob. (1.1)-(1.2) with the initial data

$$\bar{u}(T_m) = u(T_m),\ \bar{u}'(T_m) = u'(T_m),$$

with $\eta > 0$ independent of $m \in \mathbb{N}$. By $T_m \to T_\infty$, we can get $T_m + \eta > T_\infty$ for $m \in \mathbb{N}$ sufficiently large. It is clear to see that the function $U(t)$ with

$$U(t) = \begin{cases} u(t), & 0 \le t \le T_m, \\ \bar{u}(t), & T_m \le t \le T_m + \eta, \end{cases}$$

is a weak solution of Prob. (1.1)-(1.3) on $[0, T_m + \eta]$, $T_m + \eta > T_\infty$, we obtain a contradiction to the maximality of T_∞. Thus, (3.28) holds. Theorem 3.1 is proved. □

4 Exponential decay

In this section, we make the following assumptions.

$(\bar{H}_1)$ $f \in L^\infty\left(\mathbb{R}_+; L^2\right) \cap L^1\left(\mathbb{R}_+; L^2\right) \cap C^1([0,1] \times \mathbb{R}_+)$;

$(\bar{H}_2)$ $B_1, B_2 \in C^1(\mathbb{R}_+)$ and there exist the positive constants b_{i*}, χ_{1*}, $\bar{b}_{21}$ such that

(i) $B_i(y) \ge b_{i*} > 0, \forall y \ge 0, i = 1, 2,$

(ii) $yB_1(y) \ge \chi_{1*} \int_0^y B_1(z)dz, \forall y \ge 0,$

(iii) $B_2(y) \le \bar{b}_{21} B_1(y), \forall y \ge 0;$

$(\bar{H}_3)$ $B_3 \in C^1([0,1] \times \mathbb{R}_+)$ and there exist the positive constants b_{3*}, b_3^*, σ_3 such that

(i) $b_{3*} \le B_3(x,t) \le b_3^*, \forall (x,t) \in [0,1] \times \mathbb{R}_+,$

(ii) $-\sigma_3 \le B_3'(x,t) \le 0, \forall (x,t) \in [0,1] \times \mathbb{R}_+;$

$(\bar{H}_4)$ There exist $\mathcal{F} \in C^2(\mathbb{R}^2; \mathbb{R})$ and the constants $p, q, \alpha, \beta, \tilde{d}_1, d_2, \bar{d}_2 > 0$, with $q > 2$, $p \ge \beta > 2$, such that

(i) $\frac{\partial \mathcal{F}}{\partial u}(u,v) = F(u,v)$, $\frac{\partial \mathcal{F}}{\partial v}(u,v) = -\mathcal{H}(u,v)$ for all $(u,v) \in \mathbb{R}^2$,

(ii) $uF(u,v) - v\mathcal{H}(u,v) \le d_2 \mathcal{F}(u,v)$ for all $(u,v) \in \mathbb{R}^2$;

(iii) $\mathcal{F}_1(u,v) \equiv \mathcal{F}(u,v) + \tilde{d}_1 |v|^p \le \bar{d}_2 |u|^q \left(1 + |u|^\alpha + |v|^\beta\right)$ for all $(u,v) \in \mathbb{R}^2$;

$(\bar{H}_5)$ $\chi_{1*} > \dfrac{d_2}{p}$ with d_2 as in $(\bar{H}_4, (ii))$.

We can give the examples of the functions F, $\mathcal{H}$ satisfying $(\bar{H}_4)$, as follows

Example 4.1. $F(u,v)=\left(\dfrac{\alpha}{\beta}|u|^{\alpha-2}|v|^{\beta}+|u|^{q-2}\right)u$, $\mathcal{H}(u,v)=\left(|v|^{p-2}-|u|^{\alpha}|v|^{\beta-2}\right)v$, where $\alpha,\,\beta,\,p,\,q>2$ are the constants, with $p>\max\{q,2\beta,\alpha+\beta\}$.

We see that $(\bar{H}_4)$ holds, indeed, we consider $\mathcal{F}\in C^2(\mathbb{R}^2;\mathbb{R})$ defined by $\mathcal{F}(u,v)=-\dfrac{1}{p}|v|^p+\dfrac{1}{\beta}|u|^{\alpha}|v|^{\beta}+\dfrac{1}{q}|u|^q$, then we have

$$\begin{aligned}\frac{\partial\mathcal{F}}{\partial u}(u,v)=&F(u,v),\ \frac{\partial\mathcal{F}}{\partial v}(u,v)=-\mathcal{H}(u,v)\text{ for all }(u,v)\in\mathbb{R}^2;\\ uF(u,v)-v\mathcal{H}(u,v)=&-|v|^p+\left(\tfrac{\alpha+\beta}{\beta}\right)|u|^{\alpha}|v|^{\beta}+|u|^q\\ \le&d_2\mathcal{F}(u,v)\text{ for all }(u,v)\in\mathbb{R}^2,\end{aligned}$$

where $\max\{q,\alpha+\beta\}<d_2<p$.

On the other hand, $(\bar{H}_4,(iii))$ also holds, indeed,

If $q\ge\alpha$, then

$$\begin{aligned}\mathcal{F}(u,v)+\frac{1}{p}|v|^p=&\frac{1}{\beta}|u|^{\alpha}|v|^{\beta}+\frac{1}{q}|u|^q\le\max\{1/\beta,1/q\}|u|^{\alpha}\left[|v|^{\beta}+|u|^{q-\alpha}\right]\\ \le&\max\{1/\beta,1/q\}|u|^{\alpha}\left(1+|u|^q+|v|^{\beta}\right)\\ \le&\bar{d}_2|u|^{\alpha}\left(1+|u|^q+|v|^{2\beta}\right),\end{aligned}$$

where $\tilde{d}_1=1/p$, $\bar{d}_2=2\max\{1/\beta,1/q\}$, $\alpha>2$, $p\ge2\beta>2$.

If $q<\alpha$, then

$$\begin{aligned}\mathcal{F}(u,v)+\frac{1}{p}|v|^p=&\frac{1}{\beta}|u|^{\alpha}|v|^{\beta}+\frac{1}{q}|u|^q\\ \le&\max\{1/\beta,1/q\}|u|^q\left(|u|^{\alpha-q}|v|^{\beta}+1\right)\\ \le&\max\{1/\beta,1/q\}|u|^q\left(|u|^{\alpha-q}|v|^{\beta}+1\right)\\ \le&\max\{1/\beta,1/q\}|u|^q\left(\frac{|u|^{2\alpha-2q}+|v|^{2\beta}}{2}+1\right)\\ \le&\max\{1/\beta,1/q\}|u|^q\left(\frac{1+|u|^{2\alpha}+|v|^{2\beta}}{2}+1\right)\\ \le&\bar{d}_2|u|^q\left(1+|u|^{2\alpha}+|v|^{2\beta}\right),\end{aligned}$$

where $\tilde{d}_1=1/p$, $\bar{d}_2=2\max\{1/\beta,1/q\}$, $q>2$, $p\ge2\beta>2$.

Example 4.2.

$$\begin{aligned}F(u,v)=&\left(|u|^{q-2}\Phi^{k_2}(u,v)-\frac{2k_1}{p}\frac{|v|^p\Phi^{k_1-1}(u,v)}{e+u^2+v^2}+\frac{2k_2}{q}\frac{|u|^q\Phi^{k_2-1}(u,v)}{e+u^2+v^2}\right)u,\\ \mathcal{H}(u,v)=&\left(|v|^{p-2}\Phi^{k_1}(u,v)-\frac{2k_2}{q}\frac{|u|^q\Phi^{k_2-1}(u,v)}{e+u^2+v^2}+\frac{2k_1}{p}\frac{|v|^p\Phi^{k_1-1}(u,v)}{e+u^2+v^2}\right)v,\end{aligned}$$

and $\Phi(u,v)=\ln\left(e+u^2+v^2\right)$, where $p,\,q>2$; $k_1,\,k_2>1$ are the constants, with $p>q+2k_2$.

We see that $(\bar{H}_4)$ holds, indeed, we consider $\mathcal{F} \in C^2(\mathbb{R}^2;\mathbb{R})$ defined by

$$\mathcal{F}(u,v) = -\frac{1}{p}|v|^p \Phi^{k_1}(u,v) + \frac{1}{q}|u|^q \Phi^{k_2}(u,v),$$

then we have

$$\frac{\partial \mathcal{F}}{\partial u}(u,v) = F(u,v),\ \frac{\partial \mathcal{F}}{\partial v}(u,v) = -\mathcal{H}(u,v) \text{ for all } (u,v) \in \mathbb{R}^2;$$

$$\begin{aligned}
& uF(u,v) - v\mathcal{H}(u,v) \\
=& -|v|^p \Phi^{k_1}(u,v) + |u|^q \Phi^{k_2}(u,v) \\
& -\frac{2k_1}{p}|v|^p \frac{u^2+v^2}{e+u^2+v^2}\Phi^{k_1-1}(u,v) + \frac{2k_2}{q}|u|^q \frac{u^2+v^2}{e+u^2+v^2}\Phi^{k_2-1}(u,v) \\
\le& -|v|^p \Phi^{k_1}(u,v) + |u|^q \Phi^{k_2}(u,v) + \frac{2k_2}{q}|u|^q \Phi^{k_2}(u,v) \\
=& -|v|^p \Phi^{k_1}(u,v) + \tfrac{q+2k_2}{q}|u|^q \Phi^{k_2}(u,v) \\
\le& d_2\left(-\frac{1}{p}|v|^p \Phi^{k_1}(u,v) + \frac{1}{q}|u|^q \Phi^{k_2}(u,v)\right) \\
=& d_2\mathcal{F}(u,v) \text{ for all } (u,v) \in \mathbb{R}^2,
\end{aligned}$$

where $q + 2k_2 < d_2 < p$.

On the other hand, $(\bar{H}_4, (iii))$ also holds, because of

$$\begin{aligned}
\mathcal{F}(u,v) + \frac{1}{p}|v|^p =& \frac{1}{p}|v|^p\left[1 - \Phi^{k_1}(u,v)\right] + \frac{1}{q}|u|^q \Phi^{k_2}(u,v) \\
\le& \frac{1}{q}|u|^q \Phi^{k_2}(u,v) = \frac{1}{q}|u|^q\left[1 + \ln\left(1 + \frac{u^2+v^2}{e}\right)\right]^{k_2} \\
\le& \frac{1}{q}|u|^q\left(1 + \frac{u^2+v^2}{e}\right)^{k_2} \\
\le& \frac{1}{q}|u|^q 3^{k_2-1}\left(1 + \left(\frac{u^2}{e}\right)^{k_2} + \left(\frac{v^2}{e}\right)^{k_2}\right) \\
\le& \bar{d}_2|u|^q\left(1 + |u|^{2k_2} + |v|^{2k_2}\right) \text{ for all } (u,v) \in \mathbb{R}^2,
\end{aligned}$$

where $\tilde{d}_1 = 1/p$, $\bar{d}_2 = 3^{k_2-1}/q$, $\alpha = \beta = 2k_2$, $q > 2$, $p \ge 2k_2 > 2$.

Now, we show the main result of this section. That is, the global weak solution u of Prob. (1.1)-(1.3) is exponential decay provided that $\tilde{E}(0)$ is small enough and $I(0) = \int_0^{\|\tilde{u}_{0x}\|^2} B_1(z)dz - p\int_0^1 \mathcal{F}_1(\tilde{u}_0(x), \tilde{u}_{0x}(x))dx > 0$, where $p > \max\{\beta, d_2/\chi_{1*}\}$.

In order to obtain the decay result, we construct the functional

$$\mathcal{L}(t) = \tilde{E}(t) + \delta\Psi(t), \tag{4.1}$$

with $\delta > 0$; $\Psi(t)$ as in Section 3 and

$$\tilde{E}(t) = \frac{1}{2}\left\|u'(t)\right\|^2 + \frac{1}{2}\left\|\sqrt{B_3(t)}u'_x(t)\right\|^2 + \frac{1}{2}\int_0^{\|u_x(t)\|^2} B_1(z)dz \tag{4.2}$$

$$+ \bar{\lambda} e^{-2\bar{k}t} \int_0^t e^{2\bar{k}s} \left\| u'(s) \right\|^2 ds + \tilde{d}_1 \left\| u_x(t) \right\|_{L^p}^p - \int_0^1 \mathcal{F}_1 \left(u(x,t), u_x(x,t) \right) dx,$$

where $\bar{\lambda}$, $\bar{k}$ are the constants, with $0 < \bar{\lambda} < \lambda$, $\bar{k} > 0$.

We rewrite $\tilde{E}(t)$ as follows

$$\begin{aligned} \tilde{E}(t) =& \frac{1}{2} \left\| u'(t) \right\|^2 + \frac{1}{2} \left\| \sqrt{B_3(t)} u_x'(t) \right\|^2 \\ &+ \left(\frac{1}{2} - \frac{1}{p} \right) \left[2\bar{\lambda} e^{-2\bar{k}t} \int_0^t e^{2\bar{k}s} \left\| u'(s) \right\|^2 ds + \int_0^{\|u_x(t)\|^2} B_1(z) dz \right] \\ &+ \tilde{d}_1 \left\| u_x(t) \right\|_{L^p}^p + \frac{1}{p} I(t), \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} I(t) =& 2\bar{\lambda} e^{-2\bar{k}t} \int_0^t e^{2\bar{k}s} \left\| u'(s) \right\|^2 ds + \int_0^{\|u_x(t)\|^2} B_1(z) dz \\ &- p \int_0^1 \mathcal{F}_1 \left(u(x,t), u_x(x,t) \right) dx. \end{aligned} \tag{4.4}$$

Then we have the following theorem.

Theorem 4.1. *Assume that* $(\bar{H}_1) - (\bar{H}_5)$ *hold. Let* $\tilde{u}_0, \tilde{u}_1 \in H_0^1 \cap H^2$ *such that* $I(0) > 0$ *and the initial energy* $E(0)$ *satisfy*

$$\eta^* = b_{1*} - p\bar{d}_2 R_*^{q-2} \left(1 + R_*^{\alpha} + \left(\frac{E_*}{\tilde{d}_1} \right)^{\frac{\beta}{p}} \right) > 0, \tag{4.5}$$

where

$$\begin{aligned} E_* =& \left(E(0) + \tfrac{1}{2} \|f\|_{L^1(\mathbb{R}_+;L^2)} \right) \exp \left(\|f\|_{L^1(\mathbb{R}_+;L^2)} \right), \\ R_*^2 =& \frac{2pE_*}{(p-2)b_{1*}}. \end{aligned} \tag{4.6}$$

Assume that

$$\|f(t)\|^2 \leq \bar{C}_0 \exp(-\bar{\gamma}_0 t) \textit{ for all } t \geq 0, \tag{4.7}$$

where $\bar{C}_0$, $\bar{\gamma}_0$ *are two positive constants. Then, there exist positive constants* $\bar{C}$, $\bar{\gamma}$ *such that*

$$\left\| u'(t) \right\|_{H_0^1}^2 + \|u_x(t)\|^2 + \|u_x(t)\|_{L^p}^p \leq \bar{C} \exp(-\bar{\gamma} t), \textit{ for all } t \geq 0. \tag{4.8}$$

Proof of Theorem 4.1. It consists of three steps.

Step 1. The estimate of $\tilde{E}'(t)$. We have

$$\begin{aligned} (i)\ \ \tilde{E}'(t) \leq& \frac{1}{2} \|f(t)\| + \frac{1}{2} \|f(t)\| \left\| u'(t) \right\|^2, \\ (ii)\ \ \tilde{E}'(t) \leq& - \left(\lambda - \bar{\lambda} - \frac{\varepsilon_1}{2} \right) \left\| u'(t) \right\|^2 - 2\bar{\lambda}\bar{k} e^{-2\bar{k}t} \int_0^t e^{2\bar{k}s} \left\| u'(s) \right\|^2 ds \\ &- b_{2*} \left\| u_x'(t) \right\|^2 + \frac{1}{2\varepsilon_1} \|f(t)\|^2, \end{aligned} \tag{4.9}$$

for all $\varepsilon_1 > 0$. Indeed, multiplying (1.1) by $u'(x,t)$ and integrating over $[0,1]$, we get

$$\begin{aligned}\tilde{E}'(t) =& -(\lambda-\bar{\lambda})\left\|u'(t)\right\|^2 - 2\bar{\lambda}\bar{k}e^{-2\bar{k}t}\int_0^t e^{2\bar{k}s}\left\|u'(s)\right\|^2 ds \\ &- \left\|u_x'(t)\right\|^2 B_2\left(\|u_x(t)\|^2\right) + \frac{1}{2}\int_0^1 B_3'(x,t)\left|u_x'(x,t)\right|^2 dx \\ &+ \langle f(t), u'(t)\rangle.\end{aligned} \tag{4.10}$$

On the other hand

$$\left|\langle f(t), u'(t)\rangle\right| \leq \frac{1}{2}\|f(t)\| + \frac{1}{2}\|f(t)\|\left\|u'(t)\right\|^2. \tag{4.11}$$

By $B_3'(x,t) \leq 0$, it follows from (4.10), (4.11), it is clear to see that $(4.9)_{(i)}$ holds. Similarly,

$$\left|\langle f(t), u'(t)\rangle\right| \leq \frac{1}{2\varepsilon_1}\|f(t)\|^2 + \frac{\varepsilon_1}{2}\left\|u'(t)\right\|^2 \text{ for all } \varepsilon_1 > 0. \tag{4.12}$$

By $B_3'(x,t) \leq 0$, (4.10) and (4.12), that $(4.9)_{(ii)}$ is true.
Step 2. The estimate of $I(t)$.
By the continuity of $I(t)$ and $I(0) > 0$, there exists $T_1 > 0$ such that

$$I(t) > 0, \ \forall t \in [0, T_1], \tag{4.13}$$

it implies that

$$\begin{aligned}\tilde{E}(t) \geq& \frac{1}{2}\left\|u'(t)\right\|^2 + \frac{b_{3*}}{2}\left\|u_x'(t)\right\|^2 + \left(\frac{p-2}{2p}\right) b_{1*}\|u_x(t)\|^2 \\ &+ \tilde{d}_1\|u_x(t)\|_{L^p}^p, \quad \forall t \in [0, T_1].\end{aligned} \tag{4.14}$$

Combining $(4.9)_i$, (4.14) and using Gronwall's inequality to get

$$\begin{aligned}\|u_x(t)\|^2 \leq& \frac{2p\tilde{E}(t)}{(p-2)b_{1*}} \leq \frac{2pE_*}{(p-2)b_{1*}} = R_*^2, \\ \|u_x(t)\|_{L^p}^p \leq& \frac{\tilde{E}(t)}{\tilde{d}_1} \leq \frac{E_*}{\tilde{d}_1}, \ \forall t \in [0, T_1].\end{aligned} \tag{4.15}$$

Hence, it follows from $(\bar{H}_4, (iii))$, (4.6), (4.15) that

$$\begin{aligned}&p\int_0^1 \mathcal{F}_1\left(u(x,t), u_x(x,t)\right) dx \\ \leq& p\bar{d}_2\int_0^1 |u(x,t)|^q\left(1 + |u(x,t)|^\alpha + |u_x(x,t)|^\beta\right) dx \\ \leq& p\bar{d}_2\|u_x(t)\|^q\left(1 + \|u_x(t)\|^\alpha + \|u_x(t)\|_{L^\beta}^\beta\right) \\ \leq& p\bar{d}_2\|u_x(t)\|^{q-2}\left(1 + \|u_x(t)\|^\alpha + \|u_x(t)\|_{L^p}^\beta\right)\|u_x(t)\|^2 \\ \leq& p\bar{d}_2 R_*^{q-2}\left[1 + R_*^\alpha + \left(\frac{E_*}{\tilde{d}_1}\right)^{\frac{\beta}{p}}\right]\|u_x(t)\|^2.\end{aligned} \tag{4.16}$$

Consequently, $I(t) \geq 2\bar{\lambda}e^{-2\bar{k}t} \int_0^t e^{2\bar{k}s} \|u'(s)\|^2 ds + \eta^* \|u_x(t)\|^2 \geq 0, \forall t \in [0, T_1]$.

Put $T_\infty = \sup\{T > 0 : I(t) > 0, \ \forall t \in [0, T]\}$. If $T_\infty < +\infty$ then the continuity of $I(t)$ leads to $I(T_\infty) \geq 0$.

If $I(T_\infty) > 0$, by the same arguments as in the above part we can deduce that there exists $\tilde{T}_\infty > T_\infty$ such that $I(t) > 0, \forall t \in [0, \tilde{T}_\infty]$. We obtain a contradiction to the definition of T_∞.

If $I(T_\infty) = 0$, it follows that

$$0 = I(T_\infty) \geq 2\bar{\lambda}e^{-2\bar{k}T_\infty} \int_0^{T_\infty} e^{2\bar{k}s} \|u'(s)\|^2 ds + \eta^* \|u_x(T_\infty)\|^2 \geq 0.$$

Therefore

$$u(T_\infty) = \int_0^{T_\infty} e^{2\bar{k}s} \|u'(s)\|^2 ds = 0.$$

By the fact that the function $s \longmapsto e^{2\bar{k}s} \|u'(s)\|^2$ is continuous on $[0, T_\infty]$ and $e^{2\bar{k}s} > 0$, $\forall s \in [0, T_\infty]$, we have

$$\int_0^{T_\infty} e^{2\bar{k}s} \|u'(s)\|^2 ds = 0,$$

it follows that $\|u'(s)\| = 0, \forall s \in [0, T_\infty]$, it means that u is a constant function on $[0, T_\infty]$. Then, $u(0) = u(T_\infty) = 0$. It leads to $I(0) = 0$. We get in contradiction with $I(0) > 0$.

Consequently, $T_\infty = +\infty$, i.e. $I(t) > 0, \forall t \geq 0$.

Step 3. Decay result.

At first, we show that there exist the positive constants $\bar{\beta}_1$, $\bar{\beta}_2$ such that

$$\bar{\beta}_1 E_1(t) \leq \mathcal{L}(t) \leq \bar{\beta}_2 E_1(t), \ \forall t \geq 0, \tag{4.17}$$

for δ is small enough, where

$$\begin{aligned} E_1(t) = &\|u'(t)\|^2 + \|u_x'(t)\|^2 + \|u_x(t)\|^2 + \|u_x(t)\|_{L^p}^p \\ &+ e^{-2\bar{k}t} \int_0^t e^{2\bar{k}s} \|u'(s)\|^2 ds + \int_0^{\|u_x(t)\|^2} B_1(z)dz + I(t). \end{aligned} \tag{4.18}$$

Indeed, we have

$$\begin{aligned} \mathcal{L}(t) = &\frac{1}{2} \|u'(t)\|^2 + \frac{1}{2} \left\|\sqrt{B_3(t)}u_x'(t)\right\|^2 + \tilde{d}_1 \|u_x(t)\|_{L^p}^p \\ &+ \frac{p-2}{2p} \left[2\bar{\lambda}e^{-2\bar{k}t} \int_0^t e^{2\bar{k}s} \|u'(s)\|^2 ds + \int_0^{\|u_x(t)\|^2} B_1(z)dz\right] + \frac{1}{p}I(t) \\ &+ \delta\langle u'(t), u(t)\rangle + \delta\langle B_3(t)u_x'(t), u_x(t)\rangle \\ &+ \frac{\delta\lambda}{2} \|u(t)\|^2 + \frac{\delta}{2} \int_0^{\|u_x(t)\|^2} B_2(z)dz. \end{aligned} \tag{4.19}$$

Therefore

$$\begin{aligned} |\langle u(t), u'(t)\rangle| \leq &\frac{1}{2} \|u_x(t)\|^2 + \frac{1}{2} \|u'(t)\|^2, \\ |\langle B_3(t)u_x'(t), u_x(t)\rangle| \leq &\frac{1}{2} b_3^* \left(\|u_x'(t)\|^2 + \|u_x(t)\|^2\right). \end{aligned} \tag{4.20}$$

Then

$$\begin{aligned}\mathcal{L}(t) \geq & \frac{1}{2}\left\|u'(t)\right\|^2+\frac{1}{2}b_{3*}\left\|u_x'(t)\right\|^2+\tilde{d}_1\left\|u_x(t)\right\|_{L^p}^p \\ & +\frac{p-2}{2p}\left[2\bar{\lambda}e^{-2\bar{k}t}\int_0^t e^{2\bar{k}s}\left\|u'(s)\right\|^2ds+\int_0^{\|u_x(t)\|^2}B_1(z)dz\right] \\ & +\frac{1}{2p}I(t)+\frac{1}{2p}I(t)-\frac{\delta}{2}\left(\|u_x(t)\|^2+\left\|u'(t)\right\|^2\right) \\ & -\frac{\delta}{2}b_3^*\left(\left\|u_x'(t)\right\|^2+\|u_x(t)\|^2\right) \\ \geq & \frac{1-\delta}{2}\left\|u'(t)\right\|^2+\frac{b_{3*}-\delta b_3^*}{2}\left\|u_x'(t)\right\|^2+\tilde{d}_1\left\|u_x(t)\right\|_{L^p}^p \\ & +\frac{p-2}{2p}\left[2\bar{\lambda}e^{-2\bar{k}t}\int_0^t e^{2\bar{k}s}\left\|u'(s)\right\|^2ds+\int_0^{\|u_x(t)\|^2}B_1(z)dz\right] \\ & +\frac{1}{2p}I(t)+\left(\frac{\eta^*}{2p}-\frac{\delta}{2}\left(1+b_3^*\right)\right)\|u_x(t)\|^2 \\ \geq & \bar{\beta}_1E_1(t),\end{aligned} \tag{4.21}$$

where δ is small enough and

$$\begin{aligned}\bar{\beta}_1 = & \min\left\{\frac{1-\delta}{2},\ \frac{b_{3*}-\delta b_3^*}{2},\ \tilde{d}_1,\ \frac{(p-2)\bar{\lambda}}{p},\right. \\ & \left.\frac{p-2}{2p},\ \frac{\eta^*}{2p}-\frac{\delta}{2}\left(1+b_3^*\right),\ \frac{1}{2p}\right\}>0, \\ 0 < & \delta<\min\left\{1,\ \frac{b_{3*}}{b_3^*},\ \frac{\eta^*}{p\left(1+b_3^*\right)}\right\}.\end{aligned} \tag{4.22}$$

Similarly, by $((\bar{H}_3), iii)$ and (4.20), we get

$$\begin{aligned}\mathcal{L}(t) \leq & \frac{1+\delta}{2}\left\|u'(t)\right\|^2+\frac{(1+\delta)\,b_3^*}{2}\left\|u_x'(t)\right\|^2+\tilde{d}_1\left\|u_x(t)\right\|_{L^p}^p \\ & +\frac{(p-2)\bar{\lambda}}{p}e^{-2\bar{k}t}\int_0^t e^{2\bar{k}s}\left\|u'(s)\right\|^2ds \\ & +\left(\frac{p-2}{2p}+\frac{\delta\bar{b}_{21}}{2}\right)\int_0^{\|u_x(t)\|^2}B_1(z)dz \\ & +\frac{1}{p}I(t)+\frac{\delta\left(1+\lambda+b_3^*\right)}{2}\|u_x(t)\|^2 \\ \leq & \bar{\beta}_2E_1(t),\end{aligned} \tag{4.23}$$

where $\bar{\beta}_2=\max\left\{\frac{1+\delta}{2},\ \frac{(1+\delta)\,b_3^*}{2},\ \tilde{d}_1,\ \frac{(p-2)\bar{\lambda}}{p},\ \frac{p-2}{2p}+\frac{\delta\bar{b}_{21}}{2},\ \frac{\delta\left(1+\lambda+b_3^*\right)}{2}\right\}>0.$

Next, we show that the functional $\Psi(t)$ satisfies

$$\Psi'(t)\leq\left\|u'(t)\right\|^2+\left(b_3^*+\frac{\sigma_3^2}{2\varepsilon_2}\right)\left\|u_x'(t)\right\|^2-\left(\chi_{1*}-\frac{d_2}{p}\right)\int_0^{\|u_x(t)\|^2}B_1(z)dz \tag{4.24}$$

$$+\frac{2d_2\bar{\lambda}}{p}e^{-2\bar{k}t}\int_0^t e^{2\bar{k}s}\left\|u'(s)\right\|^2 ds-\frac{d_2\delta_2}{p}I(t)-d_2\tilde{d}_1\left\|u_x(t)\right\|_{L^p}^p$$
$$-\left(\frac{d_2(1-\delta_2)\eta^*}{p}-\varepsilon_2\right)\left\|u_x(t)\right\|^2+\frac{1}{2\varepsilon_2}\left\|f(t)\right\|^2,$$

for all $\varepsilon_2>0$, $\delta_2\in(0,1)$. Its proof is as below.

By multiplying (1.1) by $u(x,t)$ and integrating over $[0,1]$, we obtain

$$\begin{aligned}\Psi'(t)=&\left\|u'(t)\right\|^2-\left\|u_x(t)\right\|^2B_1\left(\left\|u_x(t)\right\|^2\right)+\left\|\sqrt{B_3(t)}u_x'(t)\right\|^2\\&+\left\langle B_3'(t)u_x'(t),u_x(t)\right\rangle+\left\langle F\left(u(t),u_x(t)\right),u(t)\right\rangle\\&-\left\langle \mathcal{H}\left(u(t),u_x(t)\right),u_x(t)\right\rangle+\left\langle f(t),u(t)\right\rangle.\end{aligned}\tag{4.25}$$

Furthermore, by $(\bar{H}_4,(ii))$, we get

$$\begin{aligned}&\left\langle F\left(u(t),u_x(t)\right),u(t)\right\rangle-\left\langle \mathcal{H}\left(u(t),u_x(t)\right),u_x(t)\right\rangle\\\leq& d_2\int_0^1\mathcal{F}\left(u(x,t),u_x(x,t)\right)dx\\=&d_2\left[\int_0^1\mathcal{F}_1\left(u(x,t),u_x(x,t)\right)dx-\tilde{d}_1\left\|u_x(t)\right\|_{L^p}^p\right]\\=&\frac{d_2}{p}\left[2\bar{\lambda}e^{-2\bar{k}t}\int_0^t e^{2\bar{k}s}\left\|u'(s)\right\|^2ds+\int_0^{\left\|u_x(t)\right\|^2}B_1(z)dz-I(t)\right]\\&-d_2\tilde{d}_1\left\|u_x(t)\right\|_{L^p}^p\\=&\frac{2d_2\bar{\lambda}}{p}e^{-2\bar{k}t}\int_0^t e^{2\bar{k}s}\left\|u'(s)\right\|^2ds+\frac{d_2}{p}\int_0^{\left\|u_x(t)\right\|^2}B_1(z)dz\\&-\frac{d_2\delta_2}{p}I(t)-\frac{d_2(1-\delta_2)}{p}I(t)-d_2\tilde{d}_1\left\|u_x(t)\right\|_{L^p}^p\\\leq&\frac{2d_2\bar{\lambda}}{p}e^{-2\bar{k}t}\int_0^t e^{2\bar{k}s}\left\|u'(s)\right\|^2ds+\frac{d_2}{p}\int_0^{\left\|u_x(t)\right\|^2}B_1(z)dz\\&-\frac{d_2\delta_2}{p}I(t)-\frac{d_2(1-\delta_2)\eta^*}{p}\left\|u_x(t)\right\|^2-d_2\tilde{d}_1\left\|u_x(t)\right\|_{L^p}^p.\end{aligned}\tag{4.26}$$

By $(\bar{H}_2,(ii))$ and $(\bar{H}_3)$ we also have

$$\begin{aligned}-\left\|u_x(t)\right\|^2B_1\left(\left\|u_x(t)\right\|^2\right)\leq&-\chi_{1*}\int_0^{\left\|u_x(t)\right\|^2}B_1(z)dz,\\\left\|\sqrt{B_3(t)}u_x'(t)\right\|^2\leq&b_3^*\left\|u_x'(t)\right\|^2,\\\left\langle B_3'(t)u_x'(t),u_x(t)\right\rangle\leq&\left\|B_3'(t)u_x'(t)\right\|\left\|u_x(t)\right\|\\\leq&\frac{\sigma_3^2}{2\varepsilon_2}\left\|u_x'(t)\right\|^2+\frac{\varepsilon_2}{2}\left\|u_x(t)\right\|^2,\\\left\langle f(t),u(t)\right\rangle\leq&\frac{\varepsilon_2}{2}\left\|u_x(t)\right\|^2+\frac{1}{2\varepsilon_2}\left\|f(t)\right\|^2,\end{aligned}\tag{4.27}$$

for all $\varepsilon_2>0$, $\delta_2\in(0,1)$.

Combining (4.25) - (4.27), we get (4.24).

The estimates $(4.9)_{(ii)}$ and (4.24) give

$$\begin{aligned}\mathcal{L}'(t) \leq &-\left(\lambda-\bar{\lambda}-\frac{\varepsilon_1}{2}-\delta\right)\left\|u'(t)\right\|^2-2\bar{\lambda}\left(\bar{k}-\frac{\delta d_2}{p}\right)e^{-2\bar{k}t}\int_0^t e^{2\bar{k}s}\left\|u'(s)\right\|^2 ds \\ &-\left[b_{2*}-\delta\left(b_3^*+\frac{\sigma_3^2}{2\varepsilon_2}\right)\right]\left\|u_x'(t)\right\|^2-\delta\left(\chi_{1*}-\frac{d_2}{p}\right)\int_0^{\|u_x(t)\|^2}B_1(z)dz \\ &-\frac{\delta d_2\delta_2}{p}I(t)-\delta\left(\frac{d_2(1-\delta_2)\eta^*}{p}-\varepsilon_2\right)\|u_x(t)\|^2 \\ &-\delta d_2\tilde{d}_1\|u_x(t)\|_{L^p}^p+\frac{1}{2}\left(\frac{1}{\varepsilon_1}+\frac{\delta}{\varepsilon_2}\right)\|f(t)\|^2,\end{aligned} \tag{4.28}$$

for all $\delta, \varepsilon_1, \varepsilon_2 > 0$, $\delta_2 \in (0,1)$.

By $\lim_{\delta_2\to 0_+,\, \varepsilon_2\to 0_+}\left(\frac{d_2(1-\delta_2)\eta^*}{p}-\varepsilon_2\right)=\frac{d_2\eta^*}{p}>0$, we can choose $\varepsilon_2>0$, $\delta_2\in(0,1)$ such that

$$\theta_1=\frac{d_2(1-\delta_2)\eta^*}{p}-\varepsilon_2>0. \tag{4.29}$$

Choosing $\delta, \varepsilon_1 > 0$ such that

$$\begin{aligned}\theta_2 =&\lambda-\bar{\lambda}-\frac{\varepsilon_1}{2}-\delta>0, \\ \theta_3 =&\bar{k}-\frac{\delta d_2}{p}>0, \\ \theta_4 =&b_{2*}-\delta\left(b_3^*+\frac{\sigma_3^2}{2\varepsilon_2}\right)>0.\end{aligned} \tag{4.30}$$

On the other hand, we have

$$\frac{1}{2}\left(\frac{1}{\varepsilon_1}+\frac{\delta}{\varepsilon_2}\right)\|f(t)\|^2\leq\frac{1}{2}\left(\frac{1}{\varepsilon_1}+\frac{\delta}{\varepsilon_2}\right)\bar{C}_0e^{-\bar{\gamma}_0t}=\tilde{C}_1e^{-\bar{\gamma}_0t}, \tag{4.31}$$

where $\tilde{C}_1=\frac{1}{2}\left(\frac{1}{\varepsilon_1}+\frac{\delta}{\varepsilon_2}\right)\bar{C}_0$.

By (4.28)-(4.31), we get

$$\begin{aligned}\mathcal{L}'(t) \leq &-\bar{\beta}_3E_1(t)+\tilde{C}_1e^{-\bar{\gamma}_0t}\leq-\frac{\bar{\beta}_3}{\bar{\beta}_2}\mathcal{L}(t)+\tilde{C}_1e^{-\bar{\gamma}_0t} \\ \leq &-\bar{\gamma}\mathcal{L}(t)+\tilde{C}_1e^{-\bar{\gamma}_0t},\end{aligned} \tag{4.32}$$

where $\bar{\beta}_3=\min\left\{\delta\theta_1,\ \theta_2,\ 2\bar{\lambda}\theta_3,\ \theta_4,\ \delta\left(\chi_{1*}-\frac{d_2}{p}\right),\ \frac{\delta d_2\delta_2}{p},\ \delta d_2\tilde{d}_1\right\}$, $0<\bar{\gamma}<\min\left\{\frac{\bar{\beta}_3}{\bar{\beta}_2},\ \bar{\gamma}_0\right\}$.

And we also have

$$\mathcal{L}(t)\geq\bar{\beta}_1E_1(t)\geq\bar{\beta}_1\left[\left\|u'(t)\right\|_{H_0^1}^2+\|u_x(t)\|^2+\|u_x(t)\|_{L^p}^p\right]. \tag{4.33}$$

Therefore, Theorem 4.1 is proved completely. □

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