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High-order iterative schemes for a system of nonlinear Kirchhoff-Carrier wave equations associated with the helical flows of Maxwell fluid

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Abstract. *This paper is devoted to study of an initial-boundary value problem for the system of nonlinear Kirchhoff-Carrier wave equations associated with the helical flows of Maxwell fluid. Based on using the Faedo - Galerkin method together with constructing high order iterative schemes, the local existence and uniqueness of a weak solution are proved. Moreover, the sequence associated with high order iterative schemes here converges at a rate of high order to the unique weak solution. This result is an extension of the recent research, in which nonlinear wave equations associated with the helical flows of Maxwell fluid considered without the terms of Kirchhoff-Carrier type.*

Keywords. System of nonlinear Kirchhoff-Carrier wave equations, the helical flows of Maxwell fluid, Faedo-Galerkin method, High order iterative schemes.

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1 Introduction

In this paper, we consider the following problem for the system of nonlinear Kirchhoff-Carrier wave equations

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$$\begin{cases} u_{tt} - a_1\left(\|u(t)\|_0^2, \|u_x(t)\|_0^2\right)\left(u_{xx} + \frac{1}{x}u_x - \frac{1}{x^2}u\right) \\ \qquad = f(x,t,u,v),\, x \in \Omega = (1,R),\, 0 < t < T, \\ v_{tt} - a_2\left(\|v(t)\|_0^2, \|v_x(t)\|_0^2\right)\left(v_{xx} + \frac{1}{x}v_x\right) \\ \qquad = g(x,t,u,v),\, x \in \Omega,\, 0 < t < T, \\ u_x(1,t) - b_1 u(1,t) = v_x(1,t) = u(R,t) = v(R,t) = 0, \\ (u(x,0), v(x,0)) = (\tilde{u}_0(x), \tilde{v}_0(x)),\ (u_t(x,0), v_t(x,0)) = (\tilde{u}_1(x), \tilde{v}_1(x)), \end{cases} \tag{1.1}$$

where $b_1 > 0$, $R > 1$ are given constants and $a_1, a_2, \tilde{u}_0, \tilde{u}_1, \tilde{v}_0, \tilde{v}_1, f, g$ are given functions, $\|u(t)\|_0^2 = \int_1^R xu^2(x,t)dx$, $\|u_x(t)\|_0^2 = \int_1^R xu_x^2(x,t)dx$.

Prob. (1.1) here is studied in literature for nonlinear Kirchhoff-Carrier wave equations, it has its origin in the nonlinear vibration of an elastic string (Kirchhoff [11]), for which the associated equation is

$$\rho h u_{tt} = \left(P_0 + \frac{Eh}{2L}\int_0^L \left|\frac{\partial u}{\partial y}(y,t)\right|^2 dy\right) u_{xx}, \tag{1.2}$$

here u is the lateral deflection, L is the length of the string, h is the cross-sectional area, E is Young's modulus, ρ is the mass density, and P_0 is the initial tension. It is also related to the Carrier equation. In [4], Carrier established the equation which models vibrations of an elastic string when changes in tension are not small

$$\rho u_{tt} - \left(1 + \frac{EA}{LT_0}\int_0^L u^2 dx\right) u_{xx} = 0, \tag{1.3}$$

where $u(x,t)$ is the x - derivative of the deformation, T_0 is the tension in the rest position, E is the Young modulus, A is the cross-section of a string, L is the length of a string and ρ is the density of a material. Clearly, if properties of a material vary with x and t, then there is a hyperbolic equation of the type (Larkin [13])

$$u_{tt} - B\left(x,t,\|u(t)\|^2\right) u_{xx} = 0. \tag{1.4}$$

The Kirchhoff -Carrier equations received much attention for a long time. We refer the reader to, e.g., Cavalcanti et al. [3], Larkin [13], Long [15], Medeiros [16], Miranda et al. [17], [18], Ngoc et al. [19], [20], [23], Truong et al. [33], for many results and further references.

On the other hand, Prob. (1.1) is also studied in literature for Maxwell fluid between two infinite coaxial circular cylinders. This is an extension of the problem considered in [24], [25], and [32], in which $a_1(\cdot,\cdot)$, $a_2(\cdot,\cdot)$ are constants. It is well known that there is a great interest of theoretical and applied scientists relating to the fluid flows in the neighborhood of translating or oscillating bodies, in which, Maxwell fluid has received special attention; see for [5], [7] - [10], [27], [29] - [32] and the references therein. In [9], M. Jamil and C. Fetecau studied the following problem

$$\begin{cases} \lambda u_{tt} + u_t = \nu(u_{xx} + \frac{1}{x}u_x - \frac{1}{x^2}u),\, 1 < x < R,\, t > 0, \\ \lambda V_{tt} + V_t = \nu(V_{xx} + \frac{1}{x}V_x),\, 1 < x < R,\, t > 0, \\ u_x(1,t) - u(1,t) = \frac{F}{\mu}t,\ V_x(1,t) = \frac{G}{\mu}t,\ t > 0, \\ u(R,t) = V(R,t) = 0,\, t > 0, \\ u(x,0) = u_t(x,0) = 0,\, 1 < x < R, \\ V(x,0) = V_t(x,0) = 0,\, 1 < x < R, \end{cases} \tag{1.5}$$

where λ, μ, ν, F, G are the given constants, this is a mathematical model describing the helical flows of Maxwell fluid in the annular region between two infinite coaxial circular cylinders of radii 1 and $R > 1$. The authors have obtained an exact solution for the problem (1.5) by means of finite Hankel transforms and presented under series form in terms of Bessel functions $J_0(x)$, $Y_0(x)$, $J_1(x)$, $Y_1(x)$, $J_2(x)$ and $Y_2(x)$, satisfying all imposed initial and boundary conditions. Extending the results of M. Jamil and C. Fetecau [9], in [32], Truong et al. have established the global existence, uniqueness, regularity and decay of solutions of Prob. (1.1), where

$$\begin{aligned} f &= -\lambda_1 u_t - f_1(u,v) + F_1(x,t), \\ g &= -\lambda_2 v_t - f_2(u,v) + F_2(x,t) \end{aligned} \tag{1.6}$$

and $f_1(u,v)$, $f_2(u,v)$ have been assumed that $(f_1, f_2) = (D_1\mathcal{F}, D_2\mathcal{F})$ with $\mathcal{F}(u,v) \leq C_1\left(1+u^2+v^2\right)$, $\forall u, v \in \mathbb{R}$, $C_1 > 0$.

To the best of our knowledge, the system of equations of Kirchhoff-Carrier type associated with the helical flows of Maxwell fluid (1.1) has not been extensively studied.

Motivated and inspired by the results of [24], [25] and [32], because of mathematical context, we continue to extend the results of [9] to establish a sequence $\{(u_m, v_m)\}$ such that which converges at a rate of high order to a weak solution (u,v) of Prob. (1.1). In order to do that, we shall associate with Prob. (1.1) a recurrent nonlinear sequence $\{(u_m, v_m)\}$ defined by

$$\left\{ \begin{array}{l} \langle u_m''(t), w\rangle + a_1\left(\|u_m(t)\|_0^2, \|u_{mx}(t)\|_0^2\right) a(u_m(t), w) = \langle F_m(t), w\rangle, \\ \langle v_m''(t), \phi\rangle + a_2\left(\|v_m(t)\|_0^2, \|v_{mx}(t)\|_0^2\right) b(v_m(t), \phi) \\ \qquad\qquad\qquad = \langle G_m(t), \phi\rangle, \ \forall (w,\phi) \in V \times V, \\ (u_m(0), u_m'(0)) = (\tilde{u}_0, \tilde{u}_1), (v_m(0), v_m'(0)) = (\tilde{v}_0, \tilde{v}_1), m = 1, 2, \cdots, \end{array} \right. \tag{1.7}$$

where

$$\left\{ \begin{array}{l} a(u,w) = \langle u_x, w_x\rangle + b_1 u(1) w(1) + \langle \frac{1}{x}u, \frac{1}{x}w\rangle, \\ b(v,\phi) = \langle v_x, \phi_x\rangle, \text{ for all } u, v, w, \phi \in V, \end{array} \right. \tag{1.8}$$

$$\left\{ \begin{array}{l} F_m(x,t) = \sum\limits_{i+j\leq N-1} \frac{1}{i!j!} D_3^i D_4^j f[u_{m-1}, v_{m-1}](x,t)(u_m - u_{m-1})^i (v_m - v_{m-1})^j, \\ G_m(x,t) = \sum\limits_{i+j\leq N-1} \frac{1}{i!j!} D_3^i D_4^j g[u_{m-1}, v_{m-1}](x,t)(u_m - u_{m-1})^i (v_m - v_{m-1})^j \end{array} \right. \tag{1.9}$$

with $V = \{v \in H^1 : v(R) = 0\}$, $D_3^i D_4^j f = \frac{\partial^{i+j} f}{\partial u^i \partial v^j}$, $(i,j) \in \mathbb{Z}_+^2$.

The above scheme is established based on a high-order method for solving the operator equation $F(x) = 0$, see [26], it also has been applied in some works, for example see [22] - [25], [33] and the references therein. It is well known that Newton's method and its variants are used to solve nonlinear operator equations $F(x) = 0$ or systems of nonlinear equations. Newton's method arises naturally when replace $F(x)$ by the linear term in the Taylor series, so that with x_0 as a first approximation, by constructing an approximating sequence $\{x_n\}$ and showing its convergence, a zero of F will be obtained. The sequence $\{x_n\}$ can be very rapidly convergent to the zero x, if it is given a sufficiently close first approximation x_0 to x and provided derivatives of the function F behave nicely in a neighbourhood of x. In this case, one speaks of *convergence of order* N if $|u_{n+1} - u| \leq C\,|u_n - u|^N$ for some $C > 0$ and all large N, see [6]. Our results can be regarded as an extension and improvement of the corresponding results of [24], [25].

2 Preliminaries

In this paper, we put $\Omega = (1, R), Q_T = \Omega \times (0, T), T > 0$, and denote the norm in the space L^2 by $\|\cdot\|$. The notations of the function spaces used here, such as $L^2,\ H^1 \equiv H^1(\Omega)$ are standard and can be found in H. Brezis [2] or J.L. Lions's book [14]. On H^1, we shall use the following norm

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2\right)^{1/2}. \tag{2.1}$$

Considering the set

$$V = \{v \in H^1 : v(R) = 0\}, \tag{2.2}$$

then, V is a closed subspace of H^1 and on V two norms $\|v\|_{H^1}$ and $\|v_x\|$ are equivalent norms. We note that L^2, H^1 are the Hilbert spaces with respect to the corresponding scalar products

$$\langle u, v\rangle = \int_1^R xu(x)v(x)dx,\ \langle u, v\rangle + \langle u_x, v_x\rangle. \tag{2.3}$$

The norms in L^2 and H^1 induced by the corresponding scalar products (2.3) are denoted by $\|\cdot\|_0$ and $\|\cdot\|_1$, respectively. We note more that H^1 is continuously and densely embedded in L^2. Identifying L^2 with $(L^2)'$ (the dual of L^2), we have $H^1 \hookrightarrow L^2 \hookrightarrow (H^1)'$, therefore, the notation $\langle\cdot,\cdot\rangle$ is also used for the pairing between H^1 and $(H^1)'$.

Corresponding to the above norms and spaces, we have the following lemmas, the proofs of which can be found in the paper [32].

Lemma 2.1. *The following inequalities are fulfilled*

$$\begin{aligned} &\text{(i)}\quad \|v\| \le \|v\|_0 \le \sqrt{R}\,\|v\| \ \textit{for all } v \in L^2,\\ &\text{(ii)}\ \|v\|_{H^1} \le \|v\|_1 \le \sqrt{R}\,\|v\|_{H^1} \ \textit{for all } v \in H^1. \end{aligned} \tag{2.4}$$

Lemma 2.2. *The imbedding $H^1 \hookrightarrow C^0(\overline{\Omega})$ is compact and*

$$\|v\|_{C^0(\overline{\Omega})} \le \alpha_0 \|v\|_{H^1} \ \textit{for all } v \in H^1, \tag{2.5}$$

where $\alpha_0 = \frac{1}{\sqrt{2(R-1)}}\sqrt{1 + \sqrt{1 + 16(R-1)^2}}$.

Lemma 2.3. *The imbedding $V \hookrightarrow C^0(\overline{\Omega})$ is compact and*

$$\begin{aligned} &\text{(i)}\quad \|v\|_{C^0(\overline{\Omega})} \le \sqrt{R-1}\,\|v_x\| \le \sqrt{R-1}\,\|v_x\|_0 \ \textit{for all } v \in V,\\ &\text{(ii)}\quad \|v\|_0 \le \sqrt{\tfrac{R+1}{2}(R-1)}\,\|v_x\|_0 \ \textit{for all } v \in V,\\ &\text{(iii)}\quad \textstyle\int_1^R x\,|v(x)|^\gamma\,dx \le \frac{R^2-1}{2}\left(\sqrt{R-1}\right)^\gamma \|v_x\|_0^\gamma \ \textit{for all } v \in V,\ \forall\gamma > 0. \end{aligned} \tag{2.6}$$

We set

$$\begin{cases} a(u, w) = \langle u_x, w_x\rangle + b_1 u(1)w(1) + \langle \frac{1}{x}u, \frac{1}{x}w\rangle,\\ b(v, \phi) = \langle v_x, \phi_x\rangle \text{ for all } u,\, v,\, w,\, \phi \in V, \end{cases} \tag{2.7}$$

and

$$\begin{cases} \|v\|_a = \sqrt{a(v,v)} = \left(\|v_x\|_0^2 + b_1 v^2(1) + \left\|\frac{1}{x}v\right\|_0^2\right)^{1/2},\\ \|v\|_b = \sqrt{b(v,v)} = \|v_x\|_0\,,\ v \in V_R \end{cases} \tag{2.8}$$

with $b_1 > 0$ is given constant. Then, $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are the symmetric bilinear forms on $V \times V$. Moreover, it is not difficult to show that the following properties are true.

Lemma 2.4. *With* $a_1^* = \left[1 + \left(b_1 + \frac{R^2-1}{2}\right)(R-1)\right]^{1/2}$, $\bar{a}_1^* = \left[1 + \frac{R+1}{2}(R-1)^2\right]^{1/2}$, *the following inequalities are fulfilled*

$$\left\{ \begin{array}{ll} \text{(i)} & \|v_x\|_0 \le \|v\|_a \le a_1^* \|v_x\|_0 \text{, for all } v \in V, \\ \text{(ii)} & \|v_x\|_0 \le \|v\|_1 \le \bar{a}_1^* \|v_x\|_0 \text{, for all } v \in V. \end{array} \right. \tag{2.9}$$

Remark 2.1. On L^2, two norms $v \longmapsto \|v\|$ and $v \longmapsto \|v\|_0$ are equivalent. We also have the similar property for two norms $v \longmapsto \|v\|_{H^1}$ and $v \longmapsto \|v\|_1$ on H^1, and five norms $v \longmapsto \|v\|_{H^1}$, $v \longmapsto \|v\|_1$, $v \longmapsto \|v_x\|$, $v \longmapsto \|v_x\|_0$ and $v \longmapsto \|v\|_a$ on V.

Lemma 2.5. *There exists the Hilbert orthonormal base* $\{w_j\}$ *of* L^2 *consisting of the eigenfunctions* w_j *corresponding to the eigenvalue* $\bar{\lambda}_j$ *such that*

$$\left\{ \begin{array}{l} 0 < \bar{\lambda}_1 \le \bar{\lambda}_2 \le \cdots \le \bar{\lambda}_j \le \bar{\lambda}_{j+1} \le \cdots, \ \lim_{j \to +\infty} \bar{\lambda}_j = +\infty, \\ a(w_j, w) = \bar{\lambda}_j \langle w_j, w \rangle \text{ for all } w \in V, \ j = 1, 2, \cdots. \end{array} \right. \tag{2.10}$$

Furthermore, the sequence $\{w_j/\sqrt{\bar{\lambda}_j}\}_j$ *is the Hilbert orthonormal base of* V *with respect to the scalar product* $a(\cdot,\cdot)$.

On the other hand, w_j, $j = 1, 2, \cdots$, *satisfy the following boundary value problem*

$$\left\{ \begin{array}{l} L_1 w_j \equiv -(w_{jxx} + \frac{1}{x} w_{jx} - \frac{1}{x^2} w_j) = \bar{\lambda}_j w_j, \text{ in } (1, R), \\ w_{jx}(1) - b_1 w_j(1) = w_j(R) = 0, \ w_j \in C^\infty([1, R]). \end{array} \right. \tag{2.11}$$

The proof of Lemma 2.5 can be found in [[28], p.87, Theorem 7.7], with $H = L^2$, $V = \{v \in H^1 : v(R) = 0\}$ and $a(\cdot,\cdot)$ is defined as in (2.7). Similarly, we also obtain the following lemma.

Lemma 2.6. *There exists the Hilbert orthonormal base* $\{\phi_j\}$ *of* L^2 *consisting of the eigenfunctions* ϕ_j *corresponding to the eigenvalue* $\bar{\mu}_j$ *such that*

$$\left\{ \begin{array}{l} 0 < \bar{\mu}_1 \le \bar{\mu}_2 \le \cdots \le \bar{\mu}_j \le \bar{\mu}_{j+1} \le \cdots, \ \lim_{j \to +\infty} \bar{\mu}_j = +\infty, \\ b(\phi_j, \phi) = \bar{\mu}_j \langle \phi_j, \phi \rangle \text{ for all } \phi \in V, \ j = 1, 2, \cdots. \end{array} \right. \tag{2.12}$$

Furthermore, the sequence $\{\phi_j/\sqrt{\bar{\mu}_j}\}$ *is the Hilbert orthonormal base of* V *with respect to the scalar product* $b(\cdot,\cdot)$.

On the other hand, ϕ_j, $j = 1, 2, \cdots$, *satisfy the following boundary value problem*

$$\left\{ \begin{array}{l} L_2 \phi_j \equiv -(\phi_{jxx} + \frac{1}{x} \phi_{jx}) = \bar{\mu}_j \phi_j, \text{ in } (1, R), \\ \phi_{jx}(1) = \phi_j(R) = 0, \ \phi_j \in C^\infty([1, R]). \end{array} \right. \tag{2.13}$$

Remark 2.2. The weak formulation of the initial-boundary value problem (1.1) can be given in the following manner.

Definition. The weak solution of Prob. (1.1) is the couple of functions (u, v) such that (u, v) belongs to the set $\bar{W}_T$, where

$$\bar{W}_T = \Big\{ (u, v) \in L^\infty\left(0, T; (H^2 \cap V) \times (H^2 \cap V)\right) : (u', v') \in L^\infty(0, T; V \times V), \\ (u'', v'') \in L^\infty\left(0, T; L^2 \times L^2\right) \Big\},$$

furthermore (u,v) satisfies the following variational equation

$$\begin{cases} \langle u''(t),w\rangle + a_1[u](t)a(u(t),w) = \langle f[u,v](t),w\rangle, \\ \langle v''(t),\phi\rangle + a_2[v](t)b(v(t),\phi) = \langle g[u,v](t),\phi\rangle \end{cases} \tag{2.14}$$

for all $(w,\phi)\in V\times V$, a.e., $t\in(0,T)$, together with the initial conditions

$$\big(u(0),u'(0)\big) = (\tilde{u}_0,\tilde{u}_1)\,,\ \ \big(v(0),v'(0)\big) = (\tilde{v}_0,\tilde{v}_1)\,, \tag{2.15}$$

where $a_1[u](t) = a_1\left(\|u(t)\|_0^2, \|u_x(t)\|_0^2\right)$, $a_2[v](t) = a_2\left(\|v(t)\|_0^2, \|v_x(t)\|_0^2\right)$,
$f[u,v](x,t) = f(x,t,u(x,t),v(x,t))$,
$g[u,v](x,t) = g(x,t,u(x,t),v(x,t))$.

Remark 2.3. We remark that the set $\bar{W}_T$ defined as above has the following property (see [14])

$$\begin{aligned} &\bar{W}_T = \\ &\Big\{(u,v)\in L^\infty\left(0,T;(H^2\cap V)\times(H^2\cap V)\right)\cap C([0,T];V\times V)\cap C^1([0,T];L^2\times L^2): \\ &\qquad (u',v')\in L^\infty\left(0,T;V\times V\right)\cap C([0,T];L^2\times L^2), \\ &\qquad\qquad (u'',v'')\in L^\infty\left(0,T;L^2\times L^2\right)\Big\}. \end{aligned}$$

3 High order iterative schemes

In this section, Prob. (1.1) is considered with given constants $b_1>0$, $R>1$, and the following assumptions for given functions a_1, a_2, $\tilde{u}_0$, $\tilde{u}_1$, $\tilde{v}_0$, $\tilde{v}_1$, f, g

(A_1) $(\tilde{u}_0,\tilde{u}_1), (\tilde{v}_0,\tilde{v}_1)\in(V\cap H^2)\times V$, $\tilde{u}_{0x}(1)-b_1\tilde{u}_0(1) = \tilde{v}_{0x}(1) = 0$;

(A_2) $a_1, a_2\in C^1(\mathbb{R}_+^2)$, $a_i(y,z)\ge a_{i*}>0$, $\forall y,z\ge 0$, $i=1,2$;

(A_3) $f, g\in C([1,R]\times\mathbb{R}_+\times\mathbb{R}^2)$, such that
(i) $D_3^iD_4^jf$, $D_3^iD_4^jg\in C([1,R]\times\mathbb{R}_+\times\mathbb{R}^2)$, $1\le i+j\le N$,
(ii) $D_1D_3^iD_4^jf$, $D_3^{i+1}D_4^jf$, $D_4D_3^iD_4^jf\in C([1,R]\times\mathbb{R}_+\times\mathbb{R}^2)$, $0\le i+j\le N-1$,
(iii) $D_1D_3^iD_4^jg$, $D_3^{i+1}D_4^jg$, $D_4D_3^iD_4^jg\in C([1,R]\times\mathbb{R}_+\times\mathbb{R}^2)$, $0\le i+j\le N-1$,
(iv) $f(R,t,0,0) = g(R,t,0,0) = 0$, $\forall t\ge 0$.

We note more that, the partial derivatives of order i, $1\le i\le N$, of a function $f = f(x,t,u,v)$ with respect to the variables x,t,u,v are denoted by $D_1f = \dfrac{\partial f}{\partial x}$, $D_2f = \dfrac{\partial f}{\partial t}$, $D_3^if = \dfrac{\partial^i f}{\partial u^i}$; $D_4^jf = \dfrac{\partial^j f}{\partial v^j}$.

Consider $T^*>0$ fixed, let $T\in(0,T^*]$, we define

$$\begin{aligned} W_T = \Big\{&(u,v)\in L^\infty\left(0,T;(H^2\cap V)\times(H^2\cap V)\right): \\ &(u',v')\in L^\infty(0,T;V\times V), (u'',v'')\in L^2(0,T;L^2\times L^2)\Big\}, \end{aligned} \tag{3.1}$$

then W_T is the Banach space with norm

$$\begin{aligned} \|(u,v)\|_{W_T} = \max\Big\{&\|(u,v)\|_{L^\infty(0,T;(H^2\cap V)\times(H^2\cap V))}, \\ &\left\|(u',v')\right\|_{L^\infty(0,T;V\times V)}, \left\|(u'',v'')\right\|_{L^2(0,T;L^2\times L^2)}\Big\}. \end{aligned} \tag{3.2}$$

For $M > 0$, we put

$$W(M,T) = \left\{ v \in W_T : \|v\|_{W_T} \leq M \right\}, \tag{3.3}$$
$$W_1(M,T) = \{(u,v) \in W(M,T) : (u'', v'') \in L^\infty(0,T; L^2 \times L^2)\}.$$

Now, we construct the recurrent sequence $\{(u_m, v_m)\}$ defined by $(u_0, v_0) = (0,0)$, and suppose that

$$(u_{m-1}, v_{m-1}) \in W_1(M,T), \tag{3.4}$$

and associate with Prob. (2.14), (2.15) the following problem:

Find $(u_m, v_m) \in W_1(M,T)$ $(m \geq 1)$ which satisfies the following linear variational problem

$$\begin{cases} \langle u_m''(t), w\rangle + a_1[u_m](t) a(u_m(t), w) = \langle F_m(t), w\rangle, \\ \langle v_m''(t), \phi\rangle + a_2[v_m](t) b(v_m(t), \phi) = \langle G_m(t), \phi\rangle, \ \forall (w,\phi) \in V \times V, \\ (u_m(0), u_m'(0)) = (\tilde{u}_0, \tilde{u}_1), \, (v_m(0), v_m'(0)) = (\tilde{v}_0, \tilde{v}_1), \end{cases} \tag{3.5}$$

where

$$\begin{cases} a_1[u_m](t) = a_1\left(\|u_m(t)\|_0^2, \|u_{mx}(t)\|_0^2\right), \\ a_2[v_m](t) = a_2\left(\|v_m(t)\|_0^2, \|v_{mx}(t)\|_0^2\right), \\ F_m(x,t) = F_m[u_m, v_m](x,t) \\ \quad = \sum_{i+j \leq N-1} \frac{1}{i!j!} D_3^i D_4^j f[u_{m-1}, v_{m-1}](x,t)(u_m - u_{m-1})^i (v_m - v_{m-1})^j, \\ G_m(x,t) = G_m[u_m, v_m](x,t) \\ \quad = \sum_{i+j \leq N-1} \frac{1}{i!j!} D_3^i D_4^j g[u_{m-1}, v_{m-1}](x,t)(u_m - u_{m-1})^i (v_m - v_{m-1})^j. \end{cases} \tag{3.6}$$

Then, we have the following theorem.

Theorem 3.1. *Let $T^* > 0$ and $(A_1) - (A_3)$ hold. Then there exist positive constants $M, T > 0$ such that, for $(u_0, v_0) = (0,0)$, there exists a recurrent sequence $\{(u_m, v_m)\} \subset W_1(M,T)$ defined by* (3.5), (3.6).

Proof. The proof consists of three steps.

Step 1. The Faedo-Galerkin approximation (introduced by Lions [14]). Put

$$u_m^{(k)}(t) = \sum\nolimits_{j=1}^k c_{mj}^{(k)}(t) w_j, \ v_m^{(k)}(t) = \sum\nolimits_{j=1}^k d_{mj}^{(k)}(t)\phi_j, \tag{3.7}$$

where the coefficients $c_{mj}^{(k)}(t)$, $d_{mj}^{(k)}(t)$ satisfy the system of nonlinear differential equations

$$\begin{cases} \langle \ddot{u}_m^{(k)}(t), w_j\rangle + a_{1m}^{(k)}(t) a(u_m^{(k)}(t), w_j) = \langle F_m^{(k)}(t), w_j\rangle, \\ \langle \ddot{v}_m^{(k)}(t), \phi_j\rangle + a_{2m}^{(k)}(t) b(v_m^{(k)}(t), \phi_j) = \langle G_m^{(k)}(t), \phi_j\rangle, \, 1 \leq j \leq k, \\ (u_m^{(k)}(0), \dot{u}_m^{(k)}(0)) = (\tilde{u}_{0k}, \tilde{u}_{1k}), \, (v_m^{(k)}(0), \dot{v}_m^{(k)}(0)) = (\tilde{v}_{0k}, \tilde{v}_{1k}), \end{cases} \tag{3.8}$$

where

$$(\bar{u}_{0k}, \bar{u}_{1k}) = \quad \sum\nolimits_{j=1}^k (\alpha_j^{(k)}, \beta_j^{(k)}) w_j \to (\tilde{u}_0, \tilde{u}_1) \text{ strongly in } \left(H^2 \cap V\right) \times V, \tag{3.9}$$
$$(\bar{v}_{0k}, \bar{v}_{1k}) = \quad \sum\nolimits_{j=1}^k (\tilde{\alpha}_j^{(k)}, \tilde{\beta}_j^{(k)}) \phi_j \to (\tilde{v}_0, \tilde{v}_1) \text{ strongly in } \left(H^2 \cap V\right) \times V,$$

and

$$\begin{cases} a_{1m}^{(k)}(t) = a_1[u_m^{(k)}](t) = a_1\left(\left\|u_m^{(k)}(t)\right\|_0^2, \left\|u_{mx}^{(k)}(t)\right\|_0^2\right), \\ a_{2m}^{(k)}(t) = a_2[v_m^{(k)}](t) = a_2\left(\left\|v_m^{(k)}(t)\right\|_0^2, \left\|v_{mx}^{(k)}(t)\right\|_0^2\right), \\ F_m^{(k)}(x,t) = F_m[u_m^{(k)}, v_m^{(k)}](x,t) \\ \qquad = \sum_{i+j\le N-1} \frac{1}{i!j!} D_3^i D_4^j f[u_{m-1}, v_{m-1}](u_m^{(k)} - u_{m-1})^i (v_m^{(k)} - v_{m-1})^j, \\ G_m^{(k)}(x,t) = G_m[u_m^{(k)}, v_m^{(k)}](x,t) \\ \qquad = \sum_{i+j\le N-1} \frac{1}{i!j!} D_3^i D_4^j g[u_{m-1}, v_{m-1}](u_m^{(k)} - u_{m-1})^i (v_m^{(k)} - v_{m-1})^j. \end{cases} \tag{3.10}$$

Let us suppose that (u_{m-1}, v_{m-1}) satisfies (3.4). Then it is clear that the system (3.8) has a solution $(u_m^{(k)}, v_m^{(k)})$ on an interval $0 \le t \le T_m^{(k)} \le T$. The following estimates allow one to take constant $T_m^{(k)} = T$ for all m and k.

Step 2. A priori estimates.

First, we put

$$\begin{cases} \|f\|_{C^0(A_M)} = \sup_{(x,t,u,v)\in A_M} |f(x,t,u,v)|, \\ K_N(M,f) = \sum_{i+j\le N} \left\|D_3^i D_4^j f\right\|_{C^0(A_M)} + \sum_{i+j\le N-1} \left\|D_1 D_3^i D_4^j f\right\|_{C^0(A_M)} \\ \qquad + \sum_{i+j\le N-1} \left(\left\|D_3^{i+1} D_4^j f\right\|_{C^0(A_M)} + \left\|D_4 D_3^i D_4^j f\right\|_{C^0(A_M)}\right), \\ A_M = [1,R]\times[0,T^*]\times[-\sqrt{R-1}M, \sqrt{R-1}M]^2, \end{cases} \tag{3.11}$$

and

$$\begin{aligned} S_m^{(k)}(t) = {}& \left\|\dot{u}_m^{(k)}(t)\right\|_0^2 + \left\|\dot{u}_m^{(k)}(t)\right\|_a^2 + a_{1m}^{(k)}(t)\left(\left\|u_m^{(k)}(t)\right\|_a^2 + \left\|L_1 u_m^{(k)}(t)\right\|_0^2\right) \\ & + \left\|\dot{v}_m^{(k)}(t)\right\|_0^2 + \left\|\dot{v}_{mx}^{(k)}(t)\right\|_0^2 + a_{2m}^{(k)}(t)\left(\left\|v_{mx}^{(k)}(t)\right\|_0^2 + \left\|L_2 v_m^{(k)}(t)\right\|_0^2\right) \\ & + \int_0^t \left(\left\|\ddot{u}_m^{(k)}(s)\right\|_0^2 + \left\|\ddot{v}_m^{(k)}(s)\right\|_0^2\right) ds. \end{aligned} \tag{3.12}$$

Then, it follows from (3.8), (3.12) that

$$\begin{aligned} S_m^{(k)}(t) = {}& S_m^{(k)}(0) + 2\int_0^t \left[\langle F_m^{(k)}(s), \dot{u}_m^{(k)}(s)\rangle + \langle G_m^{(k)}(s), \dot{v}_m^{(k)}(s)\rangle\right] ds \\ & + 2\int_0^t \left[a(F_m^{(k)}(s), \dot{u}_m^{(k)}(s)) + \langle G_{mx}^{(k)}(s), \dot{v}_{mx}^{(k)}(s)\rangle\right] ds \\ & + 2\int_0^t \dot{a}_{1m}^{(k)}(s)\left(\left\|u_m^{(k)}(s)\right\|_a^2 + \left\|L_1 u_m^{(k)}(s)\right\|_0^2\right) ds \\ & + 2\int_0^t \dot{a}_{2m}^{(k)}(s)\left(\left\|v_{mx}^{(k)}(s)\right\|_0^2 + \left\|L_2 v_m^{(k)}(s)\right\|_0^2\right) ds \\ & + \int_0^t \left(\left\|\ddot{u}_m^{(k)}(s)\right\|_0^2 + \left\|\ddot{v}_m^{(k)}(s)\right\|_0^2\right) ds \ = S_m^{(k)}(0) + \sum_{j=1}^5 I_j. \end{aligned} \tag{3.13}$$

We need to estimate the terms of (3.13). First, we need the following lemma.
Lemma 3.2. *Put*

$$L_1 v \equiv -(v_{xx} + \frac{1}{x} v_x - \frac{1}{x^2} v), \tag{3.14}$$
$$L_2 v \equiv -(v_{xx} + \frac{1}{x} v_x),$$

and $\|v\|_{H^2 \cap V} = \sqrt{\|v_x\|_0^2 + \|v_{xx}\|_0^2}$, $v \in H^2 \cap V$.
Then, there exist two constants $\gamma_1, \bar{\gamma}_1, \gamma_2 > 0$ *such that*

$$\begin{array}{ll} \text{(i)} & \|L_1 v\|_0 \leq \bar{\gamma}_1 \|v\|_{H^2 \cap V}, \\ \text{(ii)} & \|L_1 v\|_0^2 + \|v\|_a^2 \geq \gamma_1 \|v\|_{H^2 \cap V}^2, \\ \text{(iii)} & \|L_2 v\|_0 \leq \sqrt{2} \|v\|_{H^2 \cap V}, \\ \text{(iv)} & \|L_2 v\|_0^2 + \|v_x\|_0^2 \geq \gamma_2 \|v\|_{H^2 \cap V}^2 \end{array} \tag{3.15}$$

for all $v \in H^2 \cap V$.
Proof. We have

$$\begin{aligned} \|L_1 v\|_0 &= \left\| v_{xx} + \frac{1}{x} v_x - \frac{1}{x^2} v \right\|_0 \\ &\leq \|v_{xx}\|_0 + \left\| \frac{1}{x} v_x \right\|_0 + \left\| \frac{1}{x^2} v \right\|_0 \\ &\leq \|v_{xx}\|_0 + \|v_x\|_0 + \|v\|_0 \\ &\leq \|v_{xx}\|_0 + \|v_x\|_0 + \sqrt{\frac{R+1}{2}} (R-1) \|v_x\|_0 \\ &\leq \max\left\{1, \sqrt{\frac{R+1}{2}} (R-1)\right\} (\|v_{xx}\|_0 + \|v_x\|_0) \\ &\leq \sqrt{2} \max\left\{1, \sqrt{\frac{R+1}{2}} (R-1)\right\} \left(\|v_{xx}\|_0^2 + \|v_x\|_0^2\right)^{1/2} \\ &\equiv \bar{\gamma}_1 \|v\|_{H^2 \cap V}, \end{aligned}$$

where $\bar{\gamma}_1 = \sqrt{2} \max\left\{1, \sqrt{\frac{R+1}{2}} (R-1)\right\}$.
Similarly, we have

$$\begin{aligned} \|L_2 v\|_0 &= \left\| v_{xx} + \frac{1}{x} v_x \right\|_0 \leq \|v_{xx}\|_0 + \left\| \frac{1}{x} v_x \right\|_0 \\ &\leq \|v_{xx}\|_0 + \|v_x\|_0 \leq \sqrt{2} \|v\|_{H^2 \cap V}. \end{aligned}$$

For all $\varepsilon \in (0, \frac{1}{2})$, $\eta > 0$, we have

$$\begin{aligned} \|L_1 v\|_0^2 &= \int_1^R x \left(v_{xx} + \frac{1}{x} v_x - \frac{1}{x^2} v \right)^2 dx \\ &= \|v_{xx}\|_0^2 + \left\| \frac{1}{x} v_x \right\|_0^2 + \left\| \frac{1}{x^2} v \right\|_0^2 \\ &\quad + 2\langle v_{xx}, \frac{1}{x} v_x \rangle - 2\langle v_{xx}, \frac{1}{x^2} v \rangle - 2\langle \frac{1}{x} v_x, \frac{1}{x^2} v \rangle. \end{aligned}$$

By

$$\begin{aligned}
2\langle v_{xx}, \frac{1}{x}v_x\rangle \leq& \varepsilon \left\| v_{xx} \right\|_0^2 + \frac{1}{\varepsilon} \left\| \frac{1}{x}v_x \right\|_0^2, \\
2\langle v_{xx}, \frac{1}{x^2}v\rangle \leq& \varepsilon \left\| v_{xx} \right\|_0^2 + \frac{1}{\varepsilon} \left\| \frac{1}{x^2}v \right\|_0^2, \\
2\langle \frac{1}{x}v_x, \frac{1}{x^2}v\rangle \leq& \varepsilon \left\| \frac{1}{x}v_x \right\|_0^2 + \frac{1}{\varepsilon} \left\| \frac{1}{x^2}v \right\|_0^2, \\
\eta \left\| v \right\|_a^2 =& \eta \left(\left\| v_x \right\|_0^2 + b_1 v^2(1) + \left\| \frac{1}{x}v \right\|_0^2 \right) \\
\geq& \eta \left(\left\| v_x \right\|_0^2 + \left\| \frac{1}{x^2}v \right\|_0^2 \right),
\end{aligned}$$

we deduce that

$$\begin{aligned}
&\left\| L_1 v \right\|_0^2 + \eta \left\| v \right\|_a^2 \\
&\geq \left\| v_{xx} \right\|_0^2 + \left\| \frac{1}{x}v_x \right\|_0^2 + \left\| \frac{1}{x^2}v \right\|_0^2 - \varepsilon \left\| v_{xx} \right\|_0^2 - \frac{1}{\varepsilon} \left\| \frac{1}{x}v_x \right\|_0^2 - \varepsilon \left\| v_{xx} \right\|_0^2 \\
&\quad - \frac{1}{\varepsilon} \left\| \frac{1}{x^2}v \right\|_0^2 - \varepsilon \left\| \frac{1}{x}v_x \right\|_0^2 - \frac{1}{\varepsilon} \left\| \frac{1}{x^2}v \right\|_0^2 + \eta \left(\left\| v_x \right\|_0^2 + \left\| \frac{1}{x^2}v \right\|_0^2 \right) \\
&= (1 - 2\varepsilon) \left\| v_{xx} \right\|_0^2 + \left(\eta + 1 - \frac{2}{\varepsilon} \right) \left\| \frac{1}{x^2}v \right\|_0^2 \\
&\quad - \left(\varepsilon + \frac{1}{\varepsilon} - 1 \right) \left\| \frac{1}{x}v_x \right\|_0^2 + \eta \left\| v_x \right\|_0^2 \\
&\geq (1 - 2\varepsilon) \left\| v_{xx} \right\|_0^2 + \left(\eta + 1 - \frac{2}{\varepsilon} \right) \left\| \frac{1}{x^2}v \right\|_0^2 + \left[\eta - \left(\varepsilon + \frac{1}{\varepsilon} - 1 \right) \right] \left\| v_x \right\|_0^2 .
\end{aligned}$$

Choosing $\eta > \max\{\frac{2}{\varepsilon} - 1, \varepsilon + \frac{1}{\varepsilon} - 1\}$ and $\eta_1 = \min\{1 - 2\varepsilon, \eta - \left(\frac{1}{\varepsilon} + \varepsilon - 1 \right)\}$, we obtain

$$\left\| L_1 v \right\|_0^2 + \eta \left\| v \right\|_a^2 \geq \eta_1 \left(\left\| v_{xx} \right\|_0^2 + \left\| v_x \right\|_0^2 \right) = \eta_1 \left\| v \right\|_{H^2 \cap V}^2 .$$

It follows that

$$\left\| L_1 v \right\|_0^2 + \left\| v \right\|_a^2 \geq \frac{\eta_1}{\max\{1, \eta\}} \left\| v \right\|_{H^2 \cap V}^2 = \gamma_1 \left\| v \right\|_{H^2 \cap V}^2 .$$

For all $\varepsilon \in (0,1)$, $\eta > 0$, we have

$$\begin{aligned}
&\|L_2 v\|_0^2 + \eta \|v_x\|_0^2 \\
&= \|v_{xx}\|_0^2 + \left\| \frac{1}{x} v_x \right\|_0^2 + 2\langle v_{xx}, \frac{1}{x} v_x \rangle + \eta \|v_x\|_0^2 \\
&\geq \|v_{xx}\|_0^2 + \left\| \frac{1}{x} v_x \right\|_0^2 - \varepsilon \|v_{xx}\|_0^2 - \frac{1}{\varepsilon} \left\| \frac{1}{x} v_x \right\|_0^2 + \eta \|v_x\|_0^2 \\
&= (1-\varepsilon) \|v_{xx}\|_0^2 - \left(\frac{1}{\varepsilon} - 1 \right) \left\| \frac{1}{x} v_x \right\|_0^2 + \eta \|v_x\|_0^2 \\
&\geq (1-\varepsilon) \|v_{xx}\|_0^2 + \left[\eta - \left(\frac{1}{\varepsilon} - 1 \right) \right] \|v_x\|_0^2 \geq \eta_2 \|v\|_{H^2 \cap V}^2 ,
\end{aligned}$$

where $\eta > \frac{1}{\varepsilon} - 1$ and $\eta_2 = \min\{1-\varepsilon, \eta - \left(\frac{1}{\varepsilon} - 1 \right)\}$. It follows that

$$\|L_2 v\|_0^2 + \|v_x\|_0^2 \geq \frac{\eta_2}{\max\{1,\eta\}} \|v\|_{H^2 \cap V}^2 = \gamma_2 \|v\|_{H^2 \cap V}^2 .$$

Lemma 3.2 is proved. □

Now, we estimate the terms $S_m^{(k)}(t)$, $S_m^{(k)}(0)$, $I_1, \cdots, I_5$ of (3.13) as follows.

Estimate of $S_m^{(k)}(t)$.

By Lemma 3.2, we deduce from (3.12) that

$$S_m^{(k)}(t) \geq \gamma_* \bar{S}_m^{(k)}(t), \tag{3.16}$$

where $\gamma_* = \min\{1, a_{1*}\gamma_1, a_{2*}\gamma_2\}$ and

$$\begin{aligned}
\bar{S}_m^{(k)}(t) = &\left\| \dot{u}_m^{(k)}(t) \right\|_0^2 + \left\| \dot{u}_m^{(k)}(t) \right\|_a^2 + \left\| \dot{v}_m^{(k)}(t) \right\|_0^2 \\
&+ \left\| \dot{v}_{mx}^{(k)}(t) \right\|_0^2 + \left\| u_m^{(k)}(t) \right\|_{H^2 \cap V}^2 + \left\| v_m^{(k)}(t) \right\|_{H^2 \cap V}^2 \\
&+ \int_0^t \left(\left\| \ddot{u}_m^{(k)}(s) \right\|_0^2 + \left\| \ddot{v}_m^{(k)}(s) \right\|_0^2 \right) ds.
\end{aligned} \tag{3.17}$$

In order to estimate the terms $I_1, \cdots, I_5$, we use the following lemma.

Lemma 3.3. *The terms* $F_m^{(k)}(x,t)$, $G_m^{(k)}(x,t)$, $F_{mx}^{(k)}(t)$, $G_{mx}^{(k)}(t)$ *are estimated as follows*

$$\begin{aligned}
&\text{(i)} \quad \left| F_m^{(k)}(x,t) \right| \leq C_0(M,f) \left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)} \right)^{N-1} \right], \\
&\text{(ii)} \quad \left| G_m^{(k)}(x,t) \right| \leq C_0(M,g) \left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)} \right)^{N-1} \right], \\
&\text{(iii)} \quad \left\| F_{mx}^{(k)}(t) \right\|_0 \leq C_1(M,f) \left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)} \right)^{N-1} \right], \\
&\text{(iv)} \quad \left\| G_{mx}^{(k)}(t) \right\|_0 \leq C_1(M,g) \left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)} \right)^{N-1} \right],
\end{aligned} \tag{3.18}$$

where $C_0(M,f)$, $C_0(M,g)$, $C_1(M,f)$ *and* $C_1(M,g)$ *are defined as follows*

$$\begin{aligned} C_0(M,f) =& K_N(M,f)\sum\nolimits_{r=0}^{N-1}\frac{[4R_1(1+M)]^r}{r!}, \\ C_0(M,g) =& K_N(M,g)\sum\nolimits_{r=0}^{N-1}\frac{[4R_1(1+M)]^r}{r!}, \\ C_1(M,f) =& \left(d_M^* + \frac{N-1}{R_1}\right)C_0(M,f), \\ C_1(M,g) =& \left(d_M^* + \frac{N-1}{R_1}\right)C_0(M,g), \\ R_1 =& \sqrt{R-1},\ d_M^* = \sqrt{\frac{R^2-1}{2}} + 2M. \end{aligned} \tag{3.19}$$

Proof. We rewrite $F_m^{(k)}(x,t)$ as follows

$$F_m^{(k)}(x,t) = \sum_{i+j\le N-1}\frac{1}{i!j!}D_3^iD_4^jf[u_{m-1},v_{m-1}](x,t)A_m^{(k)}(i,j,x,t), \tag{3.20}$$

where

$$A_m^{(k)}(i,j,x,t) = (u_m^{(k)} - u_{m-1})^i(v_m^{(k)} - v_{m-1})^j.$$

By using the inequalities

$$\begin{aligned} |u_{m-1}(x,t)| \le& \|u_{m-1}(t)\|_{C^0(\overline{\Omega})} \le R_1\|\nabla u_{m-1}(t)\|_0 \le R_1M, \\ |v_{m-1}(x,t)| \le& R_1M, \\ \left|u_m^{(k)}(x,t)\right| \le& \left\|u_m^{(k)}(t)\right\|_{C^0(\overline{\Omega})} \le R_1\left\|u_{mx}^{(k)}(t)\right\|_0 \\ \le& R_1\left\|u_m^{(k)}(t)\right\|_{H^2\cap V} \le R_1\sqrt{\bar{S}_m^{(k)}(t)}, \\ \left|v_m^{(k)}(x,t)\right| \le& R_1\sqrt{\bar{S}_m^{(k)}(t)}, \\ a \le& 1 + a^p,\ (a+b)^p \le 2^{p-1}(a^p+b^p),\ \forall a,b\ge 0,\ \forall p\ge 1, \end{aligned}$$

$$\begin{aligned} \left|A_m^{(k)}(i,j,x,t)\right| =& \left(\left|u_m^{(k)}\right| + |u_{m-1}|\right)^i\left(\left|v_m^{(k)}\right| + |v_{m-1}|\right)^j \\ \le& R_1^{i+j}\left(M + \sqrt{\bar{S}_m^{(k)}(t)}\right)^{i+j} \\ \le& R_1^{i+j}(1+M)^{i+j}\left(1 + \sqrt{\bar{S}_m^{(k)}(t)}\right)^{i+j} \\ \le& R_1^{i+j}(1+M)^{i+j}2^{i+j-1}\left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{i+j}\right] \\ \le& [2R_1(1+M)]^{i+j}\left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right] \end{aligned}$$

for all $i, j \in \mathbb{Z}_+, 1 \le i+j \le N-1$, it follows that

$$\begin{aligned}
&\left|F_m^{(k)}(x,t)\right| \\
&\le |f[u_{m-1}, v_{m-1}](x,t)| + \sum_{1\le i+j\le N-1} \frac{1}{i!j!}\left|D_3^i D_4^j f[u_{m-1}, v_{m-1}]\right| \left|A_m^{(k)}(i,j,x,t)\right| \\
&\le K_N(M,f) + K_N(M,f) \sum_{1\le i+j\le N-1} \frac{1}{i!j!}\left[2R_1(1+M)\right]^{i+j} \left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right].
\end{aligned}$$

It is known that $\sum_{i+j=r} \frac{1}{i!j!} = \frac{2^r}{r!}$, hence

$$\begin{aligned}
\sum_{1\le i+j\le N-1} \frac{1}{i!j!}\left[2R_1(1+M)\right]^{i+j} &= \sum_{r=1}^{N-1}\sum_{i+j=r}\frac{1}{i!j!}\left[2R_1(1+M)\right]^{r} \\
&= \sum_{r=1}^{N-1} \frac{\left[4R_1(1+M)\right]^r}{r!},
\end{aligned}$$

we deduce that

$$\left|F_m^{(k)}(x,t)\right| \le C_0(M,f)\left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right], \tag{3.21}$$

where $C_0(M,f)$ is defined as in (3.19).

Similar to $F_m^{(k)}(x,t)$, we also have the estimate of $G_m^{(k)}(x,t)$ as in (3.18)(ii).

We have

$$\begin{aligned}
F_{mx}^{(k)}(x,t) =& \frac{\partial}{\partial x} f[u_{m-1}, v_{m-1}] \\
&+ \sum_{1\le i+j\le N-1} \frac{1}{i!j!}\left[\frac{\partial}{\partial x}\left(D_3^i D_4^j f[u_{m-1}, v_{m-1}]\right)\right] A_m^{(k)}(i,j,x,t) \\
&+ \sum_{1\le i+j\le N-1} \frac{1}{i!j!} D_3^i D_4^j f[u_{m-1}, v_{m-1}] A_{mx}^{(k)}(i,j,x,t) \\
=& \frac{\partial}{\partial x} f[u_{m-1}, v_{m-1}] + J_1^* + J_2^*.
\end{aligned} \tag{3.22}$$

We shall estimate the terms $\frac{\partial}{\partial x} f[u_{m-1}, v_{m-1}]$, J_1^*, J_2^* on the right-hand side of (3.22) as follows.

We have

$$\begin{aligned}
&\left\|\frac{\partial}{\partial x} f[u_{m-1}, v_{m-1}]\right\|_0 \\
&= \|D_1 f[u_{m-1}, v_{m-1}] + D_3 f[u_{m-1}, v_{m-1}]\nabla u_{m-1} + D_4 f[u_{m-1}, v_{m-1}]\nabla v_{m-1}\|_0 \\
&\le K_N(M,f)\left[\sqrt{\frac{R^2-1}{2}} + \|\nabla u_{m-1}\|_0 + \|\nabla v_{m-1}\|_0\right] \\
&\le K_N(M,f)\left[\sqrt{\frac{R^2-1}{2}} + 2M\right] = K_N(M,f) d_M^*.
\end{aligned} \tag{3.23}$$

Similarly

$$\left\| \frac{\partial}{\partial x} \left(D_3^i D_4^j f[u_{m-1}, v_{m-1}] \right) \right\|_0 \leq K_N(M,f) d_M^*,$$

so

$$\begin{aligned}
\|J_1^*\|_0 \leq & \sum_{1\leq i+j\leq N-1} \frac{1}{i!j!} \left\| \frac{\partial}{\partial x} \left(D_3^i D_4^j f[u_{m-1}, v_{m-1}] \right) A_m^{(k)}(i,j,t) \right\|_0 \qquad (3.24) \\
\leq & \sum_{1\leq i+j\leq N-1} \frac{1}{i!j!} \left\| \frac{\partial}{\partial x} \left(D_3^i D_4^j f[u_{m-1}, v_{m-1}] \right) \right\|_0 \left[2R_1 (1+M)\right]^{i+j} \left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)} \right)^{N-1} \right] \\
\leq & K_N(M,f) d_M^* \sum_{1\leq i+j\leq N-1} \frac{1}{i!j!} \left[2R_1 (1+M)\right]^{i+j} \left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)} \right)^{N-1} \right] \\
= & K_N(M,f) d_M^* \sum\nolimits_{r=1}^{N-1} \frac{\left[4R_1 (1+M)\right]^r}{r!} \left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)} \right)^{N-1} \right].
\end{aligned}$$

For all $i, j \in \mathbb{Z}_+, 1 \leq i+j \leq N-1$, we have

$$\begin{aligned}
\left\| A_{mx}^{(k)}(i,j,t) \right\|_0 \leq & \left\| i(u_m^{(k)} - u_{m-1})^{i-1} (u_{mx}^{(k)} - \nabla u_{m-1})(v_m^{(k)} - v_{m-1})^j \right\|_0 \qquad (3.25) \\
& + \left\| j(u_m^{(k)} - u_{m-1})^i (v_m^{(k)} - v_{m-1})^{j-1} (v_{mx}^{(k)} - \nabla v_{m-1}) \right\|_0 \\
\leq & \left\| i \left(\left| u_m^{(k)} \right| + |u_{m-1}| \right)^{i-1} \left(\left| v_m^{(k)} \right| + |v_{m-1}| \right)^j \left(\left| u_{mx}^{(k)} \right| + |\nabla u_{m-1}| \right) \right\|_0 \\
& + \left\| j \left(\left| u_m^{(k)} \right| + |u_{m-1}| \right)^i \left(\left| v_m^{(k)} \right| + |v_{m-1}| \right)^{j-1} \left(\left| v_{mx}^{(k)} \right| + |\nabla v_{m-1}| \right) \right\|_0 \\
\leq & i R_1^{i+j-1} \left(M + \sqrt{\bar{S}_m^{(k)}(t)} \right)^{i+j-1} \left\| \left| u_{mx}^{(k)} \right| + |\nabla u_{m-1}| \right\|_0 \\
& + j R_1^{i+j-1} \left(M + \sqrt{\bar{S}_m^{(k)}(t)} \right)^{i+j-1} \left\| \left| v_{mx}^{(k)} \right| + |\nabla v_{m-1}| \right\|_0 \\
\leq & (i+j) R_1^{i+j-1} \left(M + \sqrt{\bar{S}_m^{(k)}(t)} \right)^{i+j} \\
\leq & (i+j) R_1^{i+j-1} (1+M)^{i+j} \left(1 + \sqrt{\bar{S}_m^{(k)}(t)} \right)^{i+j} \\
\leq & (i+j) R_1^{i+j-1} (1+M)^{i+j} 2^{i+j-1} \left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)} \right)^{i+j} \right] \\
\leq & \frac{1}{R_1} (N-1) \left[2R_1 (1+M)\right]^{i+j} \left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)} \right)^{N-1} \right].
\end{aligned}$$

Hence, we deduce from (3.25) that

$$\begin{aligned}\|J_2^*\|_0 \leq & \sum_{1\leq i+j\leq N-1} \frac{1}{i!j!} \left\| D_3^i D_4^j f[u_{m-1}, v_{m-1}] A_{mx}^{(k)}(i,j,t) \right\|_0 \\ \leq & K_N(M,f) \sum_{1\leq i+j\leq N-1} \frac{1}{i!j!} \left\| A_{mx}^{(k)}(i,j,t) \right\|_0 \\ \leq & (N-1)\frac{1}{R_1} K_N(M,f) \sum_{1\leq i+j\leq N-1} \frac{1}{i!j!} \left[2R_1(1+M)\right]^{i+j} \left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right] \\ = & (N-1)\frac{1}{R_1} K_N(M,f) \sum_{r=1}^{N-1} \frac{\left[4R_1(1+M)\right]^r}{r!} \left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right].\end{aligned} \tag{3.26}$$

Combining (3.22), (3.23), (3.24) and (3.26), we obtain

$$\begin{aligned}\left\| F_{mx}^{(k)}(t) \right\|_0 \leq & \left\| \frac{\partial}{\partial x} f[u_{m-1}, v_{m-1}] \right\|_0 + \|J_1^*\|_0 + \|J_2^*\|_0 \\ \leq & K_N(M,f) d_M^* \\ & + K_N(M,f) d_M^* \sum\nolimits_{r=1}^{N-1} \frac{\left[4R_1(1+M)\right]^r}{r!} \left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right] \\ & + (N-1)\frac{1}{R_1} K_N(M,f) \sum_{r=1}^{N-1} \frac{\left[4R_1(1+M)\right]^r}{r!} \left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right] \\ \leq & C_1(M,f) \left[1 + \left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right],\end{aligned} \tag{3.27}$$

where $C_1(M,f)$ is defined as in (3.19).

Similar to $\left\| F_{mx}^{(k)}(t) \right\|_0$, we also have the estimate of $\left\| G_{mx}^{(k)}(t) \right\|_0$ as in (3.18)(iv). Lemma 3.3 is proved completely. □

Estimate of I_1. By the Cauchy inequality, we deduce from (3.18) (i), (ii) that

$$\begin{aligned}I_1 = & 2\int_0^t \left[\langle F_m^{(k)}(s), \dot{u}_m^{(k)}(s)\rangle + \langle G_m^{(k)}(s), \dot{v}_m^{(k)}(s)\rangle \right] ds \\ \leq & 2\int_0^t \left[\left\| F_m^{(k)}(s) \right\|_0 \left\| \dot{u}_m^{(k)}(s) \right\|_0 + \left\| G_m^{(k)}(s) \right\|_0 \left\| \dot{v}_m^{(k)}(s) \right\|_0 \right] ds \\ \leq & 2\sqrt{\frac{R^2-1}{2}} \left(C_0(M,f) + C_0(M,g)\right) \int_0^t \left[1 + \left(\sqrt{\bar{S}_m^{(k)}(s)}\right)^{N-1}\right] \sqrt{\bar{S}_m^{(k)}(s)} ds \\ \leq & 4\sqrt{\frac{R^2-1}{2}} \left(C_0(M,f) + C_0(M,g)\right) \int_0^t \left[1 + \left(\bar{S}_m^{(k)}(s)\right)^{N-1}\right] ds.\end{aligned} \tag{3.28}$$

Estimate of I_2. Similar to I_1, we have

$$\begin{aligned} I_2 =&2\int_0^t \left[a(F_m^{(k)}(s),\dot{u}_m^{(k)}(s))+\langle G_{mx}^{(k)}(s),\dot{v}_{mx}^{(k)}(s)\rangle\right]ds \\ \le&2\int_0^t \left[\left\|F_m^{(k)}(s)\right\|_a\left\|\dot{u}_m^{(k)}(s)\right\|_a+\left\|G_{mx}^{(k)}(s)\right\|_0\left\|\dot{v}_{mx}^{(k)}(s)\right\|_0\right]ds \\ =&2\int_0^t \left[a_1^*\left\|F_{mx}^{(k)}(s)\right\|_0+\left\|G_{mx}^{(k)}(s)\right\|_0\right]\sqrt{\bar{S}_m^{(k)}(s)}ds \\ \le&2\left(a_1^*C_1(M,f)+C_1(M,g)\right)\int_0^t\left[1+\left(\sqrt{\bar{S}_m^{(k)}(s)}\right)^{N-1}\right]\sqrt{\bar{S}_m^{(k)}(s)}ds \\ \le&4\left(a_1^*C_1(M,f)+C_1(M,g)\right)\int_0^t\left[1+\left(\bar{S}_m^{(k)}(s)\right)^{N-1}\right]ds. \end{aligned} \tag{3.29}$$

Estimate of I_3. Note that, with $a\in C(\mathbb{R}_+^2;\mathbb{R})$, if we set

$$\varPhi_{[a]}(r)=\begin{cases}\sup\limits_{0\le y+z\le r}|a(y,z)|, & r>0,\\ |a(0,0)|, & r=0,\end{cases}$$

then $\varPhi_{[a]}\in C(\mathbb{R}_+;\mathbb{R}_+)$ and this function is nondecreasing such that

$$|a(y,z)|\le\varPhi_{[a]}(y+z),\ \text{for all } z\ge 0.$$

This property was proved in [21].

Based on the above property of the function $\varPhi_{[a]}$, we can estimate the terms $\dot{a}_{1m}^{(k)}(t)$, $\dot{a}_{2m}^{(k)}(t)$ as follows

Lemma 3.4. *The terms* $\dot{a}_{1m}^{(k)}(t)$, $\dot{a}_{2m}^{(k)}(t)$ *are estimated as follows*

$$\begin{aligned} &\text{(i)} && \left|\dot{a}_{1m}^{(k)}(t)\right|\le\frac{1}{2}\bar{a}_1^{*2}\hat{\varPhi}_1\left(\bar{S}_m^{(k)}(t)\right)\bar{S}_m^{(k)}(t),\\ &\text{(ii)} && \left|\dot{a}_{2m}^{(k)}(t)\right|\le\frac{1}{2}\bar{a}_1^{*2}\hat{\varPhi}_2\left(\bar{S}_m^{(k)}(t)\right)\bar{S}_m^{(k)}(t), \end{aligned} \tag{3.30}$$

where $\hat{\varPhi}_1(S)$ *and* $\hat{\varPhi}_2(S)$ *are defined as follows*

$$\begin{aligned} \hat{\varPhi}_1(S)=&\ \varPhi_{[D_1a_1]}\left(\bar{a}_1^{*2}S\right)+\varPhi_{[D_2a_1]}\left(\bar{a}_1^{*2}S\right),\\ \hat{\varPhi}_2(S)=&\ \varPhi_{[D_1a_2]}\left(\bar{a}_1^{*2}S\right)+\varPhi_{[D_2a_2]}\left(\bar{a}_1^{*2}S\right),\ \forall S\ge 0. \end{aligned} \tag{3.31}$$

Proof. From the above inequality, we obtain

$$\begin{aligned} \left|D_1a_1[u_m^{(k)}](t)\right|=&\left|D_1a_1\left(\left\|u_m^{(k)}(t)\right\|_0^2,\left\|u_{mx}^{(k)}(t)\right\|_0^2\right)\right|\\ \le&\varPhi_{[D_1a_1]}\left(\left\|u_m^{(k)}(t)\right\|_0^2+\left\|u_{mx}^{(k)}(t)\right\|_0^2\right)\\ =&\varPhi_{[D_1a_1]}\left(\left\|u_m^{(k)}(t)\right\|_1^2\right)\le\varPhi_{[D_1a_1]}\left(\bar{a}_1^{*2}\left\|u_{mx}^{(k)}(t)\right\|_0^2\right)\\ \le&\varPhi_{[D_1a_1]}\left(\bar{a}_1^{*2}\bar{S}_m^{(k)}(t)\right). \end{aligned}$$

Similarly

$$\left|D_2a_1[u_m^{(k)}](t)\right| \leq \Phi_{[D_2a_1]}\left(\bar{a}_1^{*2}\bar{S}_m^{(k)}(t)\right).$$

By using the formula

$$\dot{a}_{1m}^{(k)}(t) = D_1a_1[u_m^{(k)}](t)\langle u_m^{(k)}(t), \dot{u}_m^{(k)}(t)\rangle + D_2a_1[u_m^{(k)}](t)\langle u_{mx}^{(k)}(t), \dot{u}_{mx}^{(k)}(t)\rangle,$$

we obtain the inequality

$$\begin{aligned}
\left|\dot{a}_{1m}^{(k)}(t)\right| \leq & \left|D_1a_1[u_m^{(k)}](t)\right| \left\|u_m^{(k)}(t)\right\|_0 \left\|\dot{u}_m^{(k)}(t)\right\|_0 \\
& + \left|D_2a_1[u_m^{(k)}](t)\right| \left\|u_{mx}^{(k)}(t)\right\|_0 \left\|\dot{u}_{mx}^{(k)}(t)\right\|_0 \\
\leq & \Phi_{[D_1a_1]}\left(\bar{a}_1^{*2}\bar{S}_m^{(k)}(t)\right) \left\|u_m^{(k)}(t)\right\|_0 \left\|\dot{u}_m^{(k)}(t)\right\|_0 \\
& + \Phi_{[D_2a_1]}\left(\bar{a}_1^{*2}\bar{S}_m^{(k)}(t)\right) \left\|u_{mx}^{(k)}(t)\right\|_0 \left\|\dot{u}_{mx}^{(k)}(t)\right\|_0 \\
\leq & \left[\Phi_{[D_1a_1]}\left(\bar{a}_1^{*2}\bar{S}_m^{(k)}(t)\right) + \Phi_{[D_2a_1]}\left(\bar{a}_1^{*2}\bar{S}_m^{(k)}(t)\right)\right] \\
& \times \left(\left\|u_m^{(k)}(t)\right\|_0 \left\|\dot{u}_m^{(k)}(t)\right\|_0 + \left\|u_{mx}^{(k)}(t)\right\|_0 \left\|\dot{u}_{mx}^{(k)}(t)\right\|_0\right) \\
\leq & \hat{\Phi}_1\left(\bar{S}_m^{(k)}(t)\right) \left\|u_m^{(k)}(t)\right\|_1 \left\|\dot{u}_m^{(k)}(t)\right\|_1 \\
\leq & \hat{\Phi}_1\left(\bar{S}_m^{(k)}(t)\right) \bar{a}_1^{*2} \left\|u_{mx}^{(k)}(t)\right\|_0 \left\|\dot{u}_{mx}^{(k)}(t)\right\|_0 \\
\leq & \frac{1}{2}\bar{a}_1^{*2}\hat{\Phi}_1\left(\bar{S}_m^{(k)}(t)\right)\bar{S}_m^{(k)}(t).
\end{aligned}$$

Similar to $\left|\dot{a}_{1m}^{(k)}(t)\right|$, we also have a estimate $\left|\dot{a}_{2m}^{(k)}(t)\right|$ as in (3.30)(ii), (3.31)$_2$.
Lemma 3.4 is proved completely. □
Now, we continue to estimate I_2 as follows
By the estimate

$$\begin{aligned}
\left\|u_m^{(k)}(t)\right\|_a^2 + \left\|L_1u_m^{(k)}(t)\right\|_0^2 \leq & a_1^{*2}\left\|u_{mx}^{(k)}(t)\right\|_0^2 + \bar{\gamma}_1^2\left\|u_m^{(k)}(t)\right\|_{H^2\cap V}^2 \\
\leq & \left(a_1^{*2} + \bar{\gamma}_1^2\right)\bar{S}_m^{(k)}(t),
\end{aligned}$$

we deduce from (3.30)(i), that

$$\begin{aligned}
I_3 = & 2\int_0^t \dot{a}_{1m}^{(k)}(s)\left(\left\|u_m^{(k)}(s)\right\|_a^2 + \left\|L_1u_m^{(k)}(s)\right\|_0^2\right)ds \\
\leq & \bar{a}_1^{*2}\left(a_1^{*2} + \bar{\gamma}_1^2\right)\int_0^t \hat{\Phi}_1\left(\bar{S}_m^{(k)}(s)\right)\left(\bar{S}_m^{(k)}(s)\right)^2 ds.
\end{aligned} \tag{3.32}$$

Estimate of I_4. Similar to I_3, from (3.30)(ii) we also have the estimate of I_4 as follows

$$\begin{aligned}
I_4 = & 2\int_0^t \dot{a}_{2m}^{(k)}(s)\left(\left\|v_{mx}^{(k)}(s)\right\|_0^2 + \left\|L_2v_m^{(k)}(s)\right\|_0^2\right)ds \\
\leq & 3\bar{a}_1^{*2}\int_0^t \hat{\Phi}_2\left(\bar{S}_m^{(k)}(s)\right)\left(\bar{S}_m^{(k)}(s)\right)^2 ds.
\end{aligned} \tag{3.33}$$

Estimate of I_5. Eq. $(3.8)_1$ is rewritten as follows

$$\langle \ddot{u}_m^{(k)}(t), w_j\rangle + a_{1m}^{(k)}(t)\langle L_1 u_m^{(k)}(t), w_j\rangle = \langle F_m^{(k)}(t), w_j\rangle,\ 1 \le j \le k.$$

Then, it follows after replacing w_j with $\ddot{u}_m^{(k)}(t)$, that

$$\begin{aligned}
\left\|\ddot{u}_m^{(k)}(t)\right\|_0^2 =& - a_{1m}^{(k)}(t)\langle L_1 u_m^{(k)}(t), \ddot{u}_m^{(k)}(t)\rangle + \langle F_m^{(k)}(t), \ddot{u}_m^{(k)}(t)\rangle \\
\le& \left[a_{1m}^{(k)}(t)\left\|L_1 u_m^{(k)}(t)\right\|_0 + \left\|F_m^{(k)}(t)\right\|_0\right]\left\|\ddot{u}_m^{(k)}(t)\right\|_0 \\
\le& \left[a_{1m}^{(k)}(t)\left\|L_1 u_m^{(k)}(t)\right\|_0 + \left\|F_m^{(k)}(t)\right\|_0\right]^2 \\
\le& 2\left(a_{1m}^{(k)}(t)\right)^2\left\|L_1 u_m^{(k)}(t)\right\|_0^2 + 2\left\|F_m^{(k)}(t)\right\|_0^2 \\
\le& 2\Phi_{[a_1^2]}\left(\left\|u_m^{(k)}(t)\right\|_1^2\right)\left\|L_1 u_m^{(k)}(t)\right\|_0^2 + 2\left\|F_m^{(k)}(t)\right\|_0^2 \\
\le& 2\bar{\gamma}_1^2\Phi_3\left(\bar{S}_m^{(k)}(t)\right)\bar{S}_m^{(k)}(t) \\
&+ 2\left(R^2-1\right)C_0^2(M,f)\left[1 + \left(\bar{S}_m^{(k)}(t)\right)^{N-1}\right],
\end{aligned} \tag{3.34}$$

where $\Phi_3\left(S\right)) = \Phi_{[a_1^2]}\left(\bar{a}_1^{*2}S\right)$.
Similarly, we get

$$\begin{aligned}
\left\|\ddot{v}_m^{(k)}(t)\right\|_0^2 =& - a_{2m}^{(k)}(t)\langle L_2 v_m^{(k)}(t), \ddot{v}_m^{(k)}(t)\rangle + \langle G_m^{(k)}(t), \ddot{v}_m^{(k)}(t)\rangle \\
\le& 4\Phi_4\left(\bar{S}_m^{(k)}(t)\right)\bar{S}_m^{(k)}(t) \\
&+ 2\left(R^2-1\right)C_0^2(M,g)\left[1 + \left(\bar{S}_m^{(k)}(t)\right)^{N-1}\right],
\end{aligned} \tag{3.35}$$

where $\Phi_4\left(S\right)) = \Phi_{[a_2^2]}\left(\bar{a}_1^{*2}S\right)$.
Hence

$$\begin{aligned}
I_5 =& \int_0^t \left(\left\|\ddot{u}_m^{(k)}(s)\right\|_0^2 + \left\|\ddot{v}_m^{(k)}(s)\right\|_0^2\right) ds \\
\le& 2\left(\bar{\gamma}_1^2 + 2\right)\int_0^t \left[\Phi_3\left(\bar{S}_m^{(k)}(s)\right) + \Phi_4\left(\bar{S}_m^{(k)}(s)\right)\right]\bar{S}_m^{(k)}(s)ds \\
&+ 2\left(R^2-1\right)\left(C_0^2(M,f) + C_0^2(M,g)\right)\int_0^t \left[1 + \left(\bar{S}_m^{(k)}(s)\right)^{N-1}\right] ds.
\end{aligned} \tag{3.36}$$

On the other hand, we have

$$\begin{aligned}
S_m^{(k)}(0) =& \|\tilde{u}_{1k}\|_0^2 + \|\tilde{u}_{1k}\|_a^2 + \|\tilde{v}_{1k}\|_0^2 + \|\tilde{v}_{1kx}\|_0^2 \\
&+ a_1\left(\|\tilde{u}_{0k}\|_0^2, \|\tilde{u}_{0kx}\|_0^2\right)\left(\|\tilde{u}_{0k}\|_a^2 + \|L_1\tilde{u}_{0k}\|_0^2\right) \\
&+ a_2\left(\|\tilde{v}_{0k}\|_0^2, \|\tilde{v}_{0kx}\|_0^2\right)\left(\|\tilde{v}_{0kx}\|_0^2 + \|L_2\tilde{v}_{0k}\|_0^2\right).
\end{aligned} \tag{3.37}$$

By means of the convergences in (3.9), we deduce the existence of a constant $M > 0$ independent of k and m such that

$$S_m^{(k)}(0) \le \frac{\gamma_*}{2} M^2, \text{ for all } k \text{ and } m \in \mathbb{N}. \tag{3.38}$$

Combining (3.13), (3.16), (3.17), (3.28), (3.29), (3.32), (3.33), (3.36) and (3.38), it leads to

$$\bar{S}_m^{(k)}(t) \le \frac{1}{2} M^2 + \int_0^t \Psi_M(\bar{S}_m^{(k)}(s)) ds, \tag{3.39}$$

where

$$\begin{aligned} \Psi_M(S) =& \bar{D}_1(M)\left(1 + S^{N-1}\right) + \bar{D}_2 (1 + \hat{\Phi}_1(S) + \hat{\Phi}_2(S)) S^2 \\ &+ \bar{D}_3 \left(\Phi_3(S) + \Phi_4(S)\right) S, \\ \bar{D}_1(M) =& \frac{4}{\gamma_*}\left(\sqrt{\frac{R^2-1}{2}}\left(C_0(M,f) + C_0(M,g)\right) + \left(a_1^* C_1(M,f) + C_1(M,g)\right)\right) \\ &+ \frac{2}{\gamma_*}\left(R^2 - 1\right)\left(C_0^2(M,f) + C_0^2(M,g)\right), \\ \bar{D}_2 =& \frac{1}{\gamma_*}\left[3 + \left(a_1^{*2} + \bar{\gamma}_1^2\right)\right] \bar{a}_1^{*2}, \ \bar{D}_3 = \frac{2}{\gamma_*}\left(\bar{\gamma}_1^2 + 2\right). \end{aligned} \tag{3.40}$$

Then, by solving a nonlinear Volterra integral equation (based on the methods in [12]), we get the following lemma.

Lemma 3.5. *There exists a constant $T > 0$ depending on T_* (independent of m) such that*

$$\bar{S}_m^{(k)}(t) \le M^2, \ \forall m \in \mathbb{N}, \ \forall t \in [0, T], \tag{3.41}$$

where C_T is a constant depending only on T.

Proof. By setting $y(t) = \frac{1}{2} M^2 + \int_0^t \Psi_M(\bar{S}_m^{(k)}(s)) ds$, and Ψ_M is a continuous and strictly increasing function, we get that

$$\begin{aligned} 0 \le& \bar{S}_m^{(k)}(t) \le y(t), \ y(0) = \frac{M^2}{2}, \\ y'(t) =& \Psi_M\left(\bar{S}_m^{(k)}(t)\right) \le \Psi_M(y(t)). \end{aligned}$$

From the above inequality, we obtain

$$\mathcal{H}(y(t)) - \mathcal{H}(\frac{M^2}{2}) = \int_{\frac{M^2}{2}}^{y(t)} \frac{dz}{\Psi_M(z)} = \int_0^t \frac{y'(s) ds}{\Psi_M(y(s))} \le t,$$

where $\mathcal{H}(y) = \int_0^y \frac{dz}{\Psi_M(z)}$ is a continuous and strictly increasing function on $\mathbb{R}_+$.

We shall prove that $\mathcal{H}(\infty) = \int_0^\infty \frac{dz}{\Psi_M(z)}$ is a convergent itegral.

Indeed, by the definitions of $\Psi_M(z)$ as in (3.40), we have

$$\begin{aligned} \Psi_M(z) =& \bar{D}_1(M)\left(1 + z^{N-1}\right) + \bar{D}_2\left(1 + \hat{\Phi}_1(z) + \hat{\Phi}_2(z)\right) z^2 \\ &+ \bar{D}_3\left(\Phi_3(z) + \Phi_4(z)\right) z \\ \ge& \bar{D}_1(M) + \bar{D}_2 z^2 \ge D_{\min}\left(1 + z^2\right), \end{aligned}$$

where $D_{\min} = \min\{\bar{D}_1(M), \bar{D}_2\}$.
Hence

$$\frac{1}{\Psi_M(z)} \leq \frac{1}{D_{\min}} \frac{1}{1+z^2},\ \forall z \geq 0.$$

On the other hand, because $\int_0^\infty \frac{dz}{1+z^2}$ is convergent, we conclude that $\mathcal{H}(\infty) = \int_0^\infty \frac{dz}{\Psi_M(z)}$ is also convergent. By this, $\mathcal{H} : \mathbb{R}_+ \to [0, \mathcal{H}(\infty))$ is a continuous bijection, therefore, $\mathcal{H}^{-1} : [0, \mathcal{H}(\infty)) \to \mathbb{R}_+$ is also continuous and strictly increasing. Due to the fact that $\mathcal{H}(M^2) - \mathcal{H}(\frac{M^2}{2}) > 0$, we can choose $T \in (0, T_*]$ such that

$$0 < T \leq \mathcal{H}(M^2) - \mathcal{H}(\frac{M^2}{2}),$$

and

$$k_T = M\mu_T^{\frac{1}{N-1}} < 1,$$

where

$$\begin{aligned}
\mu_T =& 4\sqrt{T\tilde{E}_1(M,f,g)} \exp\left[T\tilde{E}_2(M,f,g,a_1,a_2)\right], \qquad (3.42)\\
\tilde{E}_1(M,f,g) =& \frac{E_2^2(M,f) + E_2^2(M,g)}{a_*},\\
\tilde{E}_2(M,f,g,a_1,a_2) =& \frac{1 + E_1(M,f) + E_1(M,g)}{a_*}\\
&+ \frac{1}{a_*}\left[\left(1 + \sqrt{2}\bar{\gamma}_1\bar{a}_1^*\right)\tilde{K}_M(a_1) + (1 + 2\bar{a}_1^*)\tilde{K}_M(a_2)\right],\\
E_1(M,f) =& \frac{K_N(M,f)}{M}\sqrt{\frac{R^2-1}{2}} \sum_{r=1}^{N-1} \frac{(2MR_1)^r}{r!},\\
E_2(M,f) =& K_N(M,f)\sqrt{\frac{R^2-1}{2}} \frac{(2R_1)^N}{N!},\\
\tilde{K}_M(a_1) =& M^2\left[\Phi_{[D_1 a_1]}\left(2M^2\right) + \Phi_{[D_2 a_1]}\left(2M^2\right)\right],\\
\tilde{K}_M(a_2) =& M^2\left[\Phi_{[D_1 a_2]}\left(2M^2\right) + \Phi_{[D_2 a_2]}\left(2M^2\right)\right],\\
a_* =& \min\{1, a_{1*}, a_{2*}\}.
\end{aligned}$$

It leads to, for all $t \in [0, T]$,

$$0 \leq \mathcal{H}(y(t)) \leq t + \mathcal{H}(\frac{M^2}{2}) \leq T + \mathcal{H}(\frac{M^2}{2}) \leq \mathcal{H}(M^2) < H(\infty).$$

Due to $\mathcal{H}^{-1} : [0, \mathcal{H}(\infty)) \to \mathbb{R}_+$ is strictly increasing, we get

$$\bar{S}_m^{(k)}(t) \leq y(t) = \mathcal{H}^{-1}\left(\mathcal{H}(y(t))\right) \leq \mathcal{H}^{-1}\left(\mathcal{H}(M^2)\right) = M^2.$$

Lemma 3.5 is proved. □
Lemma 3.5 allows one to take constant $T_m^{(k)} = T$ for all k and $m \in \mathbb{N}$.

Step 3. Limiting process. From (3.41), we deduce the existence of a subsequence of $\{(u_m^{(k)}, v_m^{(k)})\}$, denoted by the same symbol such that

$$\begin{cases} (u_m^{(k)}, v_m^{(k)}) \to (u_m, v_m) & \text{in} \quad L^\infty(0,T; \left(H^2 \cap V\right) \times \left(H^2 \cap V\right)) \text{ weak*}, \\ (\dot{u}_m^{(k)}, \dot{v}_m^{(k)}) \to (u_m', v_m') & \text{in} \quad L^\infty(0,T; V \times V) \text{ weak*}, \\ (\ddot{u}_m^{(k)}, \ddot{v}_m^{(k)}) \to (u_m'', v_m'') & \text{in} \quad L^2(0,T; L^2 \times L^2) \text{ weak}, \\ (u_m, v_m) \in W(M,T). \end{cases} \tag{3.43}$$

By the compactness lemma of Aubin-Lions [1], we can deduce from $(3.43)_{1,2,3}$ the existence of a subsequence still denoted by $\{(u_m^{(k)}, v_m^{(k)})\}$, such that

$$\begin{cases} (u_m^{(k)}, v_m^{(k)}) \to (u_m, v_m) & \text{strongly in} \quad C([0,T]; V \times V), \\ (\dot{u}_m^{(k)}, \dot{v}_m^{(k)}) \to (u_m', v_m') & \text{strongly in} \quad C([0,T]; L^2 \times L^2). \end{cases} \tag{3.44}$$

On the other hand

$$\begin{aligned} &\left|F_m^{(k)}(x,t) - F_m(x,t)\right| \\ \le& \sum_{i+j \le N-1} \frac{1}{i!j!} \left|D_3^i D_4^j f[u_{m-1}, v_{m-1}]\right| \left|\Psi_{m,i,j}^{(k)}(x,t)\right| \\ =& K_N(M,f) \sum_{i+j \le N-1} \frac{1}{i!j!} \left|\Psi_{m,i,j}^{(k)}(x,t)\right|, \end{aligned} \tag{3.45}$$

where

$$\begin{aligned} &\Psi_{m,i,j}^{(k)}(x,t) \\ =& (u_m^{(k)} - u_{m-1})^i (v_m^{(k)} - v_{m-1})^j - (u_m - u_{m-1})^i (v_m - v_{m-1})^j \\ =& \left[(u_m^{(k)} - u_{m-1})^i - (u_m - u_{m-1})^i\right] (v_m^{(k)} - v_{m-1})^j \\ &+ (u_m - u_{m-1})^i \left[(v_m^{(k)} - v_{m-1})^j - (v_m - v_{m-1})^j\right]. \end{aligned} \tag{3.46}$$

By using the inequalities

$$\begin{aligned} |u_{m-1}| \le& \sqrt{R-1} M = R_1 M, \;\; |u_m - u_{m-1}| \le 2R_1 M, \\ \left|u_m^{(k)}\right| \le& R_1 \left\|u_{mx}^{(k)}(t)\right\|_0 \le R_1 \sqrt{\bar{S}_m^{(k)}(t)} \le R_1 M, \\ \left|u_m^{(k)} - u_{m-1}\right| \le& R_1 \left(\left\|u_{mx}^{(k)}(t)\right\|_0 + \|\nabla u_{m-1}(t)\|_0\right) \le 2R_1 M, \\ |x^\alpha - y^\alpha| \le& \alpha M_1^{\alpha-1} |x-y|, \; \forall x,y \in [-M_1, M_1], \; \forall M_1 > 0, \; \forall \alpha \in \mathbb{N}, \end{aligned}$$

we obtain

$$\begin{aligned} &\left|(u_m^{(k)} - u_{m-1})^i - (u_m - u_{m-1})^i\right| \\ \le& i M_1^{i-1} \left|u_m^{(k)} - u_m\right| \le i M_1^{i-1} R_1 \left\|u_{mx}^{(k)} - u_{mx}\right\|_0 \\ \le& \frac{i}{2M} M_1^i \left\|u_m^{(k)} - u_m\right\|_{C([0,T];V)}, \end{aligned}$$

where $M_1 = 2R_1 M$, hence

$$\left\|(u_m^{(k)} - u_{m-1})^i - (u_m - u_{m-1})^i\right\|_0 \leq \frac{i}{2M} M_1^i \sqrt{\frac{R^2-1}{2}} \left\|u_m^{(k)} - u_m\right\|_{C([0,T];V)}.$$

This implies that

$$\begin{aligned}&\left\|(u_m^{(k)} - u_{m-1})^i - (u_m - u_{m-1})^i\right\|_{C([0,T];L^2)}\\ \leq&\frac{i}{2M} M_1^i \sqrt{\frac{R^2-1}{2}} \left\|u_m^{(k)} - u_m\right\|_{C([0,T];V)}.\end{aligned}$$

Similarly, it is clear to see that

$$\begin{aligned}&\left\|(v_m^{(k)} - v_{m-1})^j - (v_m - v_{m-1})^j\right\|_{C([0,T];L^2)}\\ \leq&\frac{j}{2M} M_1^j \sqrt{\frac{R^2-1}{2}} \left\|v_m^{(k)} - v_m\right\|_{C([0,T];V)}.\end{aligned}$$

By the inequalities $|u_m - u_{m-1}|^i \leq M_1^i$, $\left|v_m^{(k)} - v_{m-1}\right|^j \leq M_1^j$, it follows that

$$\begin{aligned}&\left\|\Psi_{m,i,j}^{(k)}\right\|_{C([0,T];L^2)}\\ &\leq \frac{i+j}{2M} M_1^{i+j} \sqrt{\frac{R^2-1}{2}} \left[\left\|u_m^{(k)} - u_m\right\|_{C([0,T];V)} + \left\|v_m^{(k)} - v_m\right\|_{C([0,T];V)}\right] \to 0,\end{aligned}$$

hence

$$F_m^{(k)} \to F_m \text{ strongly in } C([0,T];L^2). \tag{3.47}$$

Similarly, by (3.44), we deduce from $(3.6)_2$ and $(3.10)_2$ that

$$G_m^{(k)} \to G_m \text{ strongly in } C([0,T];L^2). \tag{3.48}$$

We also have

$$\begin{aligned}&\left|a_{1m}^{(k)}(t) - a_1[u_m](t)\right|\\ =&\left|a_1\left(\left\|u_m^{(k)}(t)\right\|_0^2, \left\|u_{mx}^{(k)}(t)\right\|_0^2\right) - a_1\left(\|u_m(t)\|_0^2, \|u_{mx}(t)\|_0^2\right)\right|\\ \leq&\sup_{0\leq y,z\leq M^2} |D_1 a_1(y,z)| \left|\left\|u_m^{(k)}(t)\right\|_0^2 - \|u_m(t)\|_0^2\right|\\ &+ \sup_{0\leq y,z\leq M^2} |D_2 a_1(y,z)| \left|\left\|u_{mx}^{(k)}(t)\right\|_0^2 - \|u_{mx}(t)\|_0^2\right|\\ \leq&2M \sup_{0\leq y,z\leq M^2} \Phi_{[D_1 a_1]}(y+z) \left\|u_m^{(k)}(t) - u_m(t)\right\|_0\\ &+ 2M \sup_{0\leq y,z\leq M^2} \Phi_{[D_2 a_1]}(y+z) \left\|u_{mx}^{(k)}(t) - u_{mx}(t)\right\|_0\\ \leq&2M\left[\Phi_{[D_1 a_1]}\left(2M^2\right) + \Phi_{[D_2 a_1]}\left(2M^2\right)\right] \left\|u_m^{(k)} - u_m\right\|_{C([0,T];V)},\end{aligned}$$

so

$$\begin{aligned}&\sup_{0\le t\le T}\left|a_{1m}^{(k)}(t)-a_1[u_m](t)\right|\\&\le 2M\left[\Phi_{[D_1a_1]}\left(2M^2\right)+\Phi_{[D_2a_1]}\left(2M^2\right)\right]\left\|u_m^{(k)}-u_m\right\|_{C([0,T];V)}\to 0.\end{aligned}$$

Hence

$$a_{1m}^{(k)}\to a_1[u_m] \text{ strongly in } C([0,T]). \tag{3.49}$$

Similarly,

$$a_{2m}^{(k)}\to a_2[v_m] \text{ strongly in } C([0,T]). \tag{3.50}$$

Passing to limit in (3.8), we have (u_m,v_m) satisfying (3.5), (3.6) in $L^2(0,T)$.

Moreover, it follows from (3.5)-(3.8) and $(3.43)_4$ that $u_m''=-a_1[u_m](t)L_1u_m+F_m\in L^\infty(0,T;L^2)$ and $v_m''=-a_2[v_m](t)L_2v_m+G_m\in L^\infty(0,T;L^2)$, hence $(u_m,v_m)\in W_1(M,T)$, so the proof of Theorem 3.1 is completed. □

Next, we state and prove the main result in this section (Theorem 3.6 below), in which

$$W_1(T)=C([0,T];V\times V)\cap C^1([0,T];L^2\times L^2), \tag{3.51}$$

it is well known that $W_1(T)$ is a Banach space with respect to the norm (see Lions [14]):

$$\|(u,v)\|_{W_1(T)}=\|(u,v)\|_{C([0,T];V\times V)}+\left\|(u',v')\right\|_{C([0,T];L^2\times L^2)}. \tag{3.52}$$

Theorem 3.6. *Let $T^*>0$ and $(A_1)-(A_3)$ hold. Then, there exist positive constants $M,T>0$ such that*

(i) Prob. (1.1) *has a unique weak solution $(u,v)\in W_1(M,T)$.*

(ii) The recurrent sequence $\{(u_m,v_m)\}$ defined by (3.5)-(3.6) *converges to the weak solution (u,v) of Prob.* (1.1) *strongly in the space $W_1(T)$.*

Furthermore, we have the estimate

$$\|(u_m,v_m)-(u,v)\|_{W_1(T)}\le C\left(k_T\right)^{N^m},\ \forall m\in\mathbb{N}, \tag{3.53}$$

where $k_T\in(0,1)$ and C are chosen such that k_T, C depend only on T, f, g, a_1, a_2, $\tilde{u}_0$, $\tilde{u}_1$, $\tilde{v}_0$, $\tilde{v}_1$.

Proof. (a) *Existence of the solution.*

We shall prove that $\{(u_m,v_m)\}$ is a Cauchy sequence in $W_1(T)$. Let $\bar{u}_m=u_{m+1}-u_m$, $\bar{v}_m=v_{m+1}-v_m$. Then $(\bar{u}_m,\bar{v}_m)$ satisfies the variational problem

$$\begin{cases}\langle\bar{u}_m''(t),w\rangle+a_1[u_{m+1}](t)a(\bar{u}_m(t),w)\\ \qquad=-\left[a_1[u_{m+1}](t)-a_1[u_m](t)\right]\langle L_1u_m(t),w\rangle\\ \qquad\quad+\langle F_{m+1}(t)-F_m(t),w\rangle,\\ \langle\bar{v}_m''(t),\phi\rangle+a_2[v_{m+1}](t)\langle\bar{v}_{mx}(t),\phi_x\rangle\\ \qquad=-\left[a_2[v_{m+1}](t)-a_2[v_m](t)\right]\langle L_2v_m(t),\phi\rangle\\ \qquad\quad+\langle G_{m+1}(t)-G_m(t),\phi\rangle,\ \forall(w,\phi)\in V\times V,\\ (\bar{u}_m(0),\bar{v}_m(0))=(\bar{u}_m'(0),\bar{v}_m'(0))=(0,0).\end{cases} \tag{3.54}$$

Taking $(w,\phi)=(\bar{u}_m'(t),\bar{v}_m'(t))$ in (3.54), after integrating in t, we get

$$
\begin{aligned}
a_*\bar{Z}_m(t) \le & 2\int_0^t \left\langle F_{m+1}(s)-F_m(s),\bar{u}_m'(s)\right\rangle ds \\
& +2\int_0^t \left\langle G_{m+1}(s)-G_m(s),\bar{v}_m'(s)\right\rangle ds \\
& -2\int_0^t \left[a_1[u_{m+1}](s)-a_1[u_m](s)\right]\langle L_1u_m(s),\bar{u}_m'(s)\rangle ds \\
& -2\int_0^t \left[a_2[v_{m+1}](s)-a_2[v_m](s)\right]\langle L_2v_m(s),\bar{v}_m'(s)\rangle ds \\
& +2\int_0^t \left(a_1[u_{m+1}]\right)'(s)\,\|\bar{u}_m(s)\|_a^2\,ds \\
& +2\int_0^t \left(a_2[v_{m+1}]\right)'(s)\,\|\bar{v}_{mx}(s)\|_0^2\,ds \\
\equiv & \sum\nolimits_{i=1}^6 J_i,
\end{aligned} \tag{3.55}
$$

where $a_*=\min\{1,a_{1*},a_{2*}\}$ and

$$
\bar{Z}_m(t)=\left\|\bar{u}_m'(t)\right\|_0^2+\left\|\bar{v}_m'(t)\right\|_0^2+\|\bar{u}_m(t)\|_a^2+\|\bar{v}_{mx}(t)\|_0^2. \tag{3.56}
$$

We shall estimate the terms on the right-hand side of (3.55) as follows.

First integral J_1. We have

$$
\begin{aligned}
F_{m+1}(t)-F_m(t)= & f[u_m,v_m](x,t)-f[u_{m-1},v_{m-1}](x,t) \\
& +\sum_{1\le i+j\le N-1}\frac{1}{i!j!}D_3^iD_4^jf[u_m,v_m](x,t)\left(\bar{u}_m\right)^i\left(\bar{v}_m\right)^j \\
& -\sum_{1\le i+j\le N-1}\frac{1}{i!j!}D_3^iD_4^jf[u_{m-1},v_{m-1}](x,t)\left(\bar{u}_{m-1}\right)^i\left(\bar{v}_{m-1}\right)^j.
\end{aligned} \tag{3.57}
$$

By using Taylor's expansion of the function $f[u_m,v_m]=f[u_{m-1}+\bar{u}_{m-1},v_{m-1}+\bar{v}_{m-1}]$ around the point $[u_{m-1},v_{m-1}]=(x,t,u_{m-1},v_{m-1})$ up to order N, we obtain

$$
\begin{aligned}
& f[u_m,v_m]-f[u_{m-1},v_{m-1}] \\
= & \sum_{1\le i+j\le N-1}\frac{1}{i!j!}D_3^iD_4^jf[u_{m-1},v_{m-1}]\left(\bar{u}_{m-1}\right)^i\left(\bar{v}_{m-1}\right)^j+R_m[f],
\end{aligned} \tag{3.58}
$$

where

$$
R_m[f]=\sum_{i+j=N}\frac{1}{i!j!}D_3^iD_4^jf[u_{m-1}+\theta\bar{u}_{m-1},v_{m-1}+\theta\bar{v}_{m-1}]\left(\bar{u}_{m-1}\right)^i\left(\bar{v}_{m-1}\right)^j, \tag{3.59}
$$

with $0<\theta<1$.

Then, $F_{m+1}(t)-F_m(t)$ is rewritten as follows

$$
\begin{aligned}
& F_{m+1}(x,t)-F_m(x,t) \\
= & \sum_{1\le i+j\le N-1}\frac{1}{i!j!}D_3^iD_4^jf[u_m,v_m](x,t)\left(\bar{u}_m(x,t)\right)^i\left(\bar{v}_m(x,t)\right)^j+R_m[f](x,t).
\end{aligned} \tag{3.60}
$$

Thus

$$\begin{aligned} &|F_{m+1}(x,t)-F_m(x,t)| \\ \leq &K_N(M,f)\sum_{1\leq i+j\leq N-1}\frac{1}{i!j!}\left|\left(\bar{u}_m(x,t)\right)^i\left(\bar{v}_m(x,t)\right)^j\right|+\left|R_m[f](x,t)\right|. \end{aligned} \tag{3.61}$$

Estimate of $\sum_{1\leq i+j\leq N-1}\frac{1}{i!j!}\left|\left(\bar{u}_m(x,t)\right)^i\left(\bar{v}_m(x,t)\right)^j\right|$. Note that

$$\begin{aligned} \left|\left(\bar{u}_m(x,t)\right)^i\left(\bar{v}_m(x,t)\right)^j\right| \leq &R_1^{i+j}\|\bar{u}_{mx}(t)\|_0^{i-1}\|\bar{v}_{mx}(t)\|_0^j\|\bar{u}_{mx}(t)\|_0 \\ \leq &R_1^{i+j}M^{i-1}M^j\|\bar{u}_{mx}(t)\|_0 \leq R_1^{i+j}M^{i+j-1}\sqrt{\bar{Z}_m(t)}. \end{aligned} \tag{3.62}$$

Therefore, by (3.62), we obtain

$$\begin{aligned} \sum_{1\leq i+j\leq N-1}\frac{1}{i!j!}\left|\left(\bar{u}_m(x,t)\right)^i\left(\bar{v}_m(x,t)\right)^j\right| \leq &\sum_{1\leq i+j\leq N-1}\frac{1}{i!j!}R_1^{i+j}M^{i+j-1}\sqrt{\bar{Z}_m(t)} \\ = &\frac{1}{M}\sum\nolimits_{r=1}^{N-1}\frac{(2MR_1)^r}{r!}\sqrt{\bar{Z}_m(t)}. \end{aligned} \tag{3.63}$$

Estimate of $R_m[f](x,t)$. We have

$$\begin{aligned} |R_m[f](x,t)| \leq &K_N(M,f)\sum_{i+j=N}\frac{1}{i!j!}\left|\left(\bar{u}_{m-1}(x,t)\right)^i\left(\bar{v}_{m-1}(x,t)\right)^j\right| \\ \leq &K_N(M,f)\sum_{i+j=N}\frac{1}{i!j!}R_1^{i+j}\|\nabla\bar{u}_{m-1}(t)\|_0^i\|\nabla\bar{v}_{m-1}(t)\|_0^j \\ \leq &K_N(M,f)\sum_{i+j=N}\frac{1}{i!j!}R_1^{i+j}\|(\bar{u}_{m-1},\bar{v}_{m-1})\|_{W_1(T)}^{i+j} \\ = &K_N(M,f)\frac{(2R_1)^N}{N!}\|(\bar{u}_{m-1},\bar{v}_{m-1})\|_{W_1(T)}^N. \end{aligned} \tag{3.64}$$

It follows from (3.61), (3.63) and (3.64) that

$$\begin{aligned} &\|F_{m+1}(t)-F_m(t)\|_0 \\ \leq &E_1(M,f)\sqrt{\bar{Z}_m(t)}+E_2(M,f)\|(\bar{u}_{m-1},\bar{v}_{m-1})\|_{W_1(T)}^N, \end{aligned} \tag{3.65}$$

where $E_1(M,f)$, $E_2(M,f)$ are defined as in $(3.42)_{4,5}$.

Now, we can estimate the intergal J_1 as follows

$$\begin{aligned} J_1 = &2\int_0^t\left\langle F_{m+1}(s)-F_m(s),\bar{u}_m'(s)\right\rangle ds \\ \leq &2\int_0^t\left(E_1(M,f)\sqrt{\bar{Z}_m(s)}+E_2(M,f)\|(\bar{u}_{m-1},\bar{v}_{m-1})\|_{W_1(T)}^N\right)\sqrt{\bar{Z}_m(s)}ds \\ \leq &TE_2^2(M,f)\|(\bar{u}_{m-1},\bar{v}_{m-1})\|_{W_1(T)}^{2N}+(1+2E_1(M,f))\int_0^t\bar{Z}_m(s)ds. \end{aligned} \tag{3.66}$$

Second integral J_2. Similarly

$$\begin{aligned} J_2 =& 2\int_0^t \left\langle G_{m+1}(s) - G_m(s), \bar{v}_m'(s)\right\rangle ds \\ \leq& TE_2^2(M,g)\, \|(\bar{u}_{m-1}, \bar{v}_{m-1})\|_{W_1(T)}^{2N} + (1+2E_1(M,g)) \int_0^t \bar{Z}_m(s)ds, \end{aligned} \tag{3.67}$$

where

$$\begin{aligned} E_1(M,g) =& \frac{K_N(M,f)}{M}\sqrt{\frac{R^2-1}{2}} \sum\nolimits_{r=1}^{N-1} \frac{(2MR_1)^r}{r!}, \\ E_2(M,g) =& K_N(M,f)\sqrt{\frac{R^2-1}{2}} \frac{(2R_1)^N}{N!}. \end{aligned} \tag{3.68}$$

Thirth integral J_3. Note that, from the following inequalities

$$\begin{aligned} & |a_1[u_{m+1}](s) - a_1[u_m](s)| \\ =& \left| a_1\left(\|u_{m+1}(s)\|_0^2, \|\nabla u_{m+1}(s)\|_0^2\right) - a_1\left(\|u_m(s)\|_0^2, \|\nabla u_m(s)\|_0^2\right)\right| \\ \leq& 2M \sup_{0\leq y,z\leq M^2} |D_1 a_1(y,z)|\, \|u_{m+1}(s) - u_m(s)\|_0 \\ & + 2M \sup_{0\leq y,z\leq M^2} |D_2 a_1(y,z)|\, \|\nabla u_{m+1}(s) - \nabla u_m(s)\|_0 \\ =& 2\sqrt{2}\bar{a}_1^* M \left[\Phi_{[D_1 a_1]}\left(2M^2\right) + \Phi_{[D_2 a_1]}\left(2M^2\right)\right] \|\bar{u}_{mx}(s)\|_0, \end{aligned}$$

and

$$\left|\langle L_1 u_m(s), \bar{u}_m'(s)\rangle\right| \leq \bar{\gamma}_1 \|u_m(s)\|_{H^2\cap V} \left\|\bar{u}_m'(s)\right\|_0 \leq \bar{\gamma}_1 M \left\|\bar{u}_m'(s)\right\|_0,$$

we get

$$\begin{aligned} J_3 =& -2\int_0^t [a_1[u_{m+1}](s) - a_1[u_m](s)]\, \langle L_1 u_m(s), \bar{u}_m'(s)\rangle ds \\ \leq& 4\sqrt{2}\bar{\gamma}_1\bar{a}_1^* M^2 \left[\Phi_{[D_1 a_1]}\left(2M^2\right) + \Phi_{[D_2 a_1]}\left(2M^2\right)\right] \int_0^t \|\bar{u}_{mx}(s)\|_0 \left\|\bar{u}_m'(s)\right\|_0 ds \\ \leq& 2\sqrt{2}\bar{\gamma}_1\bar{a}_1^* M^2 \left[\Phi_{[D_1 a_1]}\left(2M^2\right) + \Phi_{[D_2 a_1]}\left(2M^2\right)\right] \int_0^t \bar{Z}_m(s)ds \\ =& 2\sqrt{2}\bar{\gamma}_1\bar{a}_1^* \tilde{K}_M(a_1) \int_0^t \bar{Z}_m(s)ds, \end{aligned} \tag{3.69}$$

where $\tilde{K}_M(a_1) = M^2\left[\Phi_{[D_1 a_1]}\left(2M^2\right) + \Phi_{[D_2 a_1]}\left(2M^2\right)\right]$.
Fourth integral J_4. Similarly, we deduce from

$$\begin{aligned} & |a_2[v_{m+1}](s) - a_2[v_m](s)| \\ =& \left| a_2\left(\|v_{m+1}(s)\|_0^2, \|\nabla v_{m+1}(s)\|_0^2\right) - a_2\left(\|v_m(s)\|_0^2, \|\nabla v_m(s)\|_0^2\right)\right| \\ \leq& 2\sqrt{2}\bar{a}_1^* M \left[\Phi_{[D_1 a_2]}\left(2M^2\right) + \Phi_{[D_2 a_2]}\left(2M^2\right)\right] \|\bar{v}_{mx}(s)\|_0, \end{aligned}$$

and

$$\left|\langle L_2 v_m(s), \bar{v}_m'(s)\rangle\right| \leq \sqrt{2}M \left\|\bar{v}_m'(s)\right\|_0,$$

that

$$\begin{aligned} J_4 =& -2\int_0^t [a_2[v_{m+1}](s)-a_2[v_m](s)]\,\langle L_2 v_m(s),\bar{v}_m'(s)\rangle ds \\ &\le 4\bar{a}_1^*\tilde{K}_M(a_2)\int_0^t \bar{Z}_m(s)ds, \end{aligned} \tag{3.70}$$

where $\tilde{K}_M(a_2)=M^2\left[\Phi_{[D_1a_2]}\left(2M^2\right)+\Phi_{[D_2a_2]}\left(2M^2\right)\right].$
Fifth integral J_5. By

$$\begin{aligned} &\left|(a_1[u_{m+1}])'(s)\right| \\ \le&\left|D_1a_1[u_{m+1}](s)\langle u_{m+1}(s),u_{m+1}'(s)\rangle\right|+\left|D_2a_1[u_{m+1}](s)\langle \nabla u_{m+1}(s),\nabla u_{m+1}'(s)\rangle\right| \\ \le& M^2\left(\sup_{0\le y,z\le M^2}|D_1a_1(y,z)|+\sup_{0\le y,z\le M^2}|D_2a_1(y,z)|\right) \\ \le& M^2\left[\Phi_{[D_1a_1]}\left(2M^2\right)+\Phi_{[D_2a_1]}\left(2M^2\right)\right]=\tilde{K}_M(a_1), \end{aligned}$$

we obtain

$$\begin{aligned} J_5 =& 2\int_0^t (a_1[u_{m+1}])'(s)\,\|\bar{u}_m(s)\|_a^2\,ds \\ &\le 2\tilde{K}_M(a_1)\int_0^t \|\bar{u}_m(s)\|_a^2\,ds\le 2\tilde{K}_M(a_1)\int_0^t \bar{Z}_m(s)ds. \end{aligned} \tag{3.71}$$

Sixth integral J_6. Similarly

$$J_6=2\int_0^t (a_2[v_{m+1}])'(s)\,\|\bar{v}_{mx}(s)\|_0^2\,ds\le 2\tilde{K}_M(a_2)\int_0^t \bar{Z}_m(s)ds. \tag{3.72}$$

Combining (3.55), (3.66), (3.67), (3.69)-(3.72), it leads to

$$\begin{aligned} \bar{Z}_m(t) \le& T\tilde{E}_1(M,f,g)\,\|(\bar{u}_{m-1},\bar{v}_{m-1})\|_{W_1(T)}^{2N} \\ &+2\tilde{E}_2(M,f,g,a_1,a_2)\int_0^t \bar{Z}_m(s)ds, \end{aligned} \tag{3.73}$$

where $\tilde{E}_1(M,f,g)$, $\tilde{E}_2(M,f,g,a_1,a_2)$ are defined as in $(3.42)_{2,3}$.
By Gronwall's lemma, we deduce from (3.71), that

$$\|(\bar{u}_m,\bar{v}_m)\|_{W_1(T)}\le \mu_T\,\|(\bar{u}_{m-1},\bar{v}_{m-1})\|_{W_1(T)}^N, \tag{3.74}$$

where μ_T is defined as in $(3.42)_1$, with $k_T=M\mu_T^{\frac{1}{N-1}}<1$, which implies that

$$\|(u_m,v_m)-(u_{m+p},v_{m+p})\|_{W_1(T)}\le(1-k_T)^{-1}(\mu_T)^{\frac{-1}{N-1}}(k_T)^{N^m},\ \forall m,p\in\mathbb{N}. \tag{3.75}$$

It follows that $\{(u_m,v_m)\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $(u,v)\in W_1(T)$ such that

$$(u_m,v_m)\to(u,v)\text{ strongly in }W_1(T). \tag{3.76}$$

Note that $(u_m, v_m) \in W_1(M,T)$, then there exists a subsequence $\{(u_{m_j}, v_{m_j})\}$ of $\{(u_m, v_m)\}$ such that

$$\begin{cases} (u_{m_j}, v_{m_j}) \to (u,v) & \text{in} \quad L^\infty(0,T; \left(H^2 \cap V\right) \times \left(H^2 \cap V\right)) \text{ weak*}, \\ (u'_{m_j}, v'_{m_j}) \to (u',v') & \text{in} \quad L^\infty(0,T; V \times V) \text{ weak*}, \\ (u''_{m_j}, v''_{m_j}) \to (u'',v'') & \text{in} \quad L^2(0,T; L^2 \times L^2) \text{ weak}, \\ (u,v) \in W(M,T). \end{cases} \tag{3.77}$$

By the compactness lemma of Aubin-Lions [1], we can deduce from $(3.77)_{1,2,3}$ the existence of a subsequence still denoted by $\{(u_{m_j}, v_{m_j})\}$, such that

$$\begin{cases} (u_{m_j}, v_{m_j}) \to (u,v) \text{ strongly in } C([0,T]; V \times V), \\ (u'_{m_j}, v'_{m_j}) \to (u',v') \text{ strongly in } C([0,T]; L^2 \times L^2). \end{cases} \tag{3.78}$$

We also note that

$$\begin{aligned} &\|F_m - f[u,v]\|_{C([0,T];L^2)} \\ \leq & K_N(M,f)\sqrt{\frac{R^2-1}{2}} \Big[R_1 \|(u_{m-1}, v_{m-1}) - (u,v)\|_{W_1(T)} \\ &+ \sum_{r=1}^{N-1} \frac{(2R_1)^r}{r!} \|(u_m, v_m) - (u_{m-1}, v_{m-1})\|^r_{W_1(T)} \Big]. \end{aligned} \tag{3.79}$$

Hence, from (3.76) and (3.79), it implies that

$$F_m(t) - f[u,v] \text{ strongly in } C([0,T]; L^2). \tag{3.80}$$

Similarly

$$G_m \to g[u,v] \text{ strongly in } C([0,T]; L^2). \tag{3.81}$$

On the other hand

$$\sup_{0 \leq t \leq T} |a_1[u_m](t) \to a_1[u](t)| \leq \frac{2\sqrt{2}\bar{a}_1^*}{M} \tilde{K}_M(a_1) \|u_m - u\|_{C([0,T];V)} \to 0,$$

so

$$a_1[u_m] \to a_1[u] \text{ strongly in } C([0,T]). \tag{3.82}$$

Similarly

$$a_2[v_m](t) \to a_2[v](t) \text{ strongly in } C([0,T]). \tag{3.83}$$

Finally, passing to limit in (3.5), (3.6) as $m = m_j \to \infty$, it implies from (3.76), $(3.77)_{1,2,3}$, (3.80) - (3.83) that there exists $(u,v) \in W(M,T)$ satisfying the equations

$$\begin{cases} \langle u''(t), w\rangle + a_1[u](t)a(u(t), w) = \langle f[u,v](t), w\rangle, \\ \langle v''(t), \phi\rangle + a_2[v](t)b(v(t), \phi) = \langle g[u,v](t), \phi\rangle, \end{cases} \tag{3.84}$$

for all $(w, \phi) \in V \times V$, a.e., $t \in (0,T)$, and the initial conditions

$$\big(u(0), u'(0)\big) = (\tilde{u}_0, \tilde{u}_1), \ \big(v(0), v'(0)\big) = (\tilde{v}_0, \tilde{v}_1). \tag{3.85}$$

On the other hand, from the assumption (A_2), we obtain from $(3.77)_4$, (3.80), (3.81) and (3.84), that $u'' = -a_1[u](t)L_1 u + f[u,v] \in L^\infty(0,T;L^2)$ and $v'' = -a_2[v](t)L_2 v + g[u,v] \in L^\infty(0,T;L^2)$. Thus, we have the solution $(u,v) \in W_1(M,T)$. The existence proof is completed.

(b) *Uniqueness*. By applying a similar argument as in the proof of Theorem 3.1, we can prove that the solution $(u, v) \in W_1(M, T)$ is unique.

(c) *The estimate* (3.53). Passing to the limit in (3.75) as $p \to +\infty$ for fixed m, we get (3.53).

Theorem 3.6 is proved. □

Remark 3.1. In order to construct a $N-$order iterative scheme, we need the condition (A_3) . Then, we obtain a convergent sequence at a rate of order N to a local weak solution of the problem. We note that, this condition of f can be relaxed if we only consider the existence of solution (for more detail, we refer to [24]).

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