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Boundedness of Dunkl-type fractional integral and fractional maximal operators in Dunkl-type Besov-Fofana spaces

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Abstract. *In the framework of rational Dunkl analysis on the real line, we introduce Dunkl-type Besov-Fofana spaces which include Besov-Dunkl spaces and Dunkl-type Besov-Morrey spaces. As applications, we establish the boundedness of Dunkl-type fractional integral and fractional maximal operators in these spaces.*

Keywords. Besov-Dunkl spaces, Dunkl-type Besov-Fofana spaces, Dunkl-type Besov-Morrey spaces, Dunkl-type fractional integral operator, Dunkl-type fractional maximal operator.

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1 Introduction and main results

Let $\mathbb{R}^d$ be the d-dimensional real Euclidean space and dx be the usual Lebesgue measure on $\mathbb{R}^d$. The classical Besov spaces were introduced in [4] and [5] by Besov. The underlying norm used to define these spaces was that of $L^p(dx)$ spaces. We refer to [30] and the references therein for a detailed exposition on Besov spaces. Later, the study of Besov spaces associated with some generalization of Lebesgue spaces attracted the attention of many authors. For instance, when the Lebesgue spaces are replaced by the Morrey spaces, then we obtain the Besov-Morrey spaces first defined by Kozono and Yamazaki in [19]. Properties of Besov-Morrey spaces can be found in [19], [23], [31] and [33]. Besov-Morrey spaces found important applications in the theory of non-linear partial differential equations (see for example [8,19,22]). In [17], Ho identified the condition imposed on a semi-Köthe function space so that the corresponding Besov type space is well definied. He showed that this criterion can be expressed in term of the boundedness of the Hardy-Littlewood maximal operator. As a consequence, Ho obtained a wide class of Besov type spaces called Besov-Köthe spaces. Thanks to Proposition 4.2 in [11], it is easy to see that Ho's condition is fulfilled by Fofana spaces, denoted by $(L^q,\, L^p)^\alpha(\mathbb{R}^d,\, dx)$ $(1 < q \leq \alpha \leq p \leq \infty)$,

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introduced in [12] in connection with the study of the boundedness of the fractional maximal operator of Hardy-Littlewood and of the Fourier transformation. It is proved that for $1 \leq q < \alpha$ fixed and p going from α to ∞, the spaces $(L^q,\, L^p)^\alpha(\mathbb{R}^d,\, dx)$ form a chain of distinct Banach spaces beginning with the Lebesgue space $L^\alpha(dx)$ and ending by the classical Morrey space $\mathcal{L}^{q,\,\lambda} = (L^q,\, L^\infty)^\alpha(\mathbb{R}^d,\, dx)$, where $\lambda = d(1 - \frac{q}{\alpha})$.

In recent years, generalizations of Besov spaces, Besov-Morrey spaces and Fofana spaces in the framework of rational Dunkl analysis on the real line were carried out in many papers (see for instance [1,3,6,13,15,18,20,27,28]). Recall that on the real line, the Dunkl operators are differential-difference operators associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. The harmonic analysis of the one-dimensional Dunkl operator and Dunkl transform was investigated in [2,9,13,15,32,34]. The fractional integral, the fractional maximal operator and their commutators associated with the Dunkl operator on $\mathbb{R}$ and related topics attracted great interest and have become fruitful research areas, see for instance [21,24–26] and the references therein. In [15], Guliyev and Mammadov proved the boundedness of Dunkl-type fractional integral and fractional maximal operators in Besov-Dunkl spaces. In [13], Guliyev et al. established the boundedness of Dunkl-type fractional maximal operators in Dunkl-type Besov-Morrey spaces.

In the present paper, we introduce Dunkl-type Besov-Fofana spaces on the real line. As applications, we establish the boundedness of Dunkl-type fractional integral and fractional maximal operators in these spaces. In order to state our mains results, we give some notations that will be used throughout this note. Let $k > -\frac{1}{2}$ be a fixed number and μ be the weighted Lebesgue measure on $\mathbb{R}$, given by

$$d\mu(x) = \left(2^{k+1}\Gamma(k+1)\right)^{-1} |x|^{2k+1}\, dx.$$

We denote by $L^0(\mu)$ the complex vector space of equivalence classes (modulo equality μ-almost everywhere) of complex-valued functions μ-measurable on $\mathbb{R}$. The class of locally integrable functions with respect to μ is denoted by $L^1_{loc}(\mu)$. For $1 \leq p \leq \infty$, $L^p(\mu)$ stands for the Lebesgue space associated with the measure μ. We write $\|f\|_p$ for the classical norm of $f \in L^p(\mu)$. For any subset A of $\mathbb{R}$, χ_A denotes the characteristic function of A. For $x \in \mathbb{R}$ and for $r > 0$, we set

$$B(x,\, r) = \{y \in \mathbb{R} \,:\, \max\{0,\, |x| - r\} < |y| < |x| + r\},$$

if $x \neq 0$ and

$$B_r = B(0,\, r) = (-r,\, r).$$

The letter C will be used as a generic positive constant not depending on the relevant variables. Its value may change from one occurrence to another. Let $1 \leq q,\, p \leq \infty$ and $r > 0$. For $f \in L^0(\mu)$ we define

$${}_r\|f\|_{q,p} = \begin{cases} \left\| \left[\int_{\mathbb{R}} (\tau_y\, |f|^q) \chi_{B_r}(x) d\mu(x)\right]^{\frac{1}{q}} \right\|_p & \text{if } q < \infty, \\ \left\| \left\| f \chi_{B(y,r)} \right\|_\infty \right\|_p & \text{if } q = \infty, \end{cases}$$

with the $L^p(\mu)$-norm taken with respect to the variable y. Here and in the sequel, $\tau_y (y \in \mathbb{R})$ stands for the Dunkl translation operator (see Section 2 for more details).
Let $1 \leq q \leq \alpha \leq p \leq \infty$. The Dunkl-Fofana space $(L^q, L^p)^\alpha(\mu)$ is defined as the subspace of $L^0(\mu)$ such that $\|f\|_{q,p,\alpha} < \infty$, where

$$\|f\|_{q,p,\alpha} = \sup_{r>0} (\mu(B_r))^{\frac{1}{\alpha} - \frac{1}{q} - \frac{1}{p}}\, {}_r\|f\|_{q,p}\,.$$

with the usual convention $\frac{1}{\infty}=0$. It is proved in [27] that the map $f \mapsto \|f\|_{q,p,\alpha}$ is a norm on $(L^q, L^p)^\alpha(\mu)$ and $((L^q, L^p)^\alpha(\mu), \|\cdot\|_{q,p,\alpha})$ is a complex Banach space (see [28]).
For $1 \leq \theta \leq \infty$ and $0 < s < 1$, we introduce the Dunkl-type Besov-Fofana space $B^s_\theta(L^q, L^p)^\alpha(\mu)$ defined as

$$B^s_\theta(L^q, L^p)^\alpha(\mu) = \left\{ f \in (L^q, L^p)^\alpha(\mu) : \ \|f\|_{B^s_\theta(L^q,L^p)^\alpha(\mu)} < \infty \right\},$$

where

$$\|f\|_{B^s_\theta(L^q,L^p)^\alpha(\mu)} = \|f\|_{q,p,\alpha} + \left\| \frac{\|\tau_x f(\cdot) - f(\cdot)\|_{q,p,\alpha}}{|x|^{\frac{2k+2}{\theta}+s}} \right\|_\theta .$$

Let $0 \leq \beta < 2k+2$ and $f \in L^0(\mu)$. The Dunkl-type fractional integral operator I_β $(\beta \neq 0)$ and the Dunkl-type fractional maximal operator M_β are defined as

$$I_\beta f(x) = \int_{\mathbb{R}} \tau_x f(z)|z|^{\beta-(2k+2)} d\mu(z)$$

and

$$M_\beta f(x) = \sup_{r>0} (\mu(B_r))^{\frac{\beta}{2k+2}-1} \int_{B_r} \tau_x |f|(y) d\mu(y).$$

For $\beta = 0$, the fractional maximal operator reduces to the Hardy-Littlewood maximal operator associated with the Dunkl operator, denoted by M. We shall prove the following result.

Theorem 1.1 *Let* $1 < q \leq \alpha \leq p < \infty$, $1 \leq \theta \leq \infty$, $0 < s < 1$, $0 < \beta < \frac{2k+2}{\alpha}$ *and* $f \in B^s_\theta(L^q, L^p)^\alpha(\mu)$.
Put

$$\frac{1}{\alpha^*} = \frac{1}{\alpha} - \frac{\beta}{2k+2}, \quad \frac{1}{\bar{p}} = \frac{1}{p}\big(1 - \frac{\alpha\beta}{2k+2}\big) \quad and \quad \frac{1}{\bar{q}} = \frac{1}{q}\big(1 - \frac{\alpha\beta}{2k+2}\big).$$

Then

$$\|I_\beta f\|_{B^s_\theta(L^{\bar{q}},L^{\bar{p}})^{\alpha^*}(\mu)} \leq C\|f\|_{B^s_\theta(L^q,L^p)^\alpha(\mu)}.$$

As a consequence of Theorem 1.1, we obtain the below result.

Corollary 1.1 *Let* $1 < q \leq \alpha \leq p < \infty$, $1 \leq \theta \leq \infty$, $0 < s < 1$, $0 < \beta < \frac{2k+2}{\alpha}$ *and* $f \in B^s_\theta(L^q, L^p)^\alpha(\mu)$.
Put

$$\frac{1}{\alpha^*} = \frac{1}{\alpha} - \frac{\beta}{2k+2}, \quad \frac{1}{\bar{p}} = \frac{1}{p}(1 - \frac{\alpha\beta}{2k+2}) \quad and \quad \frac{1}{\bar{q}} = \frac{1}{q}(1 - \frac{\alpha\beta}{2k+2}).$$

Then

$$\|M_\beta f\|_{B^s_\theta(L^{\bar{q}},L^{\bar{p}})^{\alpha^*}(\mu)} \leq C\|f\|_{B^s_\theta(L^q,L^p)^\alpha(\mu)}.$$

We shall also establish the following boundedness result for the Hardy-Littlewood maximal operator associated with the Dunkl operator.

Theorem 1.2 *Let* $1 < q \leq \alpha \leq p \leq \infty$, $1 \leq \theta \leq \infty$, $0 < s < 1$ *and* $f \in B^s_\theta(L^q, L^p)^\alpha(\mu)$.
Then

$$\|Mf\|_{B^s_\theta(L^q,L^p)^\alpha(\mu)} \leq C\|f\|_{B^s_\theta(L^q,L^p)^\alpha(\mu)}.$$

The remark below follows from the fact that for certain particular choices of exponents, Dunkl-type Besov-Fofana spaces reduce to Besov-Dunkl spaces or Dunkl-type Besov-Morrey spaces (see Section 3).

Remark 1.1 Note that

(1) By taking $\alpha = q$ or $\alpha = p$ in Theorem 1.1 and Corollary 1.1 we obtain Theorem 3 and Corollary 4 both established in [15].
(2) Theorem 3.2 established in [13] can be viewed as the analogue of Corollary 1.1 in the limiting case $p = \infty$ with $1 < q < \alpha < \infty$.
(3) Theorem 1.2 in the case $\alpha = q$ or $\alpha = p$ was proved in [20].
(4) Theorem 1.2 in the case $1 < q < \alpha < p = \infty$ was proved in [13].

The paper is organised as follows. In Section 2, for the sake of completeness, we give a brief review on rational Dunkl analysis on the real line. In Section 3 we recall some norm inequalities in the spaces $(L^q, L^p)^\alpha(\mu)$ and we point out some properties of the Dunkl-type Besov-Fofana spaces. Section 4 contains the proofs of Theorem 1.1, Corollary 1.1 and Theorem 1.2.

2 Some basic facts about rational Dunkl analysis on the real line

The Dunkl operator associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$ is defined by

$$\Lambda_k f(x) = \frac{df}{dx}(x) + \frac{2k+1}{x}\left(\frac{f(x) - f(-x)}{2}\right).$$

For $\lambda \in \mathbb{C}$, the Dunkl kernel denoted by $\mathfrak{E}_k(\lambda)$ (see [9]), is the only solution of the initial value problem

$$\Lambda_k f(x) = \lambda f(x),\ f(0) = 1,\ x \in \mathbb{R}.$$

It is given by the formula

$$\mathfrak{E}_k(\lambda x) = j_k(i\lambda x) + \frac{\lambda x}{2(k+1)} j_{k+1}(i\lambda x),\ x \in \mathbb{R},$$

where

$$j_k(z) = 2^k \Gamma(k+1)\frac{J_k(z)}{z^k} = \Gamma(k+1)\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{n! 2^{2n} \Gamma(n+k+1)}, \quad z \in \mathbb{C}$$

is the normalized Bessel function of the first kind and of order k. Notice that $\Lambda_{-\frac{1}{2}} = \frac{d}{dx}$ and $\mathfrak{E}_{-\frac{1}{2}}(\lambda x) = e^{\lambda x}$. It is also proved (see [29]) that $|\mathfrak{E}_k(ix)| \leq 1$ for every $x \in \mathbb{R}$.

The Dunkl kernel $\mathfrak{E}_k$ gives rise to an integral transform on $\mathbb{R}$ denoted $\mathcal{F}_k$ and called Dunkl transform (see [7]). For $f \in L^1(\mu)$,

$$\mathcal{F}_k f(\lambda) = \int_{\mathbb{R}} \mathfrak{E}_k(-i\lambda x) f(x) d\mu(x),\ \ \lambda \in \mathbb{R}.$$

We have the following properties of the Dunkl transform.

Proposition 2.1 *(See [7] or [10].)*

(1) Let $f \in L^1(\mu)$. If $\mathcal{F}_k(f)$ is in $L^1(\mu)$, then we have the following inversion formula:

$$f(x) = C\int_{\mathbb{R}} \mathfrak{E}_k(ixy)\mathcal{F}_k(f)(y) d\mu(y).$$

(2) The Dunkl transform has a unique extension to an isometric isomorphism on $L^2(\mu)$.

Let $x, y, z \in \mathbb{R}$. We put

$$W_k(x,y,z) = [1 - \sigma_{x,y,z} + \sigma_{z,x,y} + \sigma_{z,y,x}]\Delta_k(|x|,|y|,|z|)$$

where,

$$\sigma_{x,y,z} = \begin{cases} \frac{x^2+y^2-z^2}{2xy} & if \;\; x, y \in \mathbb{R} \setminus \{0\}, \\ 0 \;\; otherwise \end{cases}$$

and Δ_k is the Bessel kernel given by

$$\Delta_k(|x|,|y|,|z|) = \begin{cases} b_k \frac{[((|x|+|y|)^2-z^2)(z^2-(|x|-|y|)^2)]^{k-\frac{1}{2}}}{|xyz|^{2k}} & if \; |z| \in \mathcal{A}_{x,y}, \\ 0 \;\; otherwise, \end{cases}$$

with $b_k = 2^{1-k}\frac{(\Gamma(k+1))^2}{\sqrt{\pi}\Gamma(k+\frac{1}{2})}$ and $\mathcal{A}_{x,y} = [||x|-|y||, |x|+|y|]$.

Remark 2.1 (See [29].)

$$W_k(x,y,z) = W_k(y,x,z) = W_k(-x,z,y)$$
$$W_k(x,y,z) = W_k(-z,y,-x) = W_k(-x,-y,-z)$$

and

$$\int_{\mathbb{R}} |W_k(x,y,z)| d\mu_k(z) \leq 4.$$

In the sequel, we consider the signed measure $\nu_{x,y}$ on $\mathbb{R}$ given by

$$d\nu_{x,y}(z) = \begin{cases} W_k(x,y,z)d\mu_k(z) & if \;\; x, y \in \mathbb{R} \setminus \{0\}, \\ d\delta_x(z) & if \;\; y = 0, \\ d\delta_y(z) & if \;\; x = 0, \end{cases}$$

with $supp(\nu_{x,y}) = (-\mathcal{A}_{x,y}) \cup \mathcal{A}_{x,y}$ for all $(x,y) \in \mathbb{R} \setminus \{0\} \times \mathbb{R} \setminus \{0\}$.

Definition 2.1 *For $x, y \in \mathbb{R}$ and f a continuous function on $\mathbb{R}$, we put*

$$\tau_x f(y) = \int_{\mathbb{R}} f(z) d\nu_{x,y}(z).$$

The operators τ_x, $x \in \mathbb{R}$, are called Dunkl translation operators on $\mathbb{R}$.

For $x \in \mathbb{R}$ and $r > 0$, the map $y \mapsto \tau_x \chi_{B_r}(y)$ is supported in $B(x,r)$ and

$$0 \leq \tau_x \chi_{B_r}(y) \leq \min\left\{1, \frac{2C_\kappa}{2\kappa+1}\left(\frac{r}{|x|}\right)^{2\kappa+1}\right\}, \; y \in B(x,r),$$

as proved in [16].
Let f and g be two continuous functions on $\mathbb{R}$ with compact support. We define the generalized convolution $*_k$ of f and g by

$$f *_k g(x) = \int_{\mathbb{R}} \tau_x f(-y) g(y) d\mu(y).$$

The generalized convolution $*_k$ is associative and commutative (see [29]). We also have the below result.

Proposition 2.2 *(See Soltani [32].)*

(1) For all $x \in \mathbb{R}$, the operator τ_x extends to $L^p(\mu)$, $p \geq 1$, and

$$\|\tau_x f\|_p \leq 4\,\|f\|_p$$

for all $f \in L^p(\mu)$.

(2) Assume that $p, q, r \in [1, \infty]$ and satisfy $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Then the generalized convolution defined on $\mathcal{C}_c \times \mathcal{C}_c$, extends to a continuous map from $L^p(\mu) \times L^q(\mu)$ to $L^r(\mu)$, and we have

$$\|f *_k g\|_r \leq 4\,\|f\|_p\,\|g\|_q\,.$$

It is also proved in [14] that if $f \in L^1(\mu)$ and $g \in L^p(\mu)$, $1 \leq p < \infty$, then

$$\tau_x(f *_k g) = \tau_x f *_k g = f *_k \tau_x g, \quad x \in \mathbb{R}.$$

For any $f \in L^1_{loc}(\mu)$, the following analogue of the Lebesgue differentiation theorem holds (see [32]):

$$\lim_{r\to 0} \frac{1}{\mu(B_r)} \int_{B_r} |\tau_x f(y) - f(x)| d\mu(y) = 0 \quad \text{for a.e.} \quad x \in \mathbb{R}$$

and

$$\lim_{r\to 0} \frac{1}{\mu(B_r)} \int_{B_r} \tau_x f(y) d\mu(y) = f(x) \quad \text{for a.e.} \quad x \in \mathbb{R}.$$

We end this section by the following well-known equality:

$$\mu(B_r) = d_k r^{2k+2}, \quad r > 0,$$

where $d_k = \left(2^{k+1}(k+1)\Gamma(k+1)\right)^{-1}$.

3 Dunkl-Fofana spaces and Dunkl-type Besov-Fofana spaces

We recall below some properties of the Dunkl-Fofana spaces.

Proposition 3.1 *(See [27].) Let $1 \leq q \leq \alpha \leq p \leq \infty$.*

(1) We have

$$\|f\|_{q,p,\alpha} \leq 4^{\frac{1}{q}} \|f\|_\alpha, \quad f \in L^\alpha(\mu)$$

and consequently $L^\alpha(\mu) \subset (L^{q_1}, L^p)^\alpha(\mu)$.

(2) If $q \leq q_1 \leq \alpha \leq p$ then

$$\|f\|_{q,p,\alpha} \leq \|f\|_{q_1,p,\alpha}, \quad f \in (L^{q_1}, L^p)^\alpha$$

and consequently $(L^{q_1}, L^p)^\alpha(\mu) \subset (L^q, L^p)^\alpha(\mu)$.

(3) If $q \leq \alpha \leq p_1 \leq p$ then there exists a constant $C > 0$ such that

$$\|f\|_{q,p,\alpha} \leq C\|f\|_{q,p_1,\alpha}, \quad f \in (L^q, L^{p_1})^\alpha(\mu)$$

and consequently $(L^q, L^{p_1})^\alpha(\mu) \subset (L^q, L^p)^\alpha(\mu)$.

(4) If $\alpha \in \{p,\, q\}$ then $(L^q, L^p)^\alpha(\mu) = L^\alpha(\mu)$.

The following results which deal with norm inequalities for the Hardy-Littlewood maximal operator associated with the Dunkl operator and Dunkl-type fractional integral operators will be useful in the proofs of our mains results.

Proposition 3.2 *([27]) Let $1 < q \leq \alpha \leq p \leq \infty$. There exists a constant $C > 0$ such that*

$$\|Mf\|_{q,p,\alpha} \leq C \|f\|_{q,p,\alpha}, \quad f \in L^1_{loc}(\mu).$$

Proposition 3.3 *([28]) Let $1 < q \leq \alpha \leq p < \infty$ and $0 < \beta < \frac{2k+2}{\alpha}$. Put*

$$\frac{1}{\alpha^*} = \frac{1}{\alpha} - \frac{\beta}{2k+2}, \frac{1}{\bar{p}} = \frac{1}{p}\big(1 - \frac{\alpha\beta}{2k+2}\big) \text{ and } \frac{1}{\bar{q}} = \frac{1}{q}\big(1 - \frac{\alpha\beta}{2k+2}\big).$$

Then

$$\|I_\beta f\|_{\bar{q},\bar{p},\alpha^*} \leq C\|f\|_{q,p,\alpha}^{1-\frac{\alpha\beta}{2k+2}} \|f\|_{q,\infty,\alpha}^{\frac{\alpha\beta}{2k+2}}, \qquad f \in (L^q, L^p)^\alpha(\mu)$$

and

$$\|I_\beta f\|_{\bar{q},\bar{p},\alpha^*} \leq C\|f\|_{q,p,\alpha}, \qquad f \in (L^q, L^p)^\alpha(\mu).$$

Let $1 \leq q \leq \alpha \leq p \leq \infty$, $1 \leq \theta \leq \infty$ and $0 < s < 1$. Notice that $B^s_\theta(L^q, L^\infty)^\alpha(\mu)$ is the Dunkl-type Besov-Morrey space $B^s_{q\theta,\lambda,k}$, with $\lambda = (2k+2)(1 - \frac{q}{\alpha})$, defined in [13]. Dunkl-type Besov-Fofana spaces are also closely related to Besov-Dunkl spaces denoted by $B^s_{p\theta}(\mathbb{R})$, which consist of all functions f in $L^p(\mu)$ satisfying

$$\|f\|_{B^s_{p\theta}(\mathbb{R})} = \|f\|_p + \left\| \frac{\|\tau_x f(\cdot) - f(\cdot)\|^\theta_{L^p(\mu)}}{|x|^{\frac{2k+2}{\theta}+s}} \right\|_\theta < \infty.$$

More precisely, we have the following result.

Proposition 3.4 *Let $1 \leq q \leq \alpha \leq p \leq \infty$, $1 \leq \theta \leq \infty$ and $0 < s < 1$. Then the following assertions hold.*

(1) The space $B^s_\theta(L^q, L^p)^\alpha(\mu)$ is a complex vector subspace of $(L^q, L^p)^\alpha(\mu)$.
(2) the map $f \mapsto \|f\|_{B^s_\theta(L^q,L^p)^\alpha(\mu)}$ defines a norm on $B^s_\theta(L^q, L^p)^\alpha(\mu)$.
(3) For all $f \in B^s_{\alpha\theta}(\mathbb{R})$, $\|f\|_{B^s_\theta(L^q,L^p)^\alpha(\mu)} \leq 4^{\frac{1}{q}} \|f\|_{B^s_{\alpha\theta}(\mathbb{R})}$.
(4) If $\alpha \in \{q, p\}$ then $B^s_\theta (L^q, L^p)^\alpha (\mu) = B^s_{\alpha\theta}(\mathbb{R})$.
(5) The family of spaces $B^s_\theta (L^q, L^p)^\alpha (\mu)$ is increasing with respect to the exponent p and decreasing with respect to the exponent q.

Proof. It is an immediate consequence of Proposition 3.1 and the definitions of the spaces $B^s_\theta(L^q, L^p)^\alpha(\mu)$ and $B^s_{\alpha\theta}(\mathbb{R})$.

4 Proofs of the main results

Proof of Theorem 1.1. Let $1 < q \leq \alpha \leq p \leq \infty$, $1 \leq \theta \leq \infty$, $0 < s < 1$, $0 < \beta < \frac{2k+2}{\alpha}$ and $f \in B^s_\theta(L^q, L^p)^\alpha(\mu)$. Let $x \in \mathbb{R}$ and $y \in \mathbb{R}$. From the definition of the Dunkl-type Besov-Fofana spaces it suffices to show that there exists a constant $C > 0$ such that

$$\|\tau_x(I_\beta f) - I_\beta f\|_{\bar{q},\bar{p},\alpha^*} \leq C\|\tau_x f - f\|_{q,p,\alpha}.$$

Since it is known that Dunkl translation commutes with Dunkl-type fractional integral operator, we have

$$\begin{aligned}
&|\tau_x(I_\beta f)(y) - I_\beta f(y)| \\
&= |I_\beta(\tau_x f)(y) - I_\beta f(y)| \\
&= \left| \int_{\mathbb{R}} \tau_x f(z) \tau_{-y}|z|^{\beta-2k-2} d\mu(z) - \int_{\mathbb{R}} f(z) \tau_{-y}|z|^{\beta-2k-2} d\mu(z) \right| \\
&\le \int_{\mathbb{R}} |\tau_x f(z) - f(z)| \, \tau_{-y}|z|^{\beta-2k-2} d\mu(z) \\
&\le \int_{\mathbb{R}} \tau_y \, |\tau_x f - f| \, (z)|z|^{\beta-2k-2} d\mu(z) = I_\beta(|\tau_x f - f|)(y).
\end{aligned}$$

Hence,

$$|\tau_x(I_\beta f)(y) - I_\beta f(y)| \le I_\beta(|\tau_x f - f|)(y). \tag{4.1}$$

Taking the norm $\|\cdot\|_{\bar{q},\bar{p},\alpha^*}$ of both sides of (4.1) and applying Proposition 3.3 we get

$$\|\tau_x(I_\beta f) - I_\beta f\|_{\bar{q},\bar{p},\alpha^*} \le \|I_\beta(|\tau_x f - f|)\|_{\bar{q},\bar{p},\alpha^*} \le C\|\tau_x f - f\|_{q,p,\alpha}.$$

Thus the desired result follows.

Proof of Corollary 1.1. Let $1 < q \le \alpha \le p \le \infty$, $1 \le \theta \le \infty$, $0 < s < 1$, $0 < \beta < \frac{2k+2}{\alpha}$ and $f \in B_\theta^s(L^q, L^p)^\alpha(\mu)$. Let $x \in \mathbb{R}$. Since

$$M_\beta f(x) \le d_k^{\frac{\beta}{2k+2}-1} I_\beta|f|(x) \tag{4.2}$$

(see [28]), we end the proof by taking the norm $\|\cdot\|_{B_\theta^s(L^{\bar{q}},L^{\bar{p}})^{\alpha^*}(\mu)}$ of both sides of (4.2) and by applying Theorem 1.1.

Proof of Theorem 1.1. Let $1 < q \le \alpha \le p \le \infty$, $1 \le \theta \le \infty$, $0 < s < 1$ and $f \in B_\theta^s(L^q, L^p)^\alpha(\mu)$. Let $x, y \in \mathbb{R}$ and $r > 0$. From the definition of the Dunkl-type Besov-Fofana spaces it suffices to show that there exists a constant $C > 0$ such that

$$\|\tau_x(Mf) - Mf\|_{q,p,\alpha} \le C\|\tau_x f - f\|_{q,p,\alpha}.$$

It is known that Dunkl translation commutes with Hardy-Littlewood maximal operator associated with the Dunkl operator, that is,

$$|\tau_x(Mf)(y) - Mf(y)| = |M(\tau_x f)(y) - Mf(y)|.$$

Moreover,

$$\begin{aligned}
&\left| \int_{B_r} \tau_y(|\tau_x f(z)|) d\mu(z) - \int_{B_r} \tau_y \, |f(z)| \, d\mu(z) \right| \\
&\le \int_{\mathbb{R}} ||\tau_x f(z)| - |f(z)|| \, \tau_{-y}\chi_{B_r}(z) d\mu(z) \\
&\le \int_{\mathbb{R}} |\tau_x f(z) - f(z)| \, \tau_{-y}\chi_{B_r}(z) d\mu(z) \\
&\le \int_{B_r} \tau_y \, |\tau_x f - f| \, (z) d\mu(y) \le M(|\tau_x f - f|)(y).
\end{aligned}$$

Put

$$A = \int_{B_r} \tau_y(|\tau_x f(z)|)d\mu(z) \text{ and } B = \int_{B_r} \tau_y \,|f(z)|\, d\mu(z).$$

On the one hand, we have

$$A - B \leq M(|\tau_x f - f|)(y) \implies A \leq M(|\tau_x f - f|)(y) + B.$$

Therefore,

$$M(\tau_x f)(y) \leq M(|\tau_x f - f|)(y) + M(f)(y).$$

Hence,

$$M(\tau_x f)(y) - M(f)(y) \leq M(|\tau_x f - f|)(y). \tag{4.3}$$

On the other hand,

$$-M(|\tau_x f - f|)(y) \leq A - B \implies B \leq A + M(|\tau_x f - f|)(y).$$

Therefore,

$$M(f)(y) \leq M(\tau_x f)(y) + M(|\tau_x f - f|)(y).$$

Hence,

$$-M(|\tau_x f - f|)(y) \leq M(\tau_x f)(y) - M(f)(y). \tag{4.4}$$

It follows from (4.3) and (4.4) that

$$|M(\tau_x f)(y) - M(f)(y)| \leq M(|\tau_x f - f|)(y). \tag{4.5}$$

Taking the norm $\|\cdot\|_{q,p,\alpha}$ of both sides of (4.5) and applying Proposition 3.2 we get

$$\|\tau_x(Mf) - Mf\|_{q,p,\alpha} \leq \|M(|\tau_x f - f|)\|_{q,p,\alpha} \leq C\|\tau_x f - f\|_{q,p,\alpha}.$$

Thus the desired result follows.

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