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On basicity in Bochner space $L_p((-\pi,\pi);X)$ and Riesz property

Sabina R. Sadigova⋆

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Abstract. *The Bochner space* $L_p((0,2\pi);X)$, $1 < p < +\infty$, *is considered, in which the periodic Hilbert transform is bounded. Based on this property, it is proved that the family of subspaces* $\left\{L_p^{(k)}(X)\right\}_{k\in\mathbb{Z}}$, *generated by exponential system* $\left\{e^{ikt}\right\}_{k\in\mathbb{Z}}$, *forms a basis for* $L_p((0,2\pi);X)$. *Moreover, it is proved that this system possesses the Riesz property.*

Keywords. X-valued Hilbert transform, Riesz property, basis, Bochner space.

Mathematics Subject Classification (2010): 41A65, 30D55, 30E25

1 Introduction

The classical Hardy classes H_p of analytic functions on the unit disk of complex plane are very known and play an important role in many areas of mathematics. For more detailed information about these classes, one can refer to monographs such as [15,13,8]. Different mathematicians have provided abstract generalizations of these classes (see, e.g. [17,7,5,6,11,12,16,9]).

In this note, we consider the Bochner space $L_p(J;X)$, $1 < p < +\infty$, on the interval $J = [-\pi,\pi)$. We assume that the periodic Hilbert transform is bounded in $L_p(J;X)$, $1 < p < +\infty$. For $1 < p < +\infty$, this property is equivalent to the Banach space X having the so-called UMD (Unconditional Martingale Difference) property. We consider the sequence of subspaces $\left\{L_p^{(k)}(X)\right\}_{k\in\mathbb{Z}}$, where $L_p^{(k)}(X)$ is the subspace of $L_p(J;X)$, consisting of functions of the form $e^{ikt}x$, $x\in X$, for all $k\in\mathbb{Z}$. We prove the basicity of this system of subspaces for $L_p(J;X)$, $1<p<+\infty$. Moreover, we establish that this basis has X-valued Riesz property.

⋆ Corresponding author

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S.R. Sadigova
Institute of Mathematics and Mechanics, Ministry of Science and Education of the Republic of Azerbaijan, Baku, Azerbaijan
Odlar Yurdu University, Baku, Azerbaijan
E-mail: s_sadigova@mail.ru

2 Notations and auxiliary facts

21 Notations.

The following notations are used throughout this work: $\mathbb{N}$ will be the set of all positive integers, $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$; $\mathbb{Z}$ will be a set of all integers, $\mathbb{C}$ will stand for the field of complex numbers; B-space-Banach space; $[X;Y]$ denotes a B-space of all bounded linear operators from X to Y. Moreover, $[X] = [X;X]$; $\|\cdot\|$ is a norm in X; X^* is a dual space of X; $L[M]$ is a linear span of the set M; $\overline{M}$ is a closure of the set M; I is an identity operator; $\dot{+}$ denotes the direct sum; $\omega = \{z \in \mathbb{C} : |z| < 1\}$; $\gamma = \{z \in \mathbb{C} : |z| = 1\}$; $J \equiv [-\pi;\pi)$.

Let X be a B-space and $L_p(J;X)$, $1 < p < +\infty$, be a Bochner space equipped with the norm

$$\|f\|_{L_p(J;X)} = \left(\int_J \|f(t)\|_X^p\, dt\right)^{\frac{1}{p}}.$$

For every $k \in \mathbb{Z}$ set

$$L_p^{(k)}(X) = \left\{e^{ikt}x : x \in X\right\}.$$

It is evident that $L_p^{(k)}(X) \subset L_p(J;X)$ is a subspace (i.e., a closed subspace).

22 Basis from subspaces.

We will need some facts concerning basis from subspaces.

Let X be a B-space, and $\mathscr{X} \equiv \{X_k\}_{k\in\mathbb{N}} \subset X$ be a sequence of subspaces of X. Also, denote $\mathscr{X}_k \equiv \{X_n\}_{n\in\mathbb{N}\setminus\{k\}}$.

The system $\mathscr{X}$ *is complete in* X, *if* $\overline{L[\mathscr{X}]} = X$.

The system $\mathscr{X}$ *is minimal in* X, *if for* $\forall k \in \mathbb{N} \Rightarrow X_k \cap \overline{L[\mathscr{X}_k]} = \{0\}$.

The system $\mathscr{X}$ *forms a basis for* X *if every* $x \in X$ *has a unique expansion in the form*

$$x = \sum_{k=1}^{\infty} x_k,$$

with $x_k \in X_k$, $k \in \mathbb{N}$.

Let $P \in [X]\ \&\ P^2 = P$, be a continuous projector. Define $Q = I - P$. which is also a continuous projector. Let $Y = PX\ \&\ Y^c = QX$. The following properties hold: i) Y and Y^c are subspaces of X; ii) $Y \cap Y^c = \{0\}$; iii) $X = Y \dot{+} Y^c \Rightarrow Y$ and Y^c are complemented subspaces in X.

Conversely, if $Y_1; Y_2 \subset X$ are subspaces and $X = Y_1 \dot{+} Y_2$ holds, then this decomposition generates corresponding projectors $P;Q$ such that $I = P + Q\ \&\ PQ = QP = 0$. These projectors are called mutually disjunctive.

The projectors $\{P_k\}_{k\in\mathbb{N}} \subset [X]$ *form a basis for* X, *if they are mutually disjunctive and* $I = \sum_{k=1}^{\infty} P_k$, *i.e.* $x = \sum_{k=1}^{\infty} P_k x$, $\forall x \in X$.

The projectors $\{P_k\}_{k\in\mathbb{N}} \subset [X]$ are said to be complete in X, if the subspaces $\{X_k\}_{k\in\mathbb{N}}$ are complete in X, where $X_k = P_k X$, $k \in \mathbb{N}$. It is obvious that the projectors $\{P_k\}_{k\in\mathbb{N}} \subset [X]$ form a basis for X if and only if the subspaces $\{X_k\}_{k\in\mathbb{N}}$ form a basis for X, where $X_k = P_k X$, $k \in \mathbb{N}$.

The following basicity criterion is valid.

Theorem 2.1 *The projectors* $\{P_k\}_{k\in\mathbb{N}} \subset [X]$ *form a basis for* X *if and only if the following assertions hold:*

i) $\{P_k\}_{k\in\mathbb{N}}$ *is complete in* X*;*
ii) $\{P_k\}_{k\in\mathbb{N}}$ *are mutually disjunctive;*
iii) the projectors $S_n = \sum_{k=1}^{n} P_k$, $n \in \mathbb{N}$, *are uniformly bounded, i.e.*

$$\sup_n \|S_n\|_{[X]} < +\infty.$$

For more information about these facts and on basis properties of systems of sines, cosines, exponents and their perturbations in various function spaces one can refer to the monographs [19,2] and the works [4,18,10,3].

The set of all X-valued trigonometric polynomials $P_n : J \to X$ of the form

$$P_n(t) = \sum_{k=-n}^{n} a_k e^{ikt},\ t \in J,$$

is denoted by $\mathscr{P}(X)$, where $\{a_k\} \subset X$.

Consider the Bochner space $L_p(\gamma; X)$, which is generated by the Lebesgue linear measure dl on γ, where dl is the length element of γ. We identify the segment J and the unit circle γ by mapping $e^{it} : J \leftrightarrow \gamma$. This allow us to identify also the spaces $L_p(J; X)$ and $L_p(\gamma; X)$. And also set $f(t) =: f\left(e^{it}\right)$, for $f : \gamma \to \mathbb{C}$.

Define on $\mathscr{P}(X)$ the following multiplier operator $\mathscr{M} : \mathscr{P}(X) \to L_p(J; X)$:

$$(\mathscr{M} P)(t) = \tilde{P}(t) = -i \sum_{k\in\mathbb{Z}} sign(k)\, a_k e^{ikt},$$

where

$$P(t) = \sum_k a_k e^{ikt} \in \mathscr{P}(X),$$

and

$$sign(k) = \begin{cases} 1, & k > 0, \\ 0, & k = 0, \\ -1, & k < 0. \end{cases}$$

It is valid the following

Proposition 2.1 *Let* X *be a* B*-space. Then* $\overline{\mathscr{P}(X)} = L_p(J; X)$, $1 < p < +\infty$ *(the closure is taken in* $L_p(J; X)$*).*

Regarding this fact one can see, e.g. the monograph [14].

From Proposition 2.1 it directly follows the following

Corollary 2.1 *Let* X *be a* B*-space. The sequence of subspaces* $\left\{L_p^{(k)}(X)\right\}_{k\in\mathbb{Z}}$ *is complete in* $L_p(J; X)$, $1 \le p < +\infty$*.*

Consider the following X-valued periodic Hilbert transform

$$\left(\tilde{H} f\right)\left(e^{i\theta}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(\theta - \varphi)}{tg\frac{\varphi}{2}} d\varphi,\ \theta \in J.$$

Now, let's define the basicity of the double system $\left\{L_p^{(k)}(X)\right\}_{k\in\mathbb{Z}}$ for $L_p(J; X)$.

Definition 2.1 *We will say that the system* $\left\{L_p^{(k)}(X)\right\}_{k\in\mathbb{Z}}$ *forms a basis for* $L_p(J;X)$, $1<p<+\infty$, *if every* $f\in L_p(J;X)$ *has a unique expansion of the form*

$$f(t)=\sum_{k=0}^{\infty} f_k^{+}e^{ikt}+\sum_{k=1}^{\infty} f_k^{-}e^{-ikt},$$

with the coefficients $\{f_k^{\pm}\}\subset X$.

Also accept the following concept of Riesz property of the system $\left\{L_p^{(k)}(X)\right\}_{k\in\mathbb{Z}}$.

Definition 2.2 *We will say that the system* $\left\{L_p^{(k)}(X)\right\}_{k\in\mathbb{Z}}$ *has a Riesz property if for* $\forall f\in L_p(J;X)$, *the series*

$$R^{\pm}f=\sum_{k=0}^{+\infty}\hat{f}_{\pm k}e^{\pm ikt},$$

converges in $L_p(J;X)$, *where*

$$T_k(f)=\hat{f}_k=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-ikt}f(t)\,dt\,,\ k\in\mathbb{Z}, \tag{2.1}$$

are X*-valued Fourier coefficients of* $f(\cdot)$.

In obtaining main results we will essentially use the following

Lemma 2.1 *Let* X *be a* B*-space. Then* $\mathscr{M}P=\tilde{H}P$, $\forall P\in\mathscr{P}(X)$.

This lemma is proved completely analogously to the proof of Lemma 6.9 of monograph [1] (see, p. 162) and we will skip this proof.
From this lemma it directly follows the following obvious

Statement 2.2 *Let* X *be a* B*-space. If* $\tilde{H}\in[L_p(J;X)]$, $1<p<+\infty$, *then the multiplier operator* $\mathscr{M}$ *extends continuously on* $L_p(J;X)$: $\mathscr{M}\in[L_p(J;X)]$.

Indeed, by Proposition 2.1 the set $\mathscr{P}(X)$ is dense in $L_p(J;X)$ and therefore, the rest follows from Lemma 2.1.

3 Main results

In this section we will prove that the system $\left\{L_p^{(k)}(X)\right\}_{k\in\mathbb{Z}}$ forms a basis for $L_p(J;X)$, $1<p<+\infty$, if $\tilde{H}\in[L_p(J;X)]$.
First, let us prove the following

Theorem 3.1 *Let* X *be a* B*-space and* $\tilde{H}\in[L_p(J;X)]$, $1<p<+\infty$. *Then the system* $\left\{L_p^{(k)}(X)\right\}_{k\in\mathbb{Z}}$ *forms a basis for* $L_p(J;X)$ *in sense that (i.e. in symmetric sense)* $\forall f\in L_p(J;X)$ *has a unique expansion*

$$f(t)=\lim_{n\to\infty}\sum_{k=-n}^{n}\hat{f}_k e^{ikt}. \tag{3.1}$$

Proof. Consider

$$(S_n f)(t) = \sum_{k=-n}^{n} \hat{f}_k e^{ikt}, \; n \in \mathbb{N},$$

where $\left\{\hat{f}_k\right\}$ are X-valued Fourier coefficients of $f(\cdot)$, which are defined by (2.1). So, we have

$$S_n f = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(\tau - t) f(\tau)\, d\tau,$$

where $D_n(\cdot)$ is the Dirichlet kernel

$$D_n(t) = \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}, \; n \in \mathbb{N}.$$

From the relations

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-k)t} dt = \delta_{nk}; \; \forall n; k \in \mathbb{Z},$$

it directly follows that if the function $f(\cdot)$ has an expansion of the form (3.1), then such expansion is unique. Completeness of the system $\left\{L_p^{(k)}(X)\right\}_{k\in\mathbb{Z}}$ in $L_p(J;X)$ follows from Corollary 2.1.

To prove the theorem it is sufficient to show that the operators $\{S_n\}_{n\in\mathbb{N}} \subset [L_p(J;X)]$ are uniformly bounded. We can represent $S_n f$ in the form

$$(S_n f)(t) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{\sin(n+\frac{1}{2})(\tau-t)}{\sin\frac{\tau-t}{2}} f(\tau) d\tau$$

$$= \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{\sin n(\tau-t)\cos\frac{1}{2}(\tau-t) + \cos n(\tau-t)\sin\frac{1}{2}(\tau-t)}{\sin\frac{\tau-t}{2}} f(\tau) d\tau =$$

$$= \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{\sin n(\tau-t)}{\tan\frac{\tau-t}{2}} f(\tau) d\tau + \frac{1}{4\pi}\int_{-\pi}^{\pi} \cos n(\tau-t) f(\tau) d\tau.$$

Then, we obtain

$$\|S_n f\|_{L_p(J;X)} \le \frac{1}{4\pi}\left\|\int_{-\pi}^{\pi} \frac{\sin n(\tau-t)}{\tan\frac{\tau-t}{2}} f(\tau) d\tau\right\|_{L_p(J;X)} + \frac{1}{4\pi}\left\|\int_{-\pi}^{\pi} \cos n(\tau-t) f(\tau) d\tau\right\|_{L_p(J;X)}.$$

Using the boundedness of the periodic Hilbert transform we have

$$\|S_n f\|_{L_p(J;X)} \le c\, \|f\|_{L_p(J;X)},$$

where $c > 0$ is a constant independent of n and f. Consequently, by Theorem 2.1 we obtain the basicity (in symmetric sense) of the system $\left\{L_p^{(k)}(X)\right\}_{k\in\mathbb{Z}}$ for $L_p(J;X)$.

Theorem is proved.

The following theorem is also true.

Theorem 3.2 *Let X be a B-space and $\tilde{H} \in [L_p(J;X)]$, $1 < p < +\infty$. Then the system $\left\{L_p^{(k)}(X)\right\}_{k\in\mathbb{Z}}$ has a Riesz property.*

Proof. Define the Riesz operator R^+on $\mathscr{P}(X)$ by expression

$$\left(R^+P\right)(\tau)=\sum_{n=0}^{m}a_n\tau^n,$$

$$\forall P=\sum_{n=-m}^{m}a_n\tau^n\in\mathscr{P}(X). \tag{3.2}$$

We have

$$R^+P=\frac{1}{2}T_0(P)+\frac{1}{2}(P+i\mathscr{M}(P)),$$

where $\mathscr{M}$ is a multiplier operator and $T_k(\cdot)$ is defined by formula (2.1) (Fourier coefficients). Since, $T_0\in L_p(J;X)$ (it directly follows from expression of T_k), then it is evident that

$$R^+\in[L_p(J;X)]\Leftrightarrow\mathscr{M}\in[L_p(J;X)].$$

So, let $S_m(f)$ be m-th order partial sum of function $f\in L_p(J;X)$:

$$S_m(f)(\tau)=\sum_{n=-m}^{m}T_n(f)\tau^n\,,\;\;m\in\mathbb{N}.$$

Then for polynomials of the form (3.2) we have

$$\left(R^+P\right)(\tau)=\tau^m\left[S_m\left(\xi^{-m}P(\xi)\right)\right](\tau). \tag{3.3}$$

According to the Theorem 3.1 we have

$$\|S_m(f)\|_{L_p(J;X)}\leq c\|f\|_{L_p(J;X)}\,,\;\forall m\in\mathbb{N},\forall f\in L_p(J;X),$$

where the constant $c>0$ is independent of m and f.

Taking into account this relation, from (3.3) we obtain

$$\left\|R^+P\right\|_{L_p(J;X)}\leq c\left\|\tau^{-m}P(\tau)\right\|_{L_p(J;X)}=c\|P(\tau)\|_{L_p(J;X)}\,,\;\;\forall P\in\mathscr{P}(X).$$

Since, $\mathscr{P}(X)$ is dense in $L_p(J;X)$, then from here it follows that the operator R^+extends continuously to $L_p(J;X)$: $R^+\in[L_p(J;X)]$. In particular, we obtain that for $\forall f\in L_p(J;X)$, the series

$$R^+f=\sum_{n=0}^{\infty}\hat{f}_n\tau^n,$$

converges in $L_p(\gamma;X)$ and from the Theorem 3.1 it follows that the series

$$R^-f=\sum_{n=-\infty}^{-1}\hat{f}_n\tau^n,$$

also converges in $L_p(\gamma;X)$.

Theorem is proved.

Using the Theorems 3.1 & 3.2 it is easy to establish the validity of the following

Statement 3.3 *Let X be a B-space and $\tilde{H}\in[L_p(J;X)]$, $1<p<+\infty$. Then the system of subspaces $\left\{L_p^{(k)}(X)\right\}_{k\in\mathbb{Z}}$ forms a basis for $L_p(J;X)$.*

Note that the class of B-spaces for which $\tilde{H} \in [L_p(J;X)]$ is sufficiently wide. For example, let's remember the definition of UMD (Unconditional Martingale Difference) property.

Definition 3.1 *A B-space X is said to have the property of unconditional martingale differences (UMD property) if for all $p \in (1,\infty)$ there exists a finite constant $\beta \geq 0$ (depending on p and X) such that the following holds. Whenever $(S,\mathscr{A},\mu)$ is a σ-finite measure space, $\{\mathscr{F}_n\}_{n=0}^{N}$ is a σ-finite filtration, and $\{f_n\}_{n=0}^{N}$ is a finite martingale in $L^p(S;X)$, then for all scalars $|\varepsilon_n| = 1$, $n = 1,...,N$ we have*

$$\left\| \sum_{n=1}^{N} \varepsilon_n df_n \right\|_{L^p(S;X)} \leq \beta \left\| \sum_{n=1}^{N} df_n \right\|_{L^p(S;X)}.$$

If this condition holds, then X is said to be a UMD space.

It is very known that if X has UMD property, then $\tilde{H} \in [L_p(J;X)]$, $1 < p < +\infty$ (see, e.g. [14]). Consequently, as a particular case from Statement 3.3, we get the following

Corollary 3.1 *Let B-space X have UMD property. Then, the system $\left\{ L_p^{(k)}(X) \right\}_{k\in\mathbb{Z}}$ forms a basis for $L_p(J;X)$, $1 < p < +\infty$.*

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