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Compactness of commutators of singular integral operators in complementary Morrey spaces with variable exponent

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Abstract. *In this paper we consider complementary Morrey spaces* ${}^{\complement}\mathcal{L}^{p(\cdot),\lambda}(\Omega)$ *with variable exponent* $p(x)$ *. We prove the compactness of commutators of singular integral operators in variable exponent complementary Morrey spaces in case of unbounded sets* $\Omega \subset \mathbb{R}^n$ *.*

Keywords. singular integral operators, commutators, complementary Morrey space, BMO space.

Mathematics Subject Classification (2010): 42B20, 42B25, 42B35

1 Introduction and problem statement

The variable exponent analysis is a popular topic which continues to attract many researchers, both in view of possible applications and also because of difficulties in investigation and existing challenging problems. There is an evident increase of investigations, last two decades related to both the theory of variable exponent function spaces and operator theory in these spaces. The study of variable exponent function spaces has been stimulated by problems of elasticity, fluid dynamics, calculus of variations and differential equations with non-standard growth conditions (see [9], [26]). Various results on non-weighted and weighted boundedness in Lebesgue spaces with variable exponents $p(x)$ have been proved for maximal, singular and fractional type operators, we refer to surveying papers [11] and [28].

In 1938 C. Morrey [24] studied Morrey spaces for the first time in connection to its applications in partial differential equations. Until recently, a rapid growth has been seen in the study of Morrey type spaces because of its applications in major fields of engineering and sciences. Function spaces with non-standard growth has seen a major focus in recent times because of its wide range of applications in the area of image processing, the study of thermorheological fluids and modeling of electrorheological fluids. It would be next to impossible to give a complete account of the literature which is available to this subject.

Variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$, were introduced and studied in [3] in the Euclidean setting. In [3] the boundedness of the maximal operator was proved in variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$ under the log-condition on $p(\cdot)$, and for potential operators a Sobolev type $\mathcal{L}^{p(\cdot),\lambda(\cdot)} \to \mathcal{L}^{q(\cdot),\lambda(\cdot)}$ theorem was proved under the same log-condition in the case of bounded sets. Hästö in [19]

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used his new local-to-global approach to extend the result of [3] on the maximal operator to the case of the whole space $\mathbb{R}^n$.

The generalized variable exponent Morrey spaces were introduced and studied in [15] in the case of bounded sets. In [15] (in the case of unbounded sets [18]) the boundedness of the maximal operator, potential operators and singular integral operators in variable exponent Morrey spaces under the certain conditions were proved.

In [22] the boundedness of the maximal operator and the singular integral operator in variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}$ in the general setting of metric measure spaces was proved. In the case of constant p and λ , the results on the boundedness of potential operators and classical Calderón-Zygmund singular operators go back to [1] and [25] respectively.

We denote by [14] the local complementary Morrey spaces ${}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\mathbb{R}^n)$ with constant p , the space of all functions $f \in L^{p,loc}(\mathbb{R}^n \backslash \{x_0\})$, $r > 0$ with finite norm

$$\|f\|_{{}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\mathbb{R}^n)} = \sup_{r>0} r^{\frac{\lambda}{p'}} \|f\|_{L^p(\mathbb{R}^n \backslash B(x_0,r))}, \quad x_0 \in \mathbb{R}^n,$$

where $p' = \frac{p}{p-1}$, $1 \le p < \infty$ and $0 \le \lambda < n$. Note that ${}^{\complement}\mathcal{L}^{p,0}_{\{x_0\}}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$.

Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{\widetilde{B}(x,r)} |f(y)|dy,$$

where $\widetilde{B}(x,r) = B(x,r) \cap \Omega$.

Calderón-Zygmund singular operators

$$Tf(x) = \int_{\Omega} K(x,y)f(y)dy,$$

where $K(x,y)$ is a "standard" singular kernel, that is, a function continuous on $\{(x,y) \in \Omega \times \Omega : x \neq y\}$ and satisfying the estimates

$$|K(x,y)| \le C|x-y|^{-n} \text{ for all } x \neq y,$$

$$|K(x,y) - K(x,z)| \le C\frac{|y-z|^{\sigma}}{|x-y|^{n+\sigma}}, \quad \sigma > 0, \quad \text{if } |x-y| > 2|y-z|,$$

$$|K(x,y) - K(\xi,y)| \le C\frac{|x-\xi|^{\sigma}}{|x-y|^{n+\sigma}}, \quad \sigma > 0, \quad \text{if } |x-y| > 2|x-\xi|.$$

This paper is organized as follows. In Section 2 we provide necessary preliminaries on variable exponent Lebesgue and Morrey spaces. In Section 3 we prove compactness of commutators singular integral operators in variable exponent complementary Morrey spaces.

We use the following notation: $\mathbb{R}^n$ is the n -dimensional Euclidean space, $\Omega \subset \mathbb{R}^n$ is an open set, $\chi_E(x)$ is the characteristic function of a set $E \subseteq \mathbb{R}^n$, by c , C, c_1, c_2 etc, we denote various absolute positive constants, which may have different values even in the same line.

2 Preliminaries on variable exponent Lebesgue and Morrey spaces

In this section we refer to the book [9] for variable exponent Lebesgue spaces and give some basic definitions and facts. Let $p(\cdot)$ be a measurable function on Ω with values in $[1,\infty)$. An open set Ω is assumed to be unbounded throughout the whole paper. We mainly suppose that

$$1 < p_- \le p(x) \le p_+ < \infty, \tag{2.1}$$

where $p_- := \operatorname{ess}\inf\limits_{x\in\Omega} p(x)$, $\quad p_+ := \operatorname{ess}\sup\limits_{x\in\Omega} p(x)$. By $L^{p(\cdot)}(\Omega)$ we denote the space of all measurable functions $f(x)$ on Ω such that

$$I_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)} dx < \infty.$$

Equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\eta > 0 : \; I_{p(\cdot)}\left(\frac{f}{\eta}\right) \le 1\right\},$$

this is a Banach function space. By $p'(\cdot) = \frac{p(x)}{p(x)-1}$, $x \in \Omega$, we denote the conjugate exponent.

For the basics on variable exponent Lebesgue spaces we refer to [23], see also [2,4]. $\mathcal{P}(\Omega)$ is the set of bounded measurable functions $p : \Omega \to [1,\infty)$; $\mathcal{P}^{log}(\Omega)$ is the set of exponents $p \in \mathcal{P}(\Omega)$ satisfying the local log-condition

$$|p(x) - p(y)| \le \frac{A}{-\ln|x-y|}, \quad |x-y| \le \frac{1}{2} \;\; x, y \in \Omega, \tag{2.2}$$

where $A = A(p) > 0$ does not depend on x, y; $\mathcal{A}^{log}(\Omega)$ is the set of bounded exponents $p : \Omega \to \mathbb{R}$ satisfying the condition (2.2); $\mathbb{P}^{log}(\Omega)$ is the set of exponents $p \in \mathcal{P}^{log}(\Omega)$ with $1 < p_- \le p(x) \le p_+ < \infty$; for Ω which may be unbounded, by $\mathcal{P}_\infty(\Omega)$, $\mathcal{P}^{log}_\infty(\Omega)$, $\mathbb{P}^{log}_\infty(\Omega)$, $\mathcal{A}^{log}_\infty(\Omega)$ we denote the subsets of the above sets of exponents satisfying the decay condition (when Ω is unbounded)

$$|p(x) - p(\infty)| \le \frac{A_\infty}{\ln(2+|x|)}, \quad x \in \mathbb{R}^n, \tag{2.3}$$

where $p(\infty) = \lim\limits_{x\to\infty} p(x) > 1$.

Singular operators within the framework of the spaces with variable exponents were studied in [10].

Theorem 2.1 *([10]) Let $\Omega \subset \mathbb{R}^n$ be an unbounded open set and $p \in \mathbb{P}^{log}_\infty(\Omega)$. Then the singular integral operator T is bounded in $L^{p(\cdot)}(\Omega)$.*

We will also make use of the estimate provided by the following lemma (see [27], Corollary to Lemma 3.22).

Lemma 2.1 *([27]) Let Ω be a bounded domain and $p \in \mathcal{P}^{log}(\Omega)$ satisfy the assumption $1 \le p_- \le p(x) \le p_+ < \infty$. Let also $\sup_{x\in\Omega} \nu(x) < \infty$ and $\sup_{x\in\Omega}[n + \nu(x)p(x)] < 0$. Then*

$$\||x-\cdot|^{\nu(x)} \chi_{\Omega\setminus\widetilde{B}(x,r)}(\cdot)\|_{p(\cdot)} \le C r^{\nu(x)+\frac{n}{p(x)}}, \quad x \in \Omega, 0 < r < \ell = diam\,\Omega, \tag{2.4}$$

where C does not depend on x and r.

We will also make use of the estimate provided by the following lemma (see [9], Corollary 4.5.9).

$$\|\chi_{\widetilde{B}(x,r)}(\cdot)\|_{p(\cdot)} \le C r^{\theta_p(x,r)}, \quad x \in \Omega, \; p \in \mathbb{P}^{log}_\infty(\Omega), \tag{2.5}$$

where $\theta_p(x,r) = \begin{cases} \frac{n}{p(x)}, & r \le 1, \\ \frac{n}{p(\infty)}, & r \ge 1 \end{cases}$.

Lemma 2.2 *([18]) Let Ω be an unbounded open set, let $p \in \mathbb{P}^{log}_\infty(\Omega)$ satisfy the assumption $1 \le p_- \le p(x) \le p_+ < \infty$ and the function $\nu(x)$ satisfy the conditions $\sup_{x\in\Omega} \nu(x) < \infty$, $\inf_{x\in\Omega}[n + \nu(x)p(x)] > 0$ and additionally $\inf_{x\in\Omega}[n + \nu(x)p(\infty)] > 0$. Then*

$$\||x-\cdot|^{\nu(x)} \chi_{\widetilde{B}(x,r)}(\cdot)\|_{p(\cdot)} \le C r^{\nu(x)+\theta_p(x,r)}, \quad x \in \Omega, \; r > 0,$$

where C does not depend on x and r.

Using ideas of [18] we get the following lemma.

Lemma 2.3 *Let* Ω *be an unbounded open set, let* $p \in \mathbb{P}^{log}_{\infty}(\Omega)$ *satisfy the assumption* $1 \leq p_- \leq p(x) \leq p_+ < \infty$ *and the function* $\nu(x)$ *satisfy the assumptions of Lemma 2.1 and additionally* $\sup_{x\in\Omega}[n+\nu(x)p(\infty)] < 0$. *Then*

$$\||x-\cdot|^{\nu(x)}\chi_{\Omega\setminus\widetilde{B}(x,r)}(\cdot)\|_{p(\cdot)} \leq Cr^{\nu(x)+\theta_p(x,r)}, \quad x \in \Omega,\ r > 0, \tag{2.6}$$

where C *does not depend on* x *and* r .

Let $\lambda(x)$ be a measurable function on Ω with values in $[0,n]$. The variable Morrey space $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ is defined as the set of integrable functions f on Ω with the finite norm

$$\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} = \sup_{x\in\Omega,\, t>0} t^{-\frac{\lambda(x)}{p(x)}}\|f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}.$$

Let $M^\sharp$ be the sharp maximal function defined by

$$M^\sharp f(x) = \sup_{r>0}|B(x,r)|^{-1}\int_{\widetilde{B}(x,r)}|f(y)-f_{\widetilde{B}(x,r)}|dy,$$

where $f_{\widetilde{B}(x,t)} = |\widetilde{B}(x,t)|^{-1}\int_{\widetilde{B}(x,t)} f(z)dz$.

We define BMO space, as the set of locally integrable functions f with finite norm

$$\|f\|_{BMO} = \sup_{x\in\Omega} M^\sharp f(x) = \sup_{t>0,\ x\in\Omega}|B(x,t)|^{-1}\int_{\widetilde{B}(x,t)}|f(y)-f_{\widetilde{B}(x,t)}|dy.$$

Definition 2.1 *We define the* $BMO_{p(\cdot)}(\Omega)$ *space as the set of all locally integrable functions* f *with finite norm*

$$\|f\|_{BMO_{p(\cdot)}} = \sup_{x\in\Omega,\ r>0}\frac{\|(f(\cdot)-f_{\widetilde{B}(x,r)})\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}(\Omega)}}{\|\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}(\Omega)}}.$$

Theorem 2.2 *([20]) Let* $\Omega \subset \mathbb{R}^n$ *be an open unbounded set,* $p \in \mathbb{P}^{log}_{\infty}(\Omega)$ *, then the norms* $\|\cdot\|_{BMO_{p(\cdot)}}$ *and* $\|\cdot\|_{BMO}$ *are mutually equivalent.*

We find it convenient to introduce the variable exponent version of the local complementary space as follows.

Definition 2.2 *Let* $1 \leq p_- \leq p(x) \leq p_+ < \infty$ *,* $\lambda(x)$ *be a measurable function on* Ω *with values in* $[0,n]$ *. The complementary variable exponent Morrey space* ${}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ *is defined by the norm*

$$\|f\|_{{}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} = \sup_{x\in\Omega, t>0} t^{\frac{\lambda(x)}{p'(x)}}\|f\chi_{\Omega\setminus\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}.$$

3 Compactness of commutators of singular integral operators

In [5] the following theorem was proved.

Theorem 3.1 *Let* $p \in \mathbb{P}^{log}_{\infty}(\Omega)$ *satisfy assumption* (2.1) *and a measurable function* λ *satisfy the conditions*

$$0 \leq \lambda(x), \quad \sup_{x\in\Omega}\lambda(x) < n. \tag{3.1}$$

Then the singular integral operators T *is bounded on* ${}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$.

The commutator generated by T and a suitable function b is formally defined by

$$[b,T]f = bT(f) - T(bf).$$

Theorem 3.2 *([21]) Let* $p \in \mathbb{P}_{\infty}^{log}(\mathbb{R}^n)$ *and* $b \in BMO(\mathbb{R}^n)$ *, then the operator* $[b,T]$ *is bounded on* $L^{p(\cdot)}(\mathbb{R}^n)$.

The following theorem was proved in [5].

Theorem 3.3 *Let* $p \in \mathbb{P}_{\infty}^{log}(\Omega)$ *satisfy assumption* (2.1) *and a measurable function* λ *satisfy the condition* (3.1)*,* $b \in BMO(\Omega)$ *. Then the operator* $[b,T]$ *is bounded on* ${}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$.

In the proof of Theorem 3.5, we need the following characterization that a subset in ${}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ is a strongly pre-compact set, which is in itself interesting.

Theorem 3.4 *Let* $p \in \mathbb{P}_{\infty}^{log}(\Omega)$ *satisfy assumption* (2.1) *and a measurable function* λ *satisfy the condition* (3.1)*. Suppose* $\mathcal{W}$ *is a subset in* ${}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ *satisfying the following conditions:*
$i)$ *Norm boundedness uniformly is*

$$\sup_{f\in\mathcal{W}} \|f\|_{{}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} < \infty. \tag{3.2}$$

$ii)$ *Translation continuity uniformly is*

$$\lim_{y\longrightarrow 0} \|f(\cdot + y) - f(\cdot)\|_{{}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} = 0 \ \ for any\ f \in \mathcal{W}. \tag{3.3}$$

$iii)$ *Uniformly convergence at infinity is*

$$\lim_{\gamma\longrightarrow\infty} \left\| f\chi_{{}^{\complement}B(0,\gamma)} \right\|_{{}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} = 0 \ \ \text{for any } f \in \mathcal{W}. \tag{3.4}$$

Then $\mathcal{W}$ *is a strongly pre-compact set in* ${}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$.

Now we obtain sufficient conditions for the commutator $[b,T]$ to be a compact operator on ${}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$.

Theorem 3.5 *Let* $p \in \mathbb{P}_{\infty}^{log}(\Omega)$ *satisfy assumption* (2.1)*,* $b \in VMO(\Omega)$ *and a measurable function* λ *satisfy the condition* (3.1)*. Then the operator* $[b,T]$ *is a compact operator on* ${}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$.

Proof. We will use the method in [8]. Let $\mathcal{F}$ be the unit ball in ${}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$. By density, we only need to prove that when $b \in C_c^{\infty}(\mathbb{R}^n)$, the set $\mathcal{G} = \{[b,T]\,f : f \in \mathcal{F}\}$ is a precompact in ${}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$. By Theorem 3.4, it is sufficient to show that (3.2)-(3.4) hold uniformly in $\mathcal{G}$. Notice that $b \in C_c^{\infty}(\mathbb{R}^n)$. Applying Theorem 3.3, we have

$$\sup_{f\in\mathcal{F}} \|[b,T]\,f\|_{{}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)}$$

$$\leq C\,\|b\|_{BMO} \sup_{f\in\mathcal{F}} \|f\|_{{}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} \leq C\,\|b\|_{BMO} < \infty. \tag{3.5}$$

This shows that (3.2) holds. Next we show that (3.4) holds. To do so, we suppose that $\beta > 1$ taken so large that $supp\, b \subset \{y : |y| \leq \beta\}$. For any $0 < \varepsilon < 1$, we take $\gamma > \beta$. Below we show that for every $x \in \mathbb{R}^n$ and $r > 0$

$$\lim_{\gamma\to\infty} \left\| ([b,T]\,f)\,\chi_{{}^{\complement}B(0,\gamma)} \right\|_{{}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} = 0. \tag{3.6}$$

In fact, for any $z \in {}^{\complement}B(0,\gamma) = \{z \in \mathbb{R}^n : |z| > \gamma\}$ and every $f \in \mathcal{F}$, by Hölder inequality and Lemma 2.3, we have

$$t^{\frac{\lambda(x)}{p'(x)}} \left\| \left([b,T]f\right)\chi_{{}^{\complement}B(0,\gamma)} \right\|_{L^{p(\cdot)}({}^{\complement}\widetilde{B}(x,t))} \leq C \left\| \chi_{{}^{\complement}B(0,\gamma)} \int_{\mathbb{R}^n} \frac{|b(\cdot)-b(y)|\,|f(y)|}{|\cdot - y|^n} dy \right\|_{L^{p(\cdot)}({}^{\complement}\widetilde{B}(x,t))}$$

$$\leq C\|b\|_\infty \left\| \chi_{{}^{\complement}B(0,\gamma)} \int_{|y|\leq\beta} \frac{|f(y)|}{|\cdot - y|^n} dy \right\|_{L^{p(\cdot)}({}^{\complement}\widetilde{B}(x,t))}$$

$$\leq C\|b\|_\infty \|f\|_{{}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} \left\| \frac{1}{|\cdot|^n} \right\|_{L^{p(\cdot)}({}^{\complement}B(0,\gamma-\beta)}$$

$$\leq C\|b\|_\infty (\gamma-\beta)^{-\theta_{p'}(x,\gamma-\beta)} \|f\|_{L^{p(\cdot)}(B(0,\beta))} \leq C\varepsilon. \tag{3.7}$$

Thus, we get (3.6), which shows that (3.4) holds for $[b,T]$ in $\mathcal{G}$ uniformly.

Finally, we show that the translation continuity condition (3.3) holds for the commutator $[b,T]$ in $\mathcal{G}$ uniformly. We need to prove that, for any $0 < \varepsilon < 1/2$, if $|z|$ is sufficiently small depending only on ε, then for every $f \in \mathcal{F}$

$$\|[b,T]f(.) - [b,T]f(.+z)\|_{{}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} \leq C\varepsilon.$$

Now for $z \in \mathbb{R}^n$, we write

$$[b,T]f(x+z) - [b,T]f(x) = \int_{{}^{\complement}B\left(x,\frac{|z|}{\varepsilon}\right)} [b(x+z)-b(x)]\,K(x,y)f(y)dy$$

$$+ \int_{{}^{\complement}B\left(x,\frac{|z|}{\varepsilon}\right)} (K(x,y)-K(x+z,y))\,[b(y)-b(x+z)]\,f(y)\,dy$$

$$+ \int_{B\left(x,\frac{|z|}{\varepsilon}\right)} [b(y)-b(x)]\,K(x,y)f(y)dy$$

$$- \int_{B\left(x,\frac{|z|}{\varepsilon}\right)} [b(y)-b(x+z)]\,K(x,y)f(y)dy = J_1 + J_2 + J_3 - J_4.$$

Since $b \in C_c^\infty(\mathbb{R}^n)$, we have $|b(x)-b(x+z)| \leq C\|\nabla b\|_\infty |z|$. Then

$$\|J_1\|_{{}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} \leq C|z|\,\|Tf\|_{{}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)}$$

$$\leq C|z|\,\|f\|_{{}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} < |z|. \tag{3.8}$$

As for J_2, for every $t \in \mathbb{R}^n$ and $r > 0$, we get

$$|J_2| \leq 2\|b\|_\infty \int_{{}^{\complement}B\left(x,\frac{|z|}{\varepsilon}\right)} |K(x,y)-K(x+z,y)|\,|f(y)|dy \leq C|z|T|f|(x).$$

Using Theorem 3.1, we have

$$\|J_2\|_{{}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} \leq C\varepsilon\,\|Tf\|_{{}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} \leq C\varepsilon\,\|f\|_{{}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)}.$$

Thus, we have

$$\|J_2\|_{{}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} \leq C\varepsilon. \tag{3.9}$$

Regarding J_3, we have $|b(x)-b(y)| \leq C\|\nabla b\|_\infty |x-y|$ by $b \in C_c^\infty(\mathbb{R}^n)$. Thus,

$$|J_3| \leq C\frac{|z|}{\varepsilon} \int_{|x-y|\leq C\frac{|z|}{\varepsilon}} |K(x,y)||f(y)|dy \leq C\frac{|z|}{\varepsilon} T|f|(x).$$

By the Theorem 3.1, we have

$$\|J_3\|_{{}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} \le C\frac{|z|}{\varepsilon}\, \|T|f|\|_{{}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} \le C\frac{|z|}{\varepsilon}\, \|f\|_{{}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)}.$$

Thus,

$$\|J_3\|_{{}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} \le C\frac{|z|}{\varepsilon}. \tag{3.10}$$

Finally, by $|b(x+z)-b(y)| \le C\,\|\nabla b\|_\infty\,|x+z-y|$, we have

$$|J_4| \le C\left(\frac{|z|}{\varepsilon}+|z|\right)\int_{|x-y|\le\frac{|z|}{\varepsilon}} |K(x,y)||f(y)|dy \le C\left(\frac{|z|}{\varepsilon}+|z|\right) T|f|(x). \tag{3.11}$$

Using the same argument for J_4 and by the Theorem 3.1, it is easy to check that

$$\|J_4\|_{{}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} \le C\left(\frac{|z|}{\varepsilon}+|z|\right). \tag{3.12}$$

From (3.8), (3.9), (3.10) and (3.12) and taking $|z|$ to be sufficiently small, we can get

$$\begin{aligned} &\|[b,T]\,f\,(.) - [b,T]\,f\,(.+z)\|_{{}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} \\ &\le \|J_1\|_{{}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} + \|J_2\|_{{}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} \\ &+ \|J_3\|_{{}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} + \|J_4\|_{{}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} \le C\varepsilon. \end{aligned}$$

Therefore, we show that the translation continuity (3.3) holds for the commutator $[b,T]$ in $\mathcal{G}$ uniformly and this completes the proof of Theorem 3.5.

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