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Oscillatory properties of eigenfunctions corresponding to the negative eigenvalues of some boundary value problem with a spectral parameter in the boundary conditions

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Abstract. *In this paper we consider the boundary value problem for fourth-order ordinary differential equations with spectral parameter contained in the two of boundary conditions. We completely study the oscillation properties of solutions of the corresponding initial-boundary value problem for negative values of the spectral parameter, whence can be easily found the number of zeros of the eigenfunctions corresponding to negative eigenvalues of this problem.*

Keywords. initial-boundary value problem, eigenvalue parameter, eigenvalue, oscillatory property of eigenfunction

Mathematics Subject Classification (2010): 34B05, 34B08, 34B09, 34L10, 34L15, 47A75, 47B50, 74H45.

1 Introduction

We consider the following eigenvalue problem:

$$y^{(4)}(x) - (q(x)y'(x))' = \lambda y(x),\ x \in (0, 1), \tag{1.1}$$

$$y''(0) = y''(1) = 0, \tag{1.2}$$

$$Ty(0) - a\lambda y(0) = 0, \tag{1.3}$$

$$Ty(1) - c\lambda y(1) = 0, \tag{1.4}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, $Ty \equiv y''' - qy'$, $q(x)$ is a positive absolutely continuous function on $[0, 1]$, a and c are real constants such that $a < 0$ and $c > 0$.

Problem (1.1)-(1.4) describes free bending vibrations of a homogeneous Euler-Bernoulli beam of constant rigidity, in the cross sections of which a longitudinal force acts, and the masses are concentrated at both ends (see [6, p. 152-154]).

The spectral properties of problem (1.1)-(1.4), including the oscillatory properties of eigenfunctions and the basis properties of root functions in L_p, $1 < p < \infty$, were studied in [3]. However, it should be noted that the number of zeros contained in the interval $(0, 1)$

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of the eigenfunctions corresponding to the negative eigenvalues of the problem (1.1)-(1.4) found there is not exact.

The purpose of this paper is to clarify the number of zeros contained in the interval $(0, 1)$ of eigenfunctions corresponding to the negative eigenvalues of problem (1.1)-(1.4).

2 Preliminary

When studying the oscillatory properties of eigenfunctions of spectral problems for ordinary differential equations of the fourth order, the following statement plays an essential role.

Lemma 2.1 (see [5, Lemma 2.1]) *Let $y(x, \lambda)$ be a nontrivial solution of differential equation* (1.1) *for $\lambda > 0$. If y, y', y'', Ty are nonnegative at $x = a$ (but not all zero), then they are positive for all $x > a$. If y, $-y'$, y'', $-Ty$ are nonnegative at $x = a$ (but not all zero), then they are positive for $x < a$.*

Along with problem (1.1)-(1.4) we consider the eigenvalue problem (1.1)-(1.3) and

$$y(1) \cos \delta - Ty(1) \sin \delta = 0, \tag{2.1}$$

where $\delta \in [0, \pi/2]$. For this problem we have the following result.

Theorem 2.1 [3, Theorems 4 and 5] (see also [2]). *The eigenvalues of problem* (1.1)-(1.3), (2.1) *for $\delta = 0$ are real, simple and form an infinitely increasing sequence $\{\mu_k\}_{k=1}^{\infty}$, for $\delta = \pi/2$ are real and simple, and, except the case $a = -1$, where $\lambda = 0$ is a double eigenvalue, form an infinitely nondecreasing sequence $\{\nu_k\}_{k=1}^{\infty}$ such that*

$$\begin{array}{ll} \mu_1 < \nu_1 < 0 = \nu_2 < \mu_2 < \nu_3 < \mu_3 < \ . \ . \ . & if \ a > -1, \\ \mu_1 < 0 = \nu_1 = \nu_2 < \mu_2 < \nu_3 < \mu_3 < \ . \ . \ . & if \ a = -1, \\ \mu_1 < 0 = \nu_1 < \nu_2 < \mu_2 < \nu_3 < \mu_3 < \ . \ . \ . & if \ a < -1. \end{array} \tag{2.2}$$

Moreover, the eigenfunction $v_k(x)$, corresponding to the eigenvalue μ_k, for $k \geq 2$ has exactly $k - 1$ simple zeros in $(0, 1)$; the number of zeros belonging to the interval $(0, 1)$ of eigenfunction $v_1(x)$ can be arbitrary.

Now we consider initial-boundary value problem (1.1)-(1.3).

Theorem 2.2 [3, Theorem 6] *For each fixed $\lambda \in \mathbb{C}$ there exists a nontrivial solution $y(x, \lambda)$ of the problem* (1.1)-(1.3) *which is unique up to a constant factor.*

Remark 2.1 From the proof of Theorem 2.2 it follows that the solution $y(x, \lambda)$ of problem (1.1)-(1.3) have the following representation:

$$y(x, \lambda) = \varphi_2''(1, \lambda) \{\varphi_1(x, \lambda) + a\lambda\varphi_4(x, \lambda)\} - \left\{\varphi_1''(1, \lambda) + a\lambda\varphi_4''(1, \lambda)\right\} \varphi_2(x, \lambda),$$

where $\varphi_k(x, \lambda), k = \overline{1, 4}$, are solutions of Eq. (1.1), normalized for $x = 0$ by the Cauchy conditions

$$\varphi_k^{(s-1)}(0, \lambda) = \delta_{ks}, \ s = \overline{1, 3}, \ T\varphi_k(0, \lambda) = \delta_{k4}, \tag{2.3}$$

δ_{ks} is the Kronecker delta. Then the function $y(x, \lambda)$ is an entire function of the parameter λ for each fixed $x \in [0, 1]$, since the functions $\varphi_k(x, \lambda)$, $k = 1, 2, 3, 4$, and their derivatives are entire function of the parameter λ for each fixed $x \in [0, 1]$ (see [7, Ch. I, § 2.1]).

Remark 2.2 It is obvious that the eigenvalues μ_k and ν_k of the spectral problem (1.1)-(1.3), (2.1) for $\delta = 0$ and $\delta = \pi/2$ are zeros of the entire functions $y(1, \lambda)$ and $Ty(1, \lambda)$, respectively.

We introduce the notation: $D_k = (\mu_{k-1}, \mu_k)$, $k = 1, 2, \ldots$, where $\mu_0 = -\infty$.

Note that the function $F(\lambda) = \frac{Ty(1,\lambda)}{y(1,\lambda)}$ is defined in the domain $D_F = (\mathbb{C}\backslash\mathbb{R}) \cup \left(\bigcup_{k=1}^{\infty} D_k\right)$ for which in [3] established the following statements.

Lemma 2.2 [3, Lemmas 3-5] *The following relations hold:*

$$\frac{dF(\lambda)}{d\lambda} = \frac{1}{y^2(1,\lambda)} \left\{ \int_0^l y^2(x,\lambda)\, dx + ay^2(0,\lambda) \right\}, \; \lambda \in D_F, \tag{2.4}$$

$$\lim_{\lambda \to -\infty} F(\lambda) = -\infty, \tag{2.5}$$

$$F(\lambda) = \sum_{k=1}^{\infty} \frac{\lambda\, c_k}{\mu_k(\lambda - \mu_k)}\,, \tag{2.6}$$

where $c_k = \operatorname*{res}_{\lambda=\mu_k} F(\lambda),\; k \in \mathbb{N}$, *and* $c_1 > 0,\; c_k < 0,\; k \geq 2$.

Remark 2.3 From (2.6) we obtain

$$F''(\lambda) = 2 \sum_{k=1}^{\infty} \frac{c_k}{(\lambda - \mu_k)^3}\,,$$

which implies that $F''(\lambda) < 0$ for $\lambda \in D_2 = (\mu_1, \mu_2)$, i.e., the function $F(\lambda)$ is convex in D_2.

Lemma 2.3 *One has the following relations:*

$$F(\lambda) < 0 \; for \; \lambda \in (-\infty, \mu_1), \; \lim_{\lambda \to \mu_1 - 0} F(\lambda) = -\infty,$$

$$\lim_{\lambda \to \mu_1 + 0} F(\lambda) = +\infty, \; F(\lambda) > 0 \; for \; \lambda \in (\mu_1, \nu_1),$$

$$F(\lambda) < 0 \; for \; \lambda \in (\nu_1, 0) \; \; in \; the \; case \; \; a > -1.$$

Proof. Since μ_1 is a smallest root of the function $y(1, \lambda)$, by (2.2), Remark 2.2 and (2.5), we have

$$F(\lambda) < 0 \;\text{ for }\; \lambda \in D_1 = (-\infty, \mu_1). \tag{2.7}$$

Since μ_1 is a simple pole of the function $F(\lambda)$ it follows from (2.7) that

$$\lim_{\lambda \to \mu_1 - 0} F(\lambda) = -\infty, \;\text{ and }\; \lim_{\lambda \to \mu_1 + 0} F(\lambda) = +\infty. \tag{2.8}$$

In view of (2.2) we have $\mu_1 < \nu_1 < 0$ for $a > -1$ and $\mu_1 < \nu_1 = 0$ for $a \leq -1$. Then by Remark 2.3 and (2.8) we get

$$F(\lambda) > 0 \text{ for } \lambda \in (\mu_1, \nu_1) \;\text{ and }\; F(\lambda) < 0 \text{ for } \lambda \in (\nu_1, 0) \text{ in the case } a > -1, \tag{2.9}$$

$$F(\lambda) > 0 \text{ for } \lambda \in (\mu_1, 0) \;\text{ in the case } a \leq -1. \tag{2.10}$$

The proof of this lemma is complete.

3 Oscillation properties of solutions of the initial-boundary value problem (1.1)-(1.3)

Consider the equation

$$y(x,\lambda)=0,\ \ x\in[0,1],\ \ \lambda\in\mathbb{R}. \tag{3.1}$$

Obviously, the zeros Eq. (3.1) are functions of λ. For these zeros, in [3] the following lemma was formulated and proved.

Lemma 3.1 *The zeros in $(0,1]$ of function $y(x,\lambda)$ are simple and C^1 function of $\lambda\in\mathbb{R}$.*

The proof of this lemma for $\lambda>0$ is based on Lemma 2.1. But for $\lambda<0$ the proof of this lemma given in [3] contains a gap (see the proof of Lemma 6 there).

Now we will give a complete proof of this lemma for $\lambda<0$.
Proof of Lemma 3.1 for $\lambda<0$. Let $y(x_0,\lambda)=y'(x_0,\lambda)=0$ for some $\lambda<0$ and $x_0\in(0,1)$. Then $y(x,\lambda)$ is a solution of the initial-boundary value problem

$$y^{(4)}(x)-(q(x)y'(x))'=\lambda y(x),\ x\in(x_0,1), \tag{3.2}$$

$$y(x_0)=y'(x_0)=y''(1)=0. \tag{3.3}$$

It follows from the proof of [3, Lemma 6] that $y''(x_0,\lambda)Ty(x_0,\lambda)\neq 0$. Integrating (3.2) in the range from x_0 to 1, using the formula for integration by parts and taking into account conditions (3.3) we obtain

$$\int\limits_{x_0}^{1}\left\{y''^2(x,\lambda)+q(x)y'^2(x,\lambda)\right\}dx+Ty(1,\lambda)y(1,\lambda)=\lambda\int\limits_{x_0}^{1}y^2(x,\lambda)dx. \tag{3.4}$$

Since $\lambda<0$ and $\int\limits_{x_0}^{1}y^2(x,\lambda)dx>0$ the left hand of (3.4) takes a nonzero value. If $\lambda\in(\mu_1,\nu_1)$, then by Lemma 2.3 we have $Ty(1,\lambda)y(1,\lambda)>0$, and if $\lambda=\mu_1$ ($\lambda=\nu_1$ for $a>-1$), then $Ty(1,\mu_1)y(1,\mu_1)=0$ ($Ty(1,\nu_1)y(1,\nu_1)=0$). Consequently, the left hand of (3.4) is positive. Therefore, it follows from (3.4) that $\lambda>0$ which contradicts the condition $\lambda<0$.

By Lemma 2.3 we have

$$Ty(1,\lambda)y(1,\lambda)<0 \text{ for } \lambda<\mu_1.$$

Since the left hand of (3.4) is positive for $\lambda=\mu_1$ it follows from continuity of the left hand of (3.4) on the parameter λ that

$$\int\limits_{x_0}^{1}\left\{y''^2(x,\lambda)+q(x)y'^2(x,\lambda)\right\}dx+Ty(1,\lambda)y(1,\lambda)>0 \tag{3.5}$$

for $\lambda<\mu_1$ and close enough to μ_1. Despite the fact that $Ty(1,\lambda)y(1,\lambda)<0$ for $\lambda<\mu_1$, relation (3.5) will hold for all such λ. Indeed, otherwise for some $\lambda=\lambda^*$, the left-hand side of equality (3.4) will be equal to zero, but the right-hand side will be different from zero. Then by (3.5) it follows from (3.4) that $\lambda>0$ in contradiction with the condition $\lambda<0$.

If $\lambda\in(\nu_1,0)$ for $a>-1$, repeating the above reasoning we arrive at a contradiction.

It follows from Theorem 2.1 that $y(1,\lambda)\neq 0$ for $\lambda\in(-\infty,0)$, $\lambda\neq\mu_1$. If $y'(1,\mu_1)=0$, then μ_1 is an eigenvalue of problem

$$\begin{aligned}&y^{(4)}(x)-(q(x)y'(x))'=\lambda y(x),\ x\in(0,1),\\&y'(0)\cos\alpha-y''(0)\sin\alpha=Ty(0)-a\lambda y(0)=0,\\&y(1)=y''(1)=0,\end{aligned}$$

both for $\alpha = 0$ and $\alpha = \pi/2$. It follows from [1, Theorem 4.1, statement (i)] that the eigenvalues of this problem are real, simple and form infinitely increasing sequence $\{\zeta_k(\alpha)\}_{k=1}^{\infty}$. Moreover, by [2, Theorem 2.3] for this eigenvalues the following relation holds:

$$\zeta_1(0) < \zeta_1(\pi/2) < 0 < \zeta_2(\pi/2) < \zeta_2(0) < \zeta_3(\pi/2) < \zeta_3(0) < \ \ldots \ ,$$

which leads to a contradiction to the fact that λ is an eigenvalue of this problem both for $\alpha = 0$ and $\alpha = \pi/2$.

The smoothness of $x(\lambda)$ follows from the well-known implicit function theorem. The proof of Lemma 3.1 is complete.

Lemma 3.2 [3, Lemma 7]. *As $\lambda < \mu_1$ or $\lambda \in (\mu_1, 0)$ varies the function $y(x, \lambda)$ can lose or gain zeros only by these zeros leaving or entering the interval $[0, 1]$ through its endpoint $x = 0$.*

Now we find the number of zeros of the function $y(x, \lambda)$ contained in the interval $(0, 1)$ for $\lambda < 0$.

Lemma 3.3 [3, Lemma 10]. *Let $\lambda_0 \in (-\infty, \mu_1) \cup (\mu_1, 0)$ and $y(0, \lambda_0) \neq 0$. Then there exists $\epsilon_0 > 0$ such that for any $\lambda \in (\lambda_0 - \epsilon_0, \lambda_0 + \epsilon_0)$ the number of zeros of the function $y(x, \lambda)$ in the interval $(0, 1)$ coincide with that for the function $y(x, \lambda_0)$.*

Let $\tau(\lambda)$ be the number of zeros of the function $y(x, \lambda)$ contained in the interval $(0, 1)$.

Corollary 3.1 [3, Corollary 1]. *Let $\mu, \nu \in (-\infty,\ \mu_1)$ or $\mu, \nu \in (\mu_1, 0)$, $\mu < \nu$ and $\tau(\mu) \neq \tau(\nu)$. Then the interval (μ, ν) contain an eigenvalue of the boundary value problem*

$$\begin{array}{l} y^{(4)}(x) - (q(x)y'(x))' = \lambda y(x),\ x \in (0, 1), \\ y(0) = y''(0) = Ty(0) = y''(1) = 0. \end{array} \tag{3.6}$$

Lemma 3.4 *The real eigenvalues of problem* (3.6) *are negative.*

Proof. Let μ be the eigenvalue of problem (3.6) and $v(x)$ be the corresponding eigenfunction. If $\mu > 0$, then in view of condition $v(0) = v''(0) = Tv(0) = 0$, by Lemma 2.1, we get $v''(1) \neq 0$ which contradicts the condition $v''(1) = 0$. If $\mu = 0$, then we have $v(x) = \kappa\varphi_2(x, 0)$, where $\kappa \neq 0$ is some constant. By Lemma 2.1 for any $\lambda > 0$ we have

$$\varphi_2''(x, \lambda) > 0,\ T\varphi_2(x, \lambda) > 0 \text{ for } x \in (0, 1] \text{ and } \varphi_2'(x, \lambda) \geq 1 \text{ for } x \in [0, 1].$$

Then we get

$$\varphi_2''(x, 0) \geq 0,\ T\varphi_2(x, 0) \geq 0 \text{ and } \ \varphi_2'(x, \lambda) \geq 1 \text{ for } x \in [0, 1].$$

From the relation

$$T\varphi_2(x, 0) = \varphi_2'''(x, 0) - q(x)\varphi_2'(x, 0) \geq 0 \text{ for } x \in [0, 1],$$

we obtain $\varphi_2'''(x, 0) \geq q(x)\, \varphi_2'(x, 0) > 0$ for $x \in [0, 1]$, whence, by $\varphi_2''(0, 0) = 0$, implies that $\varphi_2''(1, 0) > 0$. Consequently, $v''(1) \neq 0$ which contradicts the condition $v''(1) = 0$. The proof of this corollary is complete.

Lemma 3.5 *The real eigenvalues of problem* (3.6) *are contained in* $(-\infty, \mu_1)$.

Proof. Let $\lambda \in [\mu_1, 0)$ be the eigenvalue of problem (3.6). Then integrating the equation in (3.6) in the range from 0 to 1, using the formula for integration by parts and taking into account boundary conditions in (3.6) we obtain

$$\int_0^1 \left\{y''^2(x,\lambda) + q(x)y'^2(x,\lambda)\right\} dx + Ty(1,\lambda)\, y(1,\lambda) = \lambda \int_0^1 y^2(x,\lambda)dx. \tag{3.7}$$

If $\lambda = \mu_1$, then $y(1, \mu_1) = 0$, and if $\lambda \in (\mu_1, \nu_1)$, then $Ty(1,\lambda)\, y(1,\lambda) > 0$. Consequently, the left hand-side of (3.7) is positive. If $a > -1$ and $\lambda = \nu_1$, then $Ty(1,\nu_1) = 0$, and if $a > -1$ and $\lambda \in (\nu_1, 0)$, then $Ty(1,\lambda)\, y(1,\lambda) < 0$. Since the left-hand side of (3.7) is positive for $\lambda = \nu_1$ in the case $a > -1$, it follows from the continuity of the left-hand side of (3.7) with respect to the parameter λ that it remains positive for all $\lambda \in (\nu_1, 0)$ in this same case. Then from (3.7) we obtain $\lambda > 0$ in contradiction with the condition $\lambda < 0$. The proof of this lemma is complete.

Corollary 3.2 *If $\lambda \in [\mu_1, 0)$, then $\tau(\lambda) = 0$.*

Proof. In view of $y(x, 0) \equiv 1$ we have $\tau(\lambda) = 0$ for all $\lambda < 0$ near 0, which by Lemma 3.5 and Corollary 3.1 implies that $\tau(\lambda) = 0$ for all $\lambda \in [\mu_1, 0)$. The proof of this corollary is complete.

Remark 3.1 Since μ_1 is a simple zero of the function $y(1,\lambda)$ by Lemma 3.5 and Corollary 3.2 it follows that there exists sufficiently small $\epsilon_1 > 0$ such that $\tau(\mu_1 - \epsilon) = 1$ for any $\epsilon \in (0, \epsilon_1)$.

Let μ be a real eigenvalue of the problem (3.6). The oscillation index of this eigenvalue which denoted by $i(\mu)$ is the difference between the number of zeros of the function $y(x,\lambda)$ for $\lambda = \mu - 0$ belonging to the interval $(0, 1)$ and the number of the same zeros for $\lambda = \mu + 0$ (see [4]).

Theorem 3.1 *The eigenvalues ξ_k, , $k = 1, 2, \ldots,$ of problem (3.6) are real and simple, contained in $(-\infty, \mu_1)$, are numbered in descending order and allow asymptotics*

$$\xi_k = -4(k + 1/4)^4\pi^4 + o(k^4),$$

and have oscillation index 1.

Proof. The proof of this theorem is similar to that of [4, Theorem 4.1] with the use of Lemma 3.5.

From the above definition, Corollary 3.2 and Remark 3.1 it follows that for $\lambda < 0$ the number of zeros of function $y(x,\lambda)$ belonging to the interval $(0, 1)$ is defined as follows:

$$\tau(\lambda) = \begin{cases} 0 & \text{if } \lambda \in [\mu_1, 0), \\ 1 + \sum\limits_{\xi_s \in (\lambda, \mu_1)} i(\xi_s) & \text{if } \lambda \in (-\infty, \mu_1). \end{cases} \tag{3.8}$$

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