Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, 44 (4), 1-17 (2024).

# On the index of a problem with an oblique derivative in weighted Sobolev space

**Sabina R. Sadigova*** · **Eminaga M. Mamedov** · **Natavan P. Nasibova**

Received: 12.01.2024 / Revised: 29.08.2024 / Accepted: 07.09.2024

**Abstract.** *The paper considers a boundary value problem with an oblique derivative for the Laplace equation in the unit ball $D$ in the complex plane. The problem is firstly studied in weighted Hardy-type classes of harmonic functions in $D$. Then, the weighted Sobolev space $W^2_{p;\nu}(D)$ with the weight function $\nu : \Gamma = \partial D \to [0, +\infty]$, is introduced into consideration. The same problem is considered in another setting in the spaces $W^2_{p;\nu}(D)$. The solution is understood in a strong sense. It is proved that if the weight belongs to the Mackenhoupt class $A_p(\Gamma)$, then this problem is Noetherian and the index is calculated. Operator corresponding to this problem, generally speaking, is not a Fredholm operator.*

**Keywords.** Laplace equation, oblique derivative, weighted space, Noetherness, index.

**Mathematics Subject Classification (2010):** 35A01, 35J05, 35K05

## 1 Introduction

The theory of boundary value problems for elliptic equations in classical formulations (i.e., with respect to Hölder and Lebesgue spaces - $L_p$-theory) is quite well developed and covered in various monographs by very famous mathematicians (see, for example, [1–8]). Due to various reasons, new Banach functional spaces appear over time, and in parallel, with respect to these spaces, one should study the problems of certain areas of mathematics such as harmonic analysis, approximation theory, the theory of differential equations, etc. These spaces include weighted Lebesgue spaces, Lebesgue spaces with variable summability exponent, Morrey spaces, grand Lebesgue spaces, Orlicz spaces, Marcinkiewicz spaces, etc. In these spaces, in comparison with other areas, the problems of harmonic analysis are quite well studied. And this, in turn, creates an acceptable opportunity to construct theories of approximation and partial differential equations with respect to these spaces. Significant results have also been obtained in this direction, and the number of such works increases with

* Corresponding author

S.R. Sadigova
Institute of Mathematics and Mechanics, Ministry of Science and Education of the Republic of Azerbaijan, Baku, Azerbaijan
E-mail: s_sadigova@mail.ru

E.M. Mamedov
Institute of Mathematics and Mechanics, Ministry of Science and Education of the Republic of Azerbaijan, Baku, Azerbaijan
E-mail: eminm62@gmail.com

N.P. Nasibova
Azerbaijan State Oil and Industry University, Baku, Azerbaijan
E-mail: natavan2008@gmail.com; natavan.nasibova@asoiu.edu.az

time (see, for example, works [9–23] and their references). These spaces have their own characteristics and in connection with this, the study of certain problems in these spaces faces certain difficulties. For example, weighted Lebesgue spaces and Lebesgue spaces with variable summability are not invariant with respect to the shift operator, Morrey and grand Lebesgue spaces are not separable, and so on. Therefore, when studying certain problems, many classical methods are not applicable and other research approaches should be involved.

Our work is devoted precisely to the above-named special case both with respect to space and with respect to the problem under consideration. This work considers a boundary value problem with an oblique derivative for the Laplace equation in the unit ball $D$ on the complex plane. First the problem is solved in weighted Hardy-type classes of harmonic functions in $D$. Then the weighted Sobolev space $W^2_{p;\nu}(D)$ with the weight function $\nu : \Gamma = \partial D \to [0, +\infty]$, is introduced into consideration. The same problem is considered in another setting in the spaces $W^2_{p;\nu}(D)$. The solution is understood in a strong sense. It is proved that if the weight belongs to the Mackenhoupt class $A_p(\Gamma)$, then this problem is Noetherian and the index is calculated. Operator corresponding to this problem, generally speaking, is not a Fredholmness.

## 2 Necessary information

First we give the following standard notation used in the work. $N$ will be se set of all positive integers, $Z_+ = \{0\} \bigcup N$; $Z$ will be a set of all integers, $C$ will stand for the field of complex numbers; $R-$are real numbers. $D = \{z \in C : |z| < 1\}$, $\Gamma = \partial D = \{z \in C : |z| = 1\}$. Denote by $|E|$ the linear measure of a measurable (according to Lebesgue) set $E \subset \Gamma$. $C_0^\infty[-\pi, \pi]$ is the space of infinitely differentiable functions with compact support in $(-\pi, \pi)$, $X^*$ is a dual space to $X$¡ $\delta_{ij}$ is the Kronecker symbol, $d\sigma$ is an element of length on $\partial D$. $[X; Y]$ is a Banach space of linear bounded operators, acting from $X$ to $Y$, $[X] = [X; X]$.

Let $\nu : \Gamma \to [0, +\infty]$ be a weight function, i.e. measurable (according to Lebesgue) and $\left|\nu^{-1}\{0; +\infty\}\right| = 0$. We will say that $\nu(\cdot)$ belongs to the Mackenhoupt weight class$A_p(\Gamma)$ $1 < p < +\infty$, if the condition

$$\sup_{E\subset\Gamma} \left(\frac{1}{|E|}\int_E \nu(t)\,dt\right)\left(\frac{1}{|E|}\int_E (\nu(t))^{-\frac{1}{p-1}}\,dt\right)^{p-1} < +\infty$$

is satisfied, where sup is taken over all measurable sets $E \subset \Gamma$. The weights from class $A_p(\Gamma)$ have the following properties.

**Statement 2.1** *Let* $\nu \in A_p(\Gamma)$ , $1 < p < +\infty$. *Then:*

*i)* $\exists p_0 \in (1, p) \; : \; \nu \in A_{p_0}(\Gamma)$*;*

*ii)* $\exists \delta > 0$ *such that the following inverse Hölder inequality holds*

$$\left(\frac{1}{|E|}\int_E \nu^{1+\delta}(x)\,dx\right)^{\frac{1}{1+\delta}} \le \frac{C}{|E|}\int_E \nu(x)\,dx,$$

*for* $\forall E \subset \Gamma$*, where* $C-$*is a constant independent of* $E$*.*

Everywhere in the future, the unit circle $\Gamma$ and the semi-interval $[-\pi, \pi)$ will be identified using the mapping $e^{it} : [-\pi, \pi) \leftrightarrow \Gamma$. Accordingly, the function $f : \Gamma \to C$ will be identified with the function $f : [-\pi, \pi) \to C \Leftrightarrow f(t) \equiv f\left(e^{it}\right)$. Also for the function $f : D \to R$, assume

$$f_r(t) = f\left(re^{it}\right), \; 0 \le r < 1, \; t \in [-\pi, \pi).$$

Let us introduce into consideration the weighted Lebesgue space $L_{p;\nu}(\Gamma) = L_{p;\nu}(-\pi,\pi)$ $1 < p < +\infty$ , defined with the norm

$$\|f\|_{p;\nu} = \left(\int_{\Gamma} |f|^p \,\nu\, |d\tau|\right)^{\frac{1}{p}} = \left(\int_{-\pi}^{\pi} |f|^p \,\nu\, dt\right)^{\frac{1}{p}}.$$

Assume $L^R_{p;\nu}(\Gamma) = ReL_{p;\nu}(\Gamma)$. Let $\alpha = (\alpha_1;\alpha_2) \in Z^2_+ -$be a multi-index and $|\alpha| = \alpha_1 + \alpha_2$. Put $\partial^\alpha u = \frac{\partial^{|\alpha|}u}{\partial x^{\alpha_1}\partial y^{\alpha_2}}$. We define a weighted Sobolev space $W^2_{p;\nu}(D)$ with the norm $\|\cdot\|_{W^2_{p;\nu}}$:

$$\|u\|_{W^2_{p;\nu}} = \sum_{|\alpha|\le 2} \|\partial^\alpha u\|_{L_{p;\nu}(D)}.$$

The following statement is true.

**Statement 2.2** *Let $\nu \in A_p(\Gamma)$ , $1 < p < +\infty$. Then: i) the continuous embedding $L_{p;\nu}(\Gamma) \subset L_1(\Gamma)$ is true; ii) $\overline{C_0^\infty[-\pi,\pi]} = L_{p;\nu}(-\pi,\pi)$ (the closure is taken in the norm of the space $L_{p;\nu}(\Gamma)$ ).*

In order to obtain the main results, we will also need the following well-known

**Statement 2.3** *The classical trigonometric system $\{1;\cos nx;\sin nx\}_{n\in N}$ forms a basis for $L_{p;\nu}(-\pi,\pi)$ , $1 < p < +\infty$, if and only if $\nu \in A_p(\Gamma)$.*

These and other facts can be found in detail in [24,25].

Denote by $\mathscr{H}(D)$ the class of all harmonic functions in $D$, i.e.

$$\mathscr{H}(D) = \{u : D \to R : \Delta u = 0 ,\ 2 \ \ D\}.$$

Put

$$h_{p;\nu} = \left\{u \in \mathscr{H}(D) :\ \|u\|_{h_{p;\nu}} < +\infty\right\},$$

where

$$\|u\|_{h_{p;\nu}} = \sup_{0\le r<1} \|u_r(\cdot)\|_{L_{p;\nu}(\Gamma)},$$

is the norm in $h_{p;\nu}$.

Along with $h_{p;\nu}$ we also introduce into consideration the class $h^{(1)}_{p;\nu}$ of harmonic functions in $D$:

$$h^{(1)}_{p;\nu} = \left\{u \in h_{p;\nu} : \frac{\partial u}{\partial r};\frac{\partial u}{\partial \varphi} \in h_{p;\nu}\right\}.$$

The norm in $h^{(1)}_{p;\nu}$ is defined by the expression

$$\|u\|_{h^{(1)}_{p;\nu}} = \|u\|_{h_{p;\nu}} + \left\|\frac{\partial u}{\partial r}\right\|_{h_{p;\nu}} + \left\|\frac{\partial u}{\partial \varphi}\right\|_{h_{p;\nu}}.$$

We will also consider the weighted Hardy class of analytical in $D$ functions. Denote the class of analytical functions in $D$ by $\mathscr{A}(D)$ . The weighted Hardy class $H^+_{p;\nu}$ is defined as follows

$$H^+_{p;\nu} = \left\{F \in \mathscr{A}(D) :\ \|F\|_{H^+_{p;\nu}} < +\infty\right\},$$

where

$$\|F\|_{H^+_{p;\nu}} = \sup_{0<r<1} \|F_r(\cdot)\|_{L_{p;\nu}(\Gamma)}.$$

It is obvious that $F \in H^+_{p;\nu} \Leftrightarrow ReF \,\&\, ImF \in h_{p;\nu}$. The following theorem is true.

**Theorem 2.4** *Let $\nu \in A_p(\Gamma)$ , $1 < p < +\infty$. Then: i) $\forall F \in H^+_{p;\nu}$ has nontangential boundary value $F^+ \in L_{p;\nu}(\Gamma)$ and the Cauchy formula*

$$F(z) = \frac{1}{2\pi i} \int_\Gamma \frac{F^+(\xi)}{\xi - z} d\xi,$$

*holds; ii) the system $\{z^n\}_{n \in Z_+}$ ($\{\tau^n\}_{n \in Z_+}$) forms a basis for $H^+_{p;\nu}$ (for $L^+_{p;\nu}(\Gamma) = H^+_{p;\nu}/_\Gamma$).*

An analogue of the Riesz theorem is also true.

**Theorem 2.5** *Let $\nu \in A_p(\Gamma)$ , $1 < p < +\infty$, and $F \in H^+_{p;\nu}$ . Then:*
*i) $\lim\limits_{r \to 1-0} \int_{-\pi}^{\pi} |F_r(t) - F^+(t)|^p \nu(t)\, dt = 0$; ii) $\lim\limits_{r \to 1-0} \int_{-\pi}^{\pi} |F_r(t)|^p \nu(t)\, dt = \int_{-\pi}^{\pi} |F^+(t)|^p \nu(t)\, dt$.*

More details about these and other results can be found, for example, in works [27–32].
From these theorems it immediately follows

**Corollary 2.1** *Let $\nu \in A_p(\Gamma)$ , $1 < p < +\infty$. Then:*
*1) $\forall u \in h_{p;\nu}$ has nontangential boundary value $u^+ \in L_{p;\nu}(\Gamma)$ and the Poisson formula*

$$u(r;\varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\varphi - \theta)\, u^+(\theta)\, d\theta$$

*holds, where $P_r(\varphi) = \frac{1-r^2}{1-2r\cos\varphi + r^2}$, is a Poisson kernel for a circle;*
*2) the following relations*

$$\lim_{r \to 1-0} \int_{-\pi}^{\pi} \left| u(r;\varphi) - u^+(\varphi) \right|^p \nu(\varphi)\, d\varphi = 0;$$

$$\lim_{r \to 1-0} \int_{-\pi}^{\pi} |u(r;\varphi)|^p \nu(\varphi)\, d\varphi = \int_{-\pi}^{\pi} \left| u^+(\varphi) \right|^p \nu(\varphi)\, d\varphi$$

*hold.*

Thus, we can conditionally write $h_{p;\nu} = ReH^+_{p;\nu} = ImH^+_{p;\nu}$. The last relation follows from the obvious fact that $f \in H^+_{p;\nu} \Leftrightarrow i\,f \in H^+_{p;\nu}$. Hence and by Theorem 2.4 it immediately follows that the system $\left\{\frac{1}{2}; Rez^n; Imz^n\right\}_{n \in N}$ is complete in $h_{p;\nu}$, if $\nu \in A_p(\Gamma)$ , $1 < p < +\infty$. Denoted by $h^+_{p;\nu}$ the restriction of the space $h_{p;\nu}$ to $\Gamma$ (i.e., we associate each function $u \in h_{p;\nu}$ with its boundary values $u^+ \in h^+_{p;\nu}$). Therefore, we have $h^+_{p;\nu} = ReL^+_{p;\nu} = ImL^+_{p;\nu}$. This directly implies that the relation

$$(T) \equiv \left\{ \frac{1}{2}; \cos n\varphi; \sin n\varphi \right\}_{n \in N} \subset h^+_{p;\nu},$$

is true and moreover, an arbitrary function $u^+ \in h^+_{p;\nu}$ can be expanded in the system $(T)$ in the norm of the space $L^R_{p;\nu}(\Gamma)$ . On the other hand, it is known that the system $(T)$ forms a basis for $L^R_{p;\nu}(\Gamma)$ and as a result we get $h^+_{p;\nu} = L^R_{p;\nu}(\Gamma)$. The operator that assigns to each function $u \in h_{p;\nu}$ its nontangential boundary values $u^+ \in h^+_{p;\nu}$, will be denoted by $\gamma^+$. As a result, we obtain that the following theorem is true.

**Theorem 2.6** *Let $\nu \in A_p(\Gamma)$, $1 < p < +\infty$. Then for $\forall f \in L^R_{p;\nu}(\Gamma)$ the Dirichlet problem*

$$\left.\begin{array}{l}\Delta u = 0 \ , \quad in \ \ D, \\ \\ \gamma^+ u = f \ , \ \ on \ \ \Gamma,\end{array}\right\} \tag{2.1}$$

*is uniquely solvable in class $h_{p;\nu}$, moreover, the estimate*

$$\|u\|_{h_{p;\nu}} \leq c \, \|f\|_{L_{p;\nu}} \tag{2.2}$$

*is valid, where $A > 0$ is a constant independent of $f$.*

Indeed, the fact that the problem (2.1) is uniquely solvable is obvious. Further, from Corollary 2.1, 2) it follows

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \left| u\left(r_n;\varphi\right) - u^+\left(\varphi\right)\right|^p \nu\left(\varphi\right) d\varphi = 0,$$

where $r_n \to 1-0$, $n \to \infty$, is some sequence. Hence it follows that $\exists \{n_1 < n_2 < ...\}$ : $u\left(r_{n_k};\varphi\right) \to u^+\left(\varphi\right)$, $k \to \infty$, a.e.. $\varphi \in (-\pi, \pi)$. Consequently

$$\left|u\left(r_{n_k};\varphi\right)\right|^p \nu\left(\varphi\right) \to \left|u^+\left(\varphi\right)\right|^p \nu\left(\varphi\right) \ , \ n \to \infty, \ \text{ a.e. } \varphi \in (-\pi, \pi) \ ,$$

and as a result, by the Fatou theorem, we obtain

$$\begin{array}{c}\int_{-\pi}^{\pi} \left|u^+\left(\varphi\right)\right|^p \nu\left(\varphi\right) \leq \sup\limits_{0<r<1} \int_{-\pi}^{\pi} \left|u\left(r;\varphi\right)\right|^p \nu\left(\varphi\right) d\varphi \, , \\ \Downarrow \\ \left\|u^+\right\|_{L_{p,\nu}(\Gamma)} \leq \|u\|_{h_{r,\nu}} \Leftrightarrow \left\|\gamma^+ u\right\|_{L_{p,\nu}(\Gamma)} \leq \|u\|_{h_{r,\varphi}} \ .\end{array}$$

It is quite obvious that $Ker \ \gamma^+ = 0$. Then it follows from Banach's theorem that $\exists \left(\gamma^+\right)^{-1} \in \left[L_{p;\nu}\left(\Gamma\right); h_{p;\nu}\right]$, i.e. estimate (2.2) is proved.

It follows directly from this theorem that the operator $\gamma^+ \in \left[h_{p;\nu}; L_{p;\nu}\left(\Gamma\right)\right]$ is an isomorphism between the spaces $h_{p;\nu}$ and $L_{p;\nu}\left(\Gamma\right)$. As a result, the following corollary is true.

**Corollary 2.2** *Let $\nu \in A_p(\Gamma)$, $1 < p < +\infty$. Then the system*

$$\left\{\frac{1}{2}; Rez^n; Imz^n\right\}_{n\in N} \equiv \left\{\frac{1}{2}; r^n \cos n\varphi; r^n \sin n\varphi\right\}_{n\in N},$$

*forms a basis for $h_{p;\nu}$. In this case, the biorthogonal system is determined by the functionals*

$$\begin{array}{l} l_n^+\left(u\right) = \frac{1}{\pi} \lim\limits_{r\to 1-0} \frac{1}{r^n} \int_{-\pi}^{\pi} u\left(r;\varphi\right) \cos n\varphi d\varphi \ , \ \forall n \in Z_+ \, , \\ l_n^-\left(u\right) = \frac{1}{\pi} \lim\limits_{r\to 1-0} \frac{1}{r^n} \int_{-\pi}^{\pi} u\left(r;\varphi\right) \sin n\varphi d\varphi \ , \ \forall n \in N \, . \end{array} \tag{2.3}$$

**Proof.** It is sufficient to prove the minimality of the system $\left\{\frac{1}{2}; Rez^n; Imz^n\right\}_{n\in N}$ in $h_{p;\nu}$. Consider the following functional

$$l_n^+\left(u\right) = \frac{1}{\pi} \lim_{r\to 1-0} \frac{1}{r^n} \int_{-\pi}^{\pi} u\left(r;\varphi\right) \cos n\varphi d\varphi \ , \ \forall n \in Z_+.$$

It is clear that

$$l_n^+\left(Rez^k\right) = \delta_{nk} \ , \ \forall n; k \in Z_+; \quad l_n^+\left(Imz^k\right) = 0 \, , \ \forall n \in Z_+, \, \forall k \in N$$

holds, where $z^k = r^k e^{ik\varphi}$ . We also have

$$\left|l_n^+ (u)\right| \le \frac{1}{\pi} \lim_{r \to 1-0} \frac{1}{r^n} \int_{-\pi}^{\pi} |u(r;\varphi)| \, d\varphi \le$$

$$\le \frac{1}{\pi} \lim_{r \to 1-0} \frac{1}{r^n} \left( \int_{-\pi}^{\pi} |u(r;\varphi)|^p \nu(\varphi) \, d\varphi \right)^{\frac{1}{p}} \left( \int_{-\pi}^{\pi} \nu^{-\frac{p'}{p}} (\varphi) \, d\varphi \right)^{\frac{1}{p'}}$$

$$\le const \lim_{r \to 1-0} \left[ \frac{1}{r^n} \sup_{0<r<1} \left( \int_{-\pi}^{\pi} |u(r;\varphi)|^p \nu(\varphi) \, d\varphi \right)^{\frac{1}{p}} \right] = const \, \|u\|_{h_{p;\nu}} .$$

It follows that $\{l_n^+\} \subset (h_{p;\nu})^*$. Similar calculations are valid for the expression $u_n^-$. Therefore, the system $\left\{l_n^+ ; \, l_{n+1}^-\right\}_{n \in Z_+}$ is biorthogonal to the basis $\left\{\frac{1}{2}; r^n \cos n\varphi; r^n \sin n\varphi\right\}_{n \in N}$ in the space $h_{p;\nu}$.

Corollary is proved.

We will also need the following

**Lemma 2.1** *Let $\nu \in A_p(\Gamma)$ , $1 < p < +\infty$, and $u \in h_{p;\nu}$. Then, if*

$$u(r;\varphi) = u_0^+ + \sum_{n=1}^{\infty} \left(u_n^+ \cos n\varphi + u_n^- \sin n\varphi\right) \, r^n$$

*is the expansion of the function $u(r;\varphi)$ on the system $\left\{\frac{1}{2}; Rez^n; Imz^n\right\}_{n \in N}$ in $h_{p;\nu}$, then the non-tangential boundary value $u^+(\varphi) \in L_{p;\nu}(\Gamma)$ has the expansion on the system $(T)$:*

$$u^+(\varphi) = u_0^+ + \sum_{n=1}^{\infty} \left(u_n^+ \cos n\varphi + u_n^- \sin n\varphi\right) .$$

Indeed, it suffices to show that

$$\frac{1}{\pi} \int_{-\pi}^{\pi} u^+(\varphi) \cos n\varphi d\varphi = u_n^+, \ \forall n \in Z_+$$

holds. Expressions for $\{u_n^-\}$ are proved similarly. So, by formula (2.3) we have

$$\left| u_n^+ - \frac{1}{\pi} \int_{-\pi}^{\pi} u^+(\varphi) \cos n\varphi d\varphi \right| = \frac{1}{\pi} \left| \lim_{r \to 1-0} \frac{1}{r^n} \int_{-\pi}^{\pi} \left[u(r;\varphi) - u^+(\varphi)\right] \cos n\varphi d\varphi \right|$$

$$\le \frac{1}{\pi} \lim_{r \to 1-0} \frac{1}{r^n} \int_{-\pi}^{\pi} \left|u(r;\varphi) - u^+(\varphi)\right| d\varphi$$

$$\le const \lim_{r \to 1-0} \frac{1}{r^n} \left( \int_{-\pi}^{\pi} \left|u(r;\varphi) - u^+(\varphi)\right|^p \nu(\varphi) \, d\varphi \right)^{\frac{1}{p}} = 0.$$

Lemma is proved.

We will also use the following well-known notions and facts. Consider the following Hardy-Littlewood maximum operator

$$(M_\Gamma f)(\xi) = \sup_{\xi \in B_r(\tau)} \frac{1}{r} \int_{B_r(\tau)} |f(\eta)| \, |d\eta| ,$$

where sup is taken over all balls $B_r(\tau)$ with center $\tau \in \Gamma$ and radius $r > 0$. The following well-known theorem is true (see, e.g., [34])

**Theorem 2.7** *Operator $M_\Gamma$ is bounded in $L_{p;\nu}(\Gamma)$ , $1 < p < +\infty$, i.e.. $M_\Gamma \in [L_{p;\nu}(\Gamma)] \Leftrightarrow \nu \in A_p(\Gamma)$.*

Denote by $\theta_0(\tau)$ a non-tangential angle of $\theta_0 \in (0, \pi)$ and with a vertex at the point $\tau \in \Gamma$. It is known that (see e.g., [33, p. 237]) there exists a constant $C_{\theta_0}$ that depends only on the angle $\theta_0$ (but independent of $\tau \in \Gamma$), for which the inequality

$$\sup_{re^{it} \in \theta_0(\tau)} \left| u\left(re^{it}\right) \right| \le C_{\theta_0} M f(\tau), \quad \text{a.e. } \tau \in \Gamma \tag{2.4}$$

holds, where $u(z)$ is the Poisson-Lebesgue integral of the function $f(\cdot)$:

$$u\left(re^{it}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t - \varphi) f(\varphi)\, d\varphi.$$

## 3 Main results

We will consider the same problem in two settings. First, we will solve the formulated problem in the weighted class of harmonic functions in $D$. In this case, the boundary condition is given by the operator $\gamma^+$. Then the same problem is solved in the weighted Sobolev space. In this case, the boundary condition is defined using the trace operator.

### 3.1. $h^{(1)}_{p;\nu}$ setting.

Let $(r; \varphi)$ , $0 \le r < 1$ , $-\pi \le \varphi < \pi$ be polar coordinates in $D$. Consider the following oblique derivative problem

$$\Delta_{r;\varphi} u = 0, \quad \text{in } D; \tag{3.1}$$

$$\gamma^+ \left( \cos\varphi \frac{\partial u}{\partial r} + \sin\varphi \frac{\partial u}{\partial \varphi} \right) \equiv \left( \cos\varphi \frac{\partial u}{\partial r} + \sin\varphi \frac{\partial u}{\partial \varphi} \right) \Big/_{r=1} = f(\varphi) \ , \ \varphi \in [-\pi, \pi), \tag{3.2}$$

where $\Delta_{r;\varphi} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}$, is the Laplace operator in polar coordinates. It is known that this boundary value problem is elliptic (see, for example, [26]). Problem (3.1), (3.2) will be solved in the space $h^{(1)}_{p;\nu}$ . So let $u \in h^{(1)}_{p;\nu}$ be the solution of this problem, where $f \in L_{p;\nu}(\Gamma)$ is some given function. Let us expand these functions in the corresponding bases

$$f(\varphi) = f_0^+ + \sum_{n=1}^{\infty} \left( f_n^+ \cos n\varphi + f_n^- \sin n\varphi \right),$$

$$u(r; \varphi) = u_0^+ + \sum_{n=1}^{\infty} \left( u_n^+ \cos n\varphi + u_n^- \sin n\varphi \right) r^n, \tag{3.3}$$

where and $\{f_n^\pm\}$ are the corresponding biorthogonal coefficients, which are defined by the expressions

$$f_n^+ = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\varphi) \cos n\varphi d\varphi \ ; \ \ f_n^- = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\varphi) \sin n\varphi d\varphi,$$

$$u_n^+ = \frac{1}{\pi} \lim_{r \to 1-0} \frac{1}{r^n} \int_{-\pi}^{\pi} u(r; \varphi) \cos n\varphi d\varphi \ ;$$

$$u_n^- = \frac{1}{\pi} \lim_{r \to 1-0} \frac{1}{r^n} \int_{-\pi}^{\pi} u(r;\varphi) \sin n\varphi d\varphi \,. \tag{3.4}$$

Represent $u(r;\varphi)$ in the form

$$u(r;\varphi) = \sum_{n=-\infty}^{+\infty} A_n r^{|n|} e^{in\varphi},$$

where

$$A_n = \begin{cases} u_0^+ \,, \; n = 0, \\ u_n^+ - iu_n^- \,, \; n \neq 0 \,. \end{cases}$$

We have

$$\frac{\partial u}{\partial r} = \sum_{n \neq 0} |n| \, A_n r^{|n|-1} e^{in\varphi}; \, \frac{\partial u}{\partial \varphi} = \sum_{n \neq 0} i \, n \, A_n r^{|n|} e^{in\varphi}.$$

By $\left(\frac{\partial u}{\partial r}\right)^+$ and $\left(\frac{\partial u}{\partial \varphi}\right)^+$ we denote the nontangential boundary values of the functions $\frac{\partial u}{\partial r}; \frac{\partial u}{\partial \varphi} \in h_{p;\nu}$ , respectively, i.e.

$$\left(\frac{\partial u}{\partial r}\right)^+ = \gamma^+ \left(\frac{\partial u}{\partial r}\right), \left(\frac{\partial u}{\partial \varphi}\right)^+ = \gamma^+ \left(\frac{\partial u}{\partial \varphi}\right).$$

Then from Lemma 2.1 we directly obtain

$$\left(\frac{\partial u}{\partial r}\right)^+ = \sum_{n \neq 0} |n| \, A_n e^{in\varphi}; \left(\frac{\partial u}{\partial \varphi}\right)^+ = \sum_{n \neq 0} i \, n \, A_n e^{in\varphi}.$$

Taking into account these expressions in the boundary condition (3.2), we have

$$\frac{e^{i\varphi} + e^{-i\varphi}}{2} \sum_{n \neq 0} |n| \, A_n e^{in\varphi} + \frac{e^{i\varphi} - e^{-i\varphi}}{2} \sum_{n \neq 0} i \, n \, A_n e^{in\varphi} = \sum_{n=-\infty}^{+\infty} c_n(f) \, e^{in\varphi},$$

where

$$c_n(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) \, e^{-in\varphi} d\varphi \;, \; \forall n \in Z_+.$$

Consequently

$$\sum_{n \neq 0} A_n \left(|n| + n\right) e^{i(n+1)\varphi} + \sum_{n \neq 0} A_n \left(|n| - n\right) e^{i(n-1)\varphi} = 2 \sum_{n=-\infty}^{+\infty} c_n(f) \, e^{in\varphi}.$$

By making the appropriate changes, as a result we have

$$\sum_{n=2}^{\infty} (n-1) \, A_{n-1} e^{in\varphi} - \sum_{n=-\infty}^{-2} (n+1) \, A_{n+1} e^{in\varphi} = \sum_{n=-\infty}^{+\infty} c_n(f) \, e^{in\varphi}. \tag{3.5}$$

This immediately implies that for the solvability of problem (3.1), (3.2), the fulfillment of the conditions

$$c_{-1}(f) = c_0(f) = c_1(f) = 0, \tag{3.6}$$

is necessary. Further, the coefficient $A_0$ is not included in the left side of relation (3.5), and hence it remains an arbitrary constant. For the remaining coefficients $\{A_n\}$ from (3.5) we obtain

$$A_n = \frac{1}{n} c_{n+1}(f)\ ,\ n \geq 1;$$

and

$$A_n = -\frac{1}{n} c_{n-1}(f)\ ,\ n \leq -1.$$

As a result, for the solution $u(r;\varphi) \in h_{p;\nu}$ of problem (3.1), (3.2), we obtain the formal expression

$$u(r;\varphi) = A_0 + \sum_{n=-\infty}^{-1} \frac{c_{n-1}(f)}{|n|} r^{|n|} e^{in\varphi} + \sum_{n=1}^{\infty} \frac{c_{n+1}(f)}{n} r^n e^{in\varphi}. \tag{3.7}$$

Let us show that the function $u(r;\varphi)$ defined by expression (3.7) is indeed a solution of problem (3.1), (3.2). It is quite obvious that $\mathit{\Delta}_{r;\varphi} u(r;\varphi) = 0\,,\ \forall re^{i\varphi} \in D$. Let us show that $u \in h^{(1)}_{p;\nu}$. For this, it is enough to prove that $\frac{\partial u}{\partial r}; \frac{\partial u}{\partial \varphi} \in h_{p;\nu}$. It is clear that $u(0;\varphi) = u_0^+ = const$. Then from the formula

$$u(r;\varphi) = \int_0^r \frac{\partial u(\rho;\varphi)}{\partial \rho} d\rho + u(0;\varphi),$$

we get that if $\frac{\partial u}{\partial r} \in h_{p;\nu}$, then $u \in h_{p;\nu}$. Thus, differentiating the expression (3.7) term by term with respect to $r$, we have

$$\frac{\partial u}{\partial r} = \sum_{n=-\infty}^{-1} c_{n-1}(f)\, r^{|n|-1} e^{in\varphi} + \sum_{n=1}^{\infty} c_{n+1}(f)\, r^{n-1} e^{in\varphi}.$$

Assume

$$u_1(r;\varphi) = \sum_{n=-\infty}^{-1} c_{n-1}(f)\ r^{|n|-1} e^{in\varphi},$$

$$u_2(r;\varphi) = \sum_{n=1}^{\infty} c_{n+1}(f)\ r^{n-1} e^{in\varphi}.$$

Let us show that $u_k \in h_{p;\nu}\,,\ k = 1, 2$. It suffices to show that the inclusion $u_2 \in h_{p;\nu}$ (the inclusion $u_1 \in h_{p;\nu}$ is proved similarly). It is clear that $\mathit{\Delta} u_2 = 0$. We have

$$u_2(r;\varphi) = e^{-i\varphi} r^{-2} \vartheta(r;\varphi),$$

where

$$\vartheta(r;\varphi) = \sum_{n=2}^{\infty} c_n(f)\ r^n e^{in\varphi}.$$

It is easy to see that the function $u_2(r;\varphi)$ tends uniformly to the function $c_2(f)\ e^{i\varphi}$ as $r \to +0$ on $\mathit{\Gamma}$. Therefore, the following relation

$$\sup_{0<r<1} \|u_2(r;\ \cdot)\|_{L_{p;\nu}(\mathit{\Gamma})} < +\infty \Leftrightarrow \sup_{0<r<1} \|\vartheta(r;\ \cdot)\|_{L_{p;\nu}(\mathit{\Gamma})} < +\infty, \tag{3.8}$$

is true. Let

$$g(\varphi) = \sum_{n=2}^{\infty} c_n(f)\ e^{in\varphi}\ ,\ \varphi \in (-\pi, \pi).$$

From the basicity of the system of exponents $\left\{ e^{in\varphi}\right\}_{n\in Z}$ in $L_{p;\nu}\left(\Gamma\right)$ , $1<p<+\infty$ , and $f\in L_{p;\nu}\left(\Gamma\right)$ it follows that $g\in L_{p;\nu}\left(\Gamma\right)$. For $\vartheta\left(r;\varphi\right)$ the following Poisson-Lebesgue formula is valid

$$\vartheta\left(r;\varphi\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r\left(\varphi-\theta\right)g\left(\theta\right)\,d\theta.$$

Taking into account estimate (2.4) and applying Theorem 2.7 to the expression $\vartheta\left(r;\varphi\right)$ we obtain

$$\left\|\vartheta\left(r;\,\cdot\,\right)\right\|_{L_{p;\nu}(\Gamma)}\leq c\left\|M_{\Gamma}g\right\|_{L_{p;\nu}(\Gamma)}\leq c\left\|g\right\|_{L_{p;\nu}(\Gamma)},$$

where $c>0$ is a constant independent of $g$ . Hence it directly follows that $\vartheta\in h_{p;\nu}$, as a result $u_1\in h_{p;\nu}$. Similarly, it is established that $u_2\in h_{p;\nu}$. Thus, the inclusion $\frac{\partial u}{\partial r}\in h_{p;\nu}$ is valid.

We also have

$$\frac{\partial u}{\partial\varphi}=-i\sum_{n=-\infty}^{-1}c_{n-1}\left(f\right)r^{|n|}e^{in\varphi}+i\sum_{n=1}^{\infty}c_{n+1}\left(f\right)r^{n}e^{in\varphi}.$$

From the same considerations it follows that $\frac{\partial u}{\partial\varphi}\in h_{p;\nu}$. So, it is established that $u\in h^{(1)}_{p;\nu}$.

And now we turn to calculating the index of problem (3.1), (3.2). Consider the operator $\gamma:h^{(1)}_{p;\nu}\to L_{p;\nu}\left(\Gamma\right)$ defined by the expression

$$\left(\gamma u\right)\left(\varphi\right)=\cos\varphi\left(\frac{\partial u}{\partial r}\right)^{+}+\sin\varphi\left(\frac{\partial u}{\partial\varphi}\right)^{+},\ \varphi\in\left(-\pi,\pi\right).$$

We have

$$\left\|\gamma u\right\|_{L_{p;\nu}(\Gamma)}\leq\left\|\cos\varphi\left(\frac{\partial u}{\partial r}\right)^{+}\right\|_{L_{p;\nu}(\Gamma)}+\left\|\sin\varphi\left(\frac{\partial u}{\partial\varphi}\right)^{+}\right\|_{L_{p;\nu}(\Gamma)}$$

$$\leq\left\|\left(\frac{\partial u}{\partial r}\right)^{+}\right\|_{L_{p;\nu}(\Gamma)}+\left\|\left(\frac{\partial u}{\partial\varphi}\right)^{+}\right\|_{L_{p;\nu}(\Gamma)}\leq/Theorem\;\;2.3/$$

$$\leq c\left(\left\|\frac{\partial u}{\partial r}\right\|_{h_{p;\nu}}+\left\|\frac{\partial u}{\partial\varphi}\right\|_{h_{p;\nu}}\right)\leq c\left\|u\right\|_{h^{(1)}_{p;\nu}}.$$

Hence it follows that $\gamma\in\left[h^{(1)}_{p;\nu};L_{p;\nu}\left(\Gamma\right)\right]$. Let us calculate the index of this operator. So, let $u\in h^{(1)}_{p;\nu}$ & $u\in Ker\gamma$, i.e. $\gamma\,u=0$. It is clear that $u\left(r;\varphi\right)$ has the representation

$$u\left(r;\varphi\right)=\sum_{n=-\infty}^{+\infty}A_nr^{|n|}e^{in\varphi}.$$

We have

$$\gamma\,u=\frac{1}{2}\left[\sum_{n=2}^{\infty}\left(n-1\right)A_{n-1}e^{in\varphi}-\sum_{n=-\infty}^{-2}\left(n+1\right)A_{n+1}e^{in\varphi}\right]=0.$$

From the basicity of the system $\left\{e^{in\varphi}\right\}_{n\in Z}$ in $L_{p;\nu}\left(\Gamma\right)$ we obtain $A_n=0\,,\forall n\neq0$. Consequently

$$u\left(r;\varphi\right)=A_0\equiv const\;\Rightarrow\dim Ker\,\gamma=1.$$

On the other hand, for the solvability of the equation $\gamma\, u = f$, the right side $f$ must satisfy three conditions of the form (3.6), where $c_k \in (L_{p;\nu}(\Gamma))^*$ , $k = -1; 0; 1$ are concrete nonzero functional. Hence it follows that, $c\mathrm{odim}\,\gamma = 3$ and as a result the index $æ(\gamma)$ of the operator $\gamma$ (in other words, of problems (3.1), (3.2)) is equal to $æ(\gamma) = \dim\gamma - c\mathrm{odim}\,\gamma = -2$. As a result, the following main theorem is proved.

**Theorem 3.1** *Let $\nu \in A_p(\Gamma)$ , $1 < p < +\infty$. Then the oblique derivative problem* (3.1),(3.2)*, with the boundary function $f \in L_{p;\nu}(\Gamma)$ is Noetherian in the class $h^{(1)}_{p;\nu}$ and its index $æ(\gamma) = -2$.*

3.2. $W^2_{p;\nu}(D)$ setting.

Let us now consider the same problem in a different setting, namely, we will look for the solution of problem (3.1), (3.2) in the weighted Sobolev space $W^2_{p;\nu}(D)$ , $1 < p < +\infty$. As before, we will assume that $\nu \in A_p(\Gamma)$ , $1 < p < +\infty$. First we define the trace operator and the trace space with respect to the space $W^1_{p;\nu}(D)$. Let $\nu \in A_p(\Gamma)$ , $1 < p < +\infty$. It is clear that a continuous embedding $W^1_{p;\nu}(D) \subset W^1_1(D)$ is true. Denote by $T_\Gamma \in \left[W^1_1(D)\,; L_1(\Gamma)\right]$ the trace operator in the sense of the space $W^1_1(D)$. Assume

$$\overset{0}{W}{}^1_{p;\nu}(D) = \left\{u \in W^1_{p;\nu}(D)\,:\, T_\Gamma u = 0\right\}.$$

$\overset{0}{W}{}^1_{p;\nu}(D)$ is a subspace of $W^1_{p;\nu}(D)$. Indeed, let $\{u_n\}_{n\in N} \subset \overset{0}{W}{}^1_{p;\nu}(D)$ be a Cauchy sequence and let $\lim\limits_n u_n = u \in W^1_{p;\nu}(D)$. We have

$$\|T_\Gamma u\|_{L_1(\Gamma)} = \lim_n \|T_\Gamma u_n\|_{L_1(\Gamma)} = 0 \Rightarrow u \in \overset{0}{W}{}^1_{p;\nu}(D).$$

Consider a factor space $\mathscr{F}^1_{p;\nu}(D) = W^1_{p;\nu}(D) \Big/ \overset{0}{W}{}^1_{p;\nu}(D)$ with factor norm

$$\|F\|_{\mathscr{F}^1_{p;\nu}(D)} = \inf_{f\in F} \|f\|_{W^1_{p;\nu}(D)}.$$

It is a Banach space.

Let us also set $T_\Gamma\left(W^1_{p;\nu}(D)\right) = W^1_{p;\nu}(\Gamma; d\sigma)$. Quite similarly to [16], the following statement is proved.

**Statement 3.2** *Let $\Omega \subset R^n$ be a bounded domain with boundary $\partial\Omega \in C^{(1)}$. Then the trace operator $T_\Gamma$ establishes an isomorphism between the linear spaces $\mathscr{F}^1_{p;\nu}(D)$ and $W^1_{p;\nu}(\Gamma; d\sigma) : T_\Gamma : \mathscr{F}^1_{p;\nu}(D) \leftrightarrow W^1_{p;\nu}(\Gamma; d\sigma)$.*

Based on this statement, we define the norm in the space $W^1_{p;\nu}(\Gamma; d\sigma)$ by the expression

$$\|g\|_{W^1_{p;\nu}(\Gamma;d\sigma)} = \left\|T^{-1}_\Gamma g\right\|_{\mathscr{F}^1_{p;\nu}(D)}\,,\ \forall g \in W^1_{p;\nu}(\Gamma; d\sigma). \tag{3.9}$$

It is clear that the space $W^1_{p;\nu}(\Gamma; d\sigma)$ is a Banach space with respect to the norm (3.9). For the function $u \in W^1_{p;\nu}(D)$, denote by $F_u$ the class $F_u \in \mathscr{F}^1_{p;\nu}(D)$, containing the element $u : u \in F_u$. So we have

$$\|T_\Gamma u\|_{W^1_{p;\nu}(\Gamma;d\sigma)} = \left\|T^{-1}_\Gamma (T_\Gamma u)\right\|_{\mathscr{F}^1_{p;\nu}(D)} = \|F_u\|_{\mathscr{F}^1_{p;\nu}(D)} = \inf_{\vartheta\in F_u} \|\vartheta\|_{W^1_{p;\nu}(D)} \le \|u\|_{W^1_{p;\nu}(D)}.$$

Hence it immediately follows that $T_\Gamma \in \left[W^1_{p;\nu}(D)\,;\, W^1_{p;\nu}(\Gamma; d\sigma)\right]$. Therefore, the following statement is true.

**Statement 3.3** *Let $\nu \in A_p(\Gamma)$ , $1 < p < +\infty$. Then the trace operator $T_\Gamma$ acts boundedly from $W^1_{p;\nu}(D)$ to the trace space $W^1_{p;\nu}(\Gamma; d\sigma)$.*

So let us consider the problem

$$\Delta_{r;\varphi} u = 0\,, \quad in \quad D, \tag{3.10}$$

$$\cos\varphi\, T_\Gamma\left(\frac{\partial u}{\partial r}\right) + \sin\varphi\, T_\Gamma\left(\frac{\partial u}{\partial \varphi}\right) = f(\varphi)\ ,\ \varphi \in (-\pi, \pi)\,, \tag{3.11}$$

where $f \in W^1_{p;\nu}(\Gamma; d\sigma)$ is a given function. Assume that $u \in W^2_{p;\nu}(D)$ is a solution to problem (3.10)-(3.11). As in the previous case, the function has the following representation

$$u = \sum_{n=-\infty}^{+\infty} A_n r^{|n|} e^{in\varphi}\,,\ re^{i\varphi} \in D. \tag{3.12}$$

Let us assume that the series (3.12) converges in the space $W^1_{p;\nu}(D)$ . Considering that $T_\Gamma\left(r^{|n|}e^{in\varphi}\right) = e^{in\varphi}$ , $\forall n \in Z$, in exactly the same way as in the previous case, from the boundary condition (3.11) we obtain the relation

$$\sum_{n=2}^{\infty}(n-1)\,A_{n-1}e^{in\varphi} - \sum_{n=-\infty}^{-2}(n+1)\,A_{n+1}e^{in\varphi} = f(\varphi)\ ,\varphi \in (-\pi, \pi)\,,$$

in which the series on the left side converge in $L_{p;\nu}(\Gamma)$. Multiplying both sides by $e^{-in\varphi}$ and integrating over $(-\pi, \pi)$, we get

$$c_{-1}(f) = c_0(f) = c_1(f) = 0, \tag{3.13}$$

where

$$A_n = \begin{cases} \frac{1}{n}c_{n+1}(f)\,,\ n \geq 1\,, \\ -\frac{1}{n}c_{n-1}(f)\,,\ n \leq -1. \end{cases}$$

Therefore, as before, we have

$$u(r;\varphi) = A_0 + \sum_{n=-\infty}^{-1}\frac{c_{n-1}(f)}{|n|}r^{|n|}e^{in\varphi} + \sum_{n=1}^{\infty}\frac{c_{n+1}(f)}{n}r^n e^{in\varphi}, \tag{3.14}$$

where $A_0 \in R$ is an arbitrary number. First, note that if $f \in L^R_{p;\nu}(\Gamma)$ is a real function, the series (3.14) represents a real-valued harmonic function in $D$ . It follows directly from the fact that in this case we have $c_{-k}(f) = \overline{c_k(f)}$ , $\forall k \in Z$.

Thus, let us consider the question of the convergence of the series (3.14) in $W^1_{p;\nu}(D)$ . We first prove the following lemma.

**Lemma 3.1** *Let $\nu \in A_p(\Gamma)$ , $1 < p < +\infty$ , and the series*

$$\sum_{n=-\infty}^{-1} c_{n-1}e^{in\varphi} + \sum_{n=1}^{\infty} A_{n+1}e^{in\varphi}, \tag{3.15}$$

*converges in $L_{p;\nu}(-\pi, \pi)$ . Then the series* (3.14) *converges in $W^1_{p;\nu}(D)$.*

**Proof.** Let all the conditions of the lemma be satisfied. Let

$$u_{nm}=\sum_{k=1}^{n}\frac{c_{-k-1}}{k}r^{k}e^{-ik\varphi}+\sum_{k=1}^{m}\frac{c_{k+1}}{k}r^{k}e^{ik\varphi}\,,\ \forall n,m\in N.$$

We have

$$\left\|u_{nm}-u_{n_1m_1}\right\|_{L_{p;\nu}(D)}\leq\sum_{k=n}^{n_1}\frac{|c_{-k-1}|}{k}\left\|r^{k}\right\|_{L_{p;\nu}(D)}+\sum_{k=m}^{m_1}\frac{|c_{k+1}|}{k}\left\|r^{k}\right\|_{L_{p;\nu}(D)}\,. \tag{3.16}$$

It is clear that $|c_k|\leq const<+\infty\,,\ \forall k\in Z$ is true.

On the other hand

$$\left\|r^{k}\right\|^{p}_{L_{p;\nu}(D)}=\int_0^1\int_{-\pi}^{\pi}\left|r^{k}\right|^{p}\nu\left(\varphi\right)\,r\,d\varphi dr=const\int_0^1 r^{kp+1}dr=\frac{const}{kp+2}\Rightarrow$$

$$\left\|r^{k}\right\|_{L_{p;\nu}(D)}=\frac{const}{(kp+2)^{\frac{1}{2}}}\leq\frac{const}{k^{\frac{1}{p}}}\,,\,\forall k\in N.$$

Taking into account this relation in (3.16), we obtain

$$\left\|u_{nm}-u_{n_1m_1}\right\|_{L_{p;\nu}(D)}\leq const\left(\sum_{k=n}^{n_1}\frac{1}{k^{1+\frac{1}{p}}}+\sum_{k=m}^{m_1}\frac{1}{k^{1+\frac{1}{p}}}\right)\rightarrow 0\,,\ n;n_1;m;m_1\rightarrow\infty.$$

It follows from here that the series (3.14) converges in $L_{p;\nu}\left(D\right)$ , and as a result $u\in L_{p;\nu}\left(D\right)$ . Let us show that $\frac{\partial u}{\partial x};\frac{\partial u}{\partial y}\in L_{p;\nu}\left(D\right)$. For this purpose, we first consider the following series

$$\Psi\left(\rho;\theta\right)=\sum_{k=1}^{\infty}\rho^{k}\left(c_k\sin k\theta+d_k\cos k\theta\right).$$

Obviously, if $\sup\limits_{k}\left\{|c_k|\,;\,|d_k|\right\}<+\infty$, then on any compact set in the unit ball this series converges uniformly, and in particular, for $\forall\rho:0\leq\rho<1$, it converges uniformly on $\Gamma$. Let

$$\psi\left(\theta\right)=\sum_{k=1}^{\infty}\left(c_k\sin k\theta+d_k\cos k\theta\right).$$

It is clear that if $\psi\in L_{p;\nu}\left(-\pi,\pi\right)$, then the function $\Psi$ can be represented as a Poisson-Lebesgue integral

$$\Psi\left(\rho;\theta\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}P_{\rho}\left(s-\theta\right)\psi\left(s\right)ds.$$

Paying attention to inequality (2.4) again, we obtain that $\exists c>0$ such that the estimate

$|\Psi\left(\rho;\theta\right)|\leq c\left(M_{\Gamma}\psi\right)\left(\theta\right)$ , a.e. $\theta\in\left(-\pi,\pi\right)$, is true. Consequently

$|\Psi\left(\rho;\theta\right)|^{p}\nu\left(\theta\right)\leq c\left(M_{\Gamma}\psi\right)^{p}\nu\left(\theta\right)$ , a.e. $\theta\in\left(-\pi,\pi\right)$.

By Theorem 2.7 we have

$$\int_{-\pi}^{\pi}|\Psi\left(\rho;\theta\right)|^{p}\nu\left(\theta\right)d\theta\leq c\int_{-\pi}^{\pi}\left(M_{\Gamma}\psi\right)^{p}\nu\left(\theta\right)d\theta\leq c\int_{-\pi}^{\pi}|\psi\left(\theta\right)|^{p}\nu\left(\theta\right)d\theta\,,\ \forall\rho\in\left(0,1\right).$$

This directly implies the following estimate

$$\left\|\Psi\right\|_{L_{p;\nu}(D)}\leq A\left\|\psi\right\|_{L_{p;\nu}(\Gamma)}\,. \tag{3.17}$$

So, in polar coordinates, partial derivatives are expressed by the formulas

$$\frac{\partial u}{\partial x} = \cos\theta \frac{\partial u}{\partial \rho} - \frac{\sin\theta}{\rho} \frac{\partial u}{\partial \theta};$$

$$\frac{\partial u}{\partial y} = \sin\theta \frac{\partial u}{\partial \rho} + \frac{\cos\theta}{\rho} \frac{\partial u}{\partial \theta}.$$

Let us represent $u(\rho;\theta)$ as

$$u(\rho;\theta) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} \frac{1}{k}\rho^k (a_k \cos k\theta + b_k \sin k\theta),$$

where

$$a_k = a_k(f) = Rec_k(f); \quad b_k = b_k(f) = Imc_k(f).$$

We have

$$\frac{\partial u}{\partial x} = \sum_{k=0}^{\infty} \rho^k (a_{k+1} \cos k\theta + b_{k+1} \sin k\theta).$$

It is quite obvious that $\frac{\partial u}{\partial x}$ is also harmonic in $D$. By the condition of the lemma, the series (3.15), as well as the series

$$\varphi(\theta) = \sum_{k=0}^{\infty} \rho^k (a_{k+1} \cos k\theta + b_{k+1} \sin k\theta),$$

converges in $L_{p;\nu}(\Gamma)$. Then, by the above reasoning, we obtain that $\frac{\partial u}{\partial x}$ has a Poisson-Lebesgue representation

$$\frac{\partial u}{\partial x} = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_\rho(s-\theta)\varphi(s)\,ds.$$

Since $\varphi \in L_{p;\nu}(\Gamma)$, it follows from here, as above, that $\frac{\partial u}{\partial x} \in L_{p;\nu}(\Gamma)$ . It is proved in exactly the same way that $\frac{\partial u}{\partial y} \in L_{p;\nu}(\Gamma)$.

Lemma is proved.

Further, we note that since the system $\{e^{in\varphi}\}_{n\in Z}$ forms a basis for $L_{p;\nu}(-\pi,\pi)$, $1 < p < +\infty$, then the series (3.15) converges in $L_{p;\nu}(-\pi,\pi)$ if and only if the series

$$f(\varphi) = \sum_{n=-\infty}^{+\infty} c_n e^{in\varphi},$$

converges in $L_{p;\nu}(-\pi,\pi)$, i.e. $f \in L_{p;\nu}(-\pi,\pi)$. As a result, we obtain that if $f \in L_{p;\nu}(-\pi,\pi)$, then the function $u(\rho;\varphi)$ defined by expression (3.14) belongs to the space $W^1_{p;\nu}(D)$.

In a completely analogous way, it is proved that if the series

$$\sum_{n=-\infty}^{+\infty} nc_n e^{in\varphi}, \tag{3.18}$$

converges in $L_{p;\nu}(-\pi,\pi)$, then the series (3.14) belongs to the space $W^2_{p;\nu}(D)$.

Thus, let us consider the question of the convergence of the series (3.18) in $L_{p;\nu}(-\pi,\pi)$. Let $f \in W^1_{p;\nu}(-\pi,\pi)$. We have

$$2\pi\, c_n(f) = \int_{-\pi}^{\pi} f(t)\, e^{int}dt = \frac{1}{in}\left( f(t)\, e^{int} \Big/_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f'(t)\, e^{int}dt \right) =$$

$$= \frac{1}{in}\left(f(-\pi) - f(\pi) - 2\pi c_n(f')\right),\ \forall n \in Z,$$

where $f' = \frac{df}{dt}$. It immediately follows from this that if $f(-\pi) = f(\pi)$, then $nc_n(f) = \frac{1}{i}c_n(f')$, and as a result it is clear that the series (3.18) converges in $L_{p;\nu}(-\pi,\pi)$. Assume

$$\widetilde{W}^1_{p;\nu}(-\pi,\pi) = \left\{ f \in W^1_{p;\nu}(-\pi,\pi) :\ f(-\pi) = f(\pi) \right\}.$$

Thus, summing up the obtained results, we come to the following conclusion.

**Theorem 3.4** *Let* $\nu \in A_p(\Gamma)$ , $1 < p < +\infty$. *Then the oblique derivative problem* (3.1), (3.2) *with the boundary function* $f \in \widetilde{W}^1_{p;\nu}(-\pi,\pi)$ *is Noetherian in the space* $W^2_{p;\nu}(D)$ *and its index is* æ $= -2$.

The fact that the index æ $= -2$ follows from the same considerations as in Theorem 3.1, since in this case the coefficient $A_0$ in expression (3.14) is arbitrary, and the functionals (3.13) are also bounded in $W^1_{p;\nu}(-\pi,\pi)$ .

Consider the same problem with a boundary function $f \in W^1_{p;\nu}(-\pi,\pi)$. Let $\delta_a$ be the Dirac functional concentrated at the point $a$. Then the condition $f(-\pi) = f(\pi)$ can be written as $\delta_\pi - \delta_{-\pi}(f) = 0$. It is clear that the functional $v = \delta_\pi - \delta_{-\pi}$ is also bounded in $W^1_{p;\nu}(-\pi,\pi)$ . Taking this circumstance into account, we obtain the following theorem.

**Theorem 3.5** *Let* $\nu \in A_p(\Gamma)$ , $1 < p < +\infty$. *Then the oblique derivative problem* (3.1), (3.2) *with the boundary function* $f \in W^1_{p;\nu}(-\pi,\pi)$ *is Noetherian in the space* $W^2_{p;\nu}(D)$ *and its index is* æ $= -3$.


## Acknowledgements

Authors would like to express their deepest gratitude to Professor B.T. Bilalov for his encouragement and valuable guidance throughout this research.

This work is supported by the Azerbaijan Science Foundation-Grant No: AEF-MQM-QA-1-2021-4(41)-8/02/1-M-02.

This work was supported by the Azerbaijan Science Foundation-Grant No: AEF-MCG-2023-1(43)-13/05/1-M-05.


## References


1. Bers, L., John, F., Schechter, M.: *Partial Differential Equations, Lectures in Applied Mathematics*, 3A. Providence, R.I.: AMS XIII, 343 p., 1979.
2. Miranda, C.: *Partial Differential Equations of Elliptic Type*, Moscow, Foreign Languages Publishing House, 1957.
3. Gilbarg, D., Trudinger, N.S.: *Elliptic Partial Differential Equations of Second Order*, Berlin-Heidelberg-New York-Tokyo: Springer-Verlag, 1983.
4. Ladyzhenskaya, O.A. Solonnikov V.A., Ural'tseva N.N.: *Linear and Quasilinear Equations of Parabolic Type*, Nauka, Moscow, 1967.

5. Mikhlin, S.G.: *Linear partial differential equations*, Visshaya shkola, Moscow, 1977 (in Russian).
6. Mikhaylov, V.P.: *Partial differential equations*, Nauka, Moscow, 1976 (in Russian).
7. Mizohata, S.: *The theory of partial differential equations*, Mir, Moscow, 1977 (in Russian).
8. Bilalov, B.T., Guseynov, Z.G.: *Basicity of a system of exponents with a piece-wise linear phase in variable spaces*, Mediterr. J. Math. **9**(3), 487–498 (2012).
9. Bilalov, B.T.: *The basis property of a perturbed system of exponentials in Morrey-type spaces*, Siberian Math. J. **60** (2), 323–350 (2019).
10. Bilalov, B.T., Ahmadov, T.M., Zeren, Y., Sadigova, S.R.: *Solution in the Small and Interior Schauder-type Estimate for the $m$-th Order Elliptic Operator in Morrey-Sobolev Spaces*, Azerbaijan J. Math. **12** (2), 190-219 (2022).
11. Bilalov, B.T., Sadigova, S.R.: *On solvability in the small of higher order elliptic equations in grand-Sobolev spaces*, Complex Var. Elliptic Equ. **63** (3), 2117-2130 (2021).
12. Bilalov, B.T., Sadigova, S.R.: *Interior Schauder-type estimates for higher-order elliptic operators in grand-Sobolev spaces*, Sahand Commun. Math. Anal. **1** (2), 129-148 (2021).
13. Bilalov, B.T., Sadigova, S.R.: *On local solvability of higher order elliptic equations in rearrangement invariant spaces*, Sibirsk. Mat. Zh. **63** (3), 516–530 (2022).
14. Bilalov, B.T., Ahmedzadeh, N.R., Garayev, T.Z.: *Some Remarks on Solvability of Dirichlet Problem for Laplace Equation in Non-standard Function Spaces*, Mediterr. J. Math. **19** (133), 1-25 (2022).
15. Bilalov, B.T., Sadigova, S.R.: *On the Fredholmness of the Dirichlet problem for a second-order elliptic equation in grand-Sobolev spaces*, Ricerche Mat. **73**, 283–322 (2024).
16. Bilalov, B.T., Zeren, Y., Sadigova, S.R., Seyma, C.: *Solvability in the small of $m$-th order elliptic equations in weighted grand Sobolev spaces*, Turkish. J. Math. **46**, 2078-2095 (2022).
17. Bilalov, B.T., Sadigova, S.R., Seyma, C.: *The Concept of a Trace and the Boundedness of the Trace Operator in Banach-Sobolev Function Spaces*, Numer. Funct. Anal. Optim. **43** (9), 1069-1094 (2022).
18. Bilalov, B.T., Sadigova, S.R., Alili, V.G.: *The Method of Boundary Value Problems in the Study of the Basis Properties of Perturbed System of Exponents in Banach Function Spaces*, Comput. Methods Funct. Theory **24**, 101–120 (2024).
19. Caso, L., D'Ambrosio, R., Softova, L.: *Generalized Morrey Spaces over Unbounded Domains*, Azerbaijan J. Math. **10** (1), 193-208 (2020).
20. Castillo, R.E., Rafeiro, H., Rojas, E.M.: *Unique Continuation of the Quasilinear Elliptic Equation on Lebesgue Spaces $L_p$* Azerbaijan J. Math. **11** (1), 136-153 (2021).
21. Mammadov, T.J.: *Strong solvability of a nonlocal problem for the Laplace equation in weighted grand Sobolev spaces*, Azerbaijan J. Math. **13** (1), 188-204 (2023).
22. Palagachev, D.K., Softova, L.G.: *Elliptic Systems in Generalized Morrey Spaces*, Azerbaijan J. Math. **11** (2), 153-162 (2021).
23. Hunt, R.A., Young, W.S.: *A weighted norm inequality for Fourier series*, Bull. Amer. Math. Soc. **80** (2), 274-277 (1974).
24. Garcia-Cuerva, J., Rubio De Francia, J.L.: *Weighted Norm Inequalities and Related Topics*, North- Holland, 1985.
25. Agmon, S., Douglis, A., Nirenberg, L.: *Estimates Close to the Boundary of the Solution of Elliptic Partial Differential Equations for General Boundary Conditions*, 1, IL, Moscow, 1962.
26. Vlasov, V.I.: *Hardy spaces, approximation issues and boundary value problems*, Eurasian Math. J. **9** (3), 85-94 (2018).

27. Vlasov, V.I., Rachkov, A.V.: *Weighted Hardy-type spaces*, Dokl. Akad. Nauk, **328** (3), 281–284 (1993); *Dokl. Math.* **47** (1), 57–61 (1993).
28. Bilalov, B.T., Huseynli, A.A., Seyidova, F.Sh.: *Riemann boundary value problems in generalized weighted Hardy spaces*, Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, **43** (2), 240–251 (2017).
29. Najafov, T.I., Nasibova, N.P.: *On the Noetherness of the Riemann Problem in Generalized Weighted Hardy Classes*, Azerbaijan J. Math. **5** (2), 109-124 (2015).
30. Sadigova, S.R.: *On Riemann problem in weighted Smirnov classes with power weight*, Rocky Mountain J. Math. **51** (3), 1007–1026 (2021).
31. Bilalov, B.T., Guliyeva, A., Sadigova, S.R.: *On Riemann problem in weighted Smirnov classes with general weight*, Acta Et Commentationes Universitatis Tartuensis De Mathematica **25** (1), 33-56 (2021).
32. Stein, E.: *Singular integrals and differential properties of functions*, Mir, Moscow, 1973.
33. Garcia-Gueva, J., Rubio De Francia, J.L.: *Weighted norm inequalities and related topics*, North-Holland, 1985.
34. Bilalov, B.T., Sadigova, S.R., Nasibova, N.P., Seyma, C.: *Fredholmness of the Dirichlet problem for 2m-th order elliptic equations in grand Sobolev spaces* (submitted)