

## On embeddings of total Morrey spaces associated with Bessel operators

Fatma A. Muslumova\* · Yagub Y. Mammadov · Jamila N. Abdullayeva

Received: 02.09.2024 / Revised: 28.01.2025 / Accepted: 08.02.2025

**Abstract.** Let  $\nu \in (0, \infty)$  and  $\Delta_\nu := -\frac{d^2}{dx^2} - \frac{2\nu+1}{x} \frac{d}{dx}$  be the Bessel operator on  $\mathbb{R}_+ := (0, \infty)$ . In this paper, we study some embeddings into the total Morrey space  $L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)$ ,  $0 \leq \lambda, \mu < 2\nu + 2$  associated with the Bessel operator on  $\mathbb{R}_+$ . These spaces generalize the Morrey space associated with the Bessel operator on  $\mathbb{R}_+$  so that  $L_{p,\lambda}(\mathbb{R}_+, dm_\nu) \equiv L_{p,\lambda,\lambda}(\mathbb{R}_+, dm_\nu)$  and the modified Morrey spaces associated with the Bessel operator on  $\mathbb{R}_+$  so that  $\tilde{L}_{p,\lambda}(\mathbb{R}_+, dm_\nu) \equiv L_{p,\lambda,0}(\mathbb{R}_+, dm_\nu)$ .

**Keywords.** Bessel operator, total Morrey space, embedding

**Mathematics Subject Classification (2010):** 42B20, 42B25, 42B35

### 1 Introduction

Let  $\nu > -1/2$  and  $\Delta_\nu$  be the Bessel operator defined by setting, for suitable functions  $f$  and  $x \in \mathbb{R}_+$ , see [5, 20]. An early work concerning the Bessel operator is from Muckenhoupt and Stein [20]. They aimed to develop a theory associated to  $\Delta_\nu$  which is parallel to the classical one associated to the Laplace operator  $\Delta$ . After that, a lot of work concerning the Bessel operators was carried out. See, for example [2, 4, 6–9, 15, 26, 27] and the references therein. Among the study of  $\Delta_\nu$ , the properties of Riesz transforms associated to  $\Delta_\nu$  defined by

$$R_{\Delta_\nu} f := \partial_x (\Delta_\nu)^{-1/2} f,$$

have been studied extensively, see for example [2, 4, 23]. Characterizations of function spaces associated to the Bessel operator  $\Delta_\nu$  were also studied by many authors. Among these,

---

\* Corresponding author

F.A. Muslumova  
Nakhchivan State University, Department of General Mathematics, Azerbaijan  
E-mail: f.muslimova@hotmail.com

Y.Y. Mammadov  
Nakhchivan Teacher Institute, Department of Mathematics and its teaching methodology, Azerbaijan  
E-mail: yagubmammadov@yahoo.com

J.N. Abdullayeva  
Azerbaijan State Pedagogical University, Azerbaijan  
Azerbaijan Academy of Labour and Social Relations  
E-mail: abdullaevajamila@64gmail.com

we point out that the Lebesgue space associated to the Bessel operator  $\Delta_\nu$  is of the form  $L_p(\mathbb{R}_+, dm_\nu)$ , where  $1 < p < \infty$ ,  $dm_\nu(x) := x^{2\nu+1} dx$ , and  $dx$  is the standard Lebesgue measure on  $\mathbb{R}_+$  (see for example [5]).

Morrey spaces, introduced by C. B. Morrey [19], play important roles in the regularity theory of PDE, including heat equations and Navier-Stokes equations. In [11] Guliyev introduce a variant of Morrey spaces called total Morrey spaces  $L_{p,\lambda,\mu}(\mathbb{R}^n)$ ,  $0 < p < \infty$ ,  $\lambda \in \mathbb{R}$  and  $\mu \in \mathbb{R}$ , see also, [1, 12–14, 18, 21, 22].

In the present work, we give basic properties of the total Morrey space  $L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)$ ,  $0 \leq \lambda, \mu < 2\nu + 2$  associated with the Bessel operator on  $\mathbb{R}_+$  and study some embeddings into the total Morrey space  $L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)$ . These spaces generalize the Morrey space associated with the Bessel operator on  $\mathbb{R}_+$  so that  $L_{p,\lambda}(\mathbb{R}_+, dm_\nu) \equiv L_{p,\lambda,\lambda}(\mathbb{R}_+, dm_\nu)$  and the modified Morrey spaces associated with the Bessel operator on  $\mathbb{R}_+$  so that  $\tilde{L}_{p,\lambda}(\mathbb{R}_+, dm_\nu) \equiv L_{p,\lambda,0}(\mathbb{R}_+, dm_\nu)$ .

The paper is organized as follows. In Section 2, we present some definitions and auxiliary results. In Section 3, we give some embeddings into the total Morrey spaces associated with the Bessel operator on  $\mathbb{R}_+$ .

Finally, we make some conventions on notation. By  $A \lesssim B$  we mean that  $A \leq CB$  with some positive constant  $C$  independent of appropriate quantities. If  $A \lesssim B$  and  $B \lesssim A$ , we write  $A \approx B$  and say that  $A$  and  $B$  are equivalent.

## 2 Preliminaries

Bessel functions were first discovered in 1732 by D. Bernoulli (1700-1782), who provided a series solution (representing a Bessel function) for the oscillatory displacements of a heavy hanging chain (see [3]). Euler later developed a series similar to that of Bernoulli, which was also a Bessel function, and Bessel's equation appeared in a 1764 article by Euler dealing with the vibrations of a circular drumhead. J. Fourier (1768-1836) also used Bessel functions in his classical treatise on heat in 1822, but it was Bessel who first recognized their special properties. Bessel functions are closely associated with problems possessing circular or cylindrical symmetry. For example, they arise in the study of free vibrations of a circular membrane. They also occur in electromagnetic theory and numerous other areas of physics and engineering (see [3]). The more complete reference about Bessel functions is the treatise of Watson [28].

Trimeche [24, 25] has pointed out how the theory of Bessel functions generates an harmonic analysis on the half line tied to the differential operator

$$\Delta_\nu := -\frac{d^2}{dx^2} - \frac{2\nu + 1}{x} \frac{d}{dx}, \quad \nu > -1/2,$$

which is referred to as the Bessel operator of index  $\nu$ .

For a real parameter  $\nu \geq -1/2$ , we consider the Bessel operator on  $\mathbb{R}_+$  :

$$\Delta_\nu := -\frac{d^2}{dx^2} - \frac{2\nu + 1}{x} \frac{d}{dx}.$$

Note that  $\Delta_{-1/2} = -\frac{d^2}{dx^2}$ .

Let  $\nu > -1/2$  be a fixed number and  $m_\nu$  be the weighted Lebesgue measure on  $\mathbb{R}_+$ , given by

$$dm_\nu(x) := (2^{\nu+1} \Gamma(\nu + 1))^{-1} x^{2\nu+1} dx.$$

For every  $1 \leq p \leq \infty$ , we denote by  $L_{p,\nu} = L_p(\mathbb{R}_+, dm_\nu)$  the spaces of complex-valued functions  $f$ , measurable on  $\mathbb{R}_+$  such that

$$\|f\|_{p,\nu} \equiv \|f\|_{L_{p,\nu}} = \left( \int_{\mathbb{R}_+} |f(x)|^p dm_\nu(x) \right)^{1/p} < \infty \quad \text{if } p \in [1, \infty),$$

and

$$\|f\|_{L_{\infty,\nu}} = \operatorname{ess\,sup}_{x \in \mathbb{R}_+} |f(x)| \quad \text{if } p = \infty.$$

For  $1 \leq p < \infty$  we denote by  $WL_{p,\nu}$ , the weak  $L_{p,\nu}$  spaces defined as the set of locally integrable functions  $f(x)$ ,  $x \in \mathbb{R}_+$  with the finite norm

$$\|f\|_{WL_{p,\nu}} = \sup_{r>0} r (m_\nu \{x \in \mathbb{R}_+ : |f(x)| > r\})^{1/p}.$$

Note that

$$L_{p,\nu} \subset WL_{p,\nu} \quad \text{and} \quad \|f\|_{WL_{p,\nu}} \leq \|f\|_{L_{p,\nu}} \quad \text{for all } f \in L_{p,\nu}.$$

Let  $B(x, t) = \{y \in \mathbb{R}_+ : y \in (\max\{0, x-t\}, x+t)\} = (x-t, x+t) \cap \mathbb{R}_+$  and  $t > 0$ . Then  $B(0, t) = (0, t)$  and  $m_\nu((0, t)) = \left(2^{\nu+1} (\nu+1) \Gamma(\nu+1)\right)^{-1} t^{2\nu+2}$ .

**Remark 2.1** We mention that  $(L_p(\mathbb{R}_+, dm_\nu), \|f\|_{L_{p,\nu}})$  is a Banach space. Moreover, since  $(\mathbb{R}_+, |\cdot|, dm_\nu)$  is a space of homogeneous type,  $(L_p(\mathbb{R}_+, dm_\nu), \|f\|_{L_{p,\nu}})$  is a Lebesgue spaces on spaces of homogeneous type in [16, 17].

### 3 Some embeddings into the total Morrey spaces

In this section, we study some embeddings into the total Morrey space  $L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)$ ,  $0 \leq \lambda, \mu < 2\nu + 2$  associated with the Bessel operator on  $\mathbb{R}_+$ .

**Definition 3.1** Let  $1 \leq p < \infty$ ,  $0 \leq \lambda \leq 2\nu + 2$  and  $[t]_1 = \min\{1, t\}$ ,  $t > 0$ . We denote by  $L_{p,\lambda}(\mathbb{R}_+, dm_\nu)$  Morrey space, by  $\tilde{L}_{p,\lambda}(\mathbb{R}_+, dm_\nu)$  the modified Morrey space and by  $L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)$  total Morrey space associated with the Bessel operator as the set of locally integrable functions  $f(x)$ ,  $x \in \mathbb{R}_+$ , with the finite norms

$$\begin{aligned} \|f\|_{L_{p,\lambda}(\mathbb{R}_+, dm_\nu)} &:= \sup_{x \in \mathbb{R}_+, t>0} \left( t^{-\lambda} \int_{B(x,t)} |f(y)|^p dm_\nu(y) \right)^{1/p}, \\ \|f\|_{\tilde{L}_{p,\lambda}(\mathbb{R}_+, dm_\nu)} &:= \sup_{x \in \mathbb{R}_+, t>0} \left( [t]_1^{-\lambda} \int_{B(x,t)} |f(y)|^p dm_\nu(y) \right)^{1/p}, \\ \|f\|_{L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)} &:= \sup_{x \in \mathbb{R}_+, t>0} \left( [t]_1^{-\lambda} [1/t]_1^\mu \int_{B(x,t)} |f(y)|^p dm_\nu(y) \right)^{1/p}, \end{aligned}$$

respectively.

If  $\min\{\lambda, \mu\} < 0$  or  $\max\{\lambda, \mu\} > 2\nu + 2$ , then  $L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu) = \Theta(\mathbb{R}_+)$ , where  $\Theta(\mathbb{R}_+)$  is the set of all functions equivalent to 0 on  $\mathbb{R}_+$ .

Note that

$$\begin{aligned} L_{p,0,0}(\mathbb{R}_+, dm_\nu) &= \tilde{L}_{p,0}(\mathbb{R}_+, dm_\nu) = L_{p,0}(\mathbb{R}_+, dm_\nu) = L_p(\mathbb{R}_+, dm_\nu), \\ L_{p,\lambda,\lambda}(\mathbb{R}_+, dm_\nu) &= L_{p,\lambda}(\mathbb{R}_+, dm_\nu), \quad L_{p,\lambda,0}(\mathbb{R}_+, dm_\nu) = \tilde{L}_{p,\lambda}(\mathbb{R}_+, dm_\nu) \\ L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu) &\subset_{\succ} L_{p,\lambda}(\mathbb{R}_+, dm_\nu) \quad \text{and} \quad \|f\|_{L_{p,\lambda}(\mathbb{R}_+, dm_\nu)} \leq \|f\|_{L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)}, \end{aligned} \quad (3.1)$$

$$L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu) \subset_{\succ} L_{p,\mu}(\mathbb{R}_+, dm_\nu) \quad \text{and} \quad \|f\|_{L_{p,\mu}(\mathbb{R}_+, dm_\nu)} \leq \|f\|_{L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)}. \quad (3.2)$$

**Definition 3.2** Let  $1 \leq p < \infty$ ,  $0 \leq \lambda \leq 2\nu + 2$ . We denote by  $WL_{p,\lambda}(\mathbb{R}_+, dm_\nu)$  weak Morrey space, by  $W\tilde{L}_{p,\lambda}(\mathbb{R}_+, dm_\nu)$  the modified weak Morrey space and by  $WL_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)$  weak total Morrey space associated with the Bessel operator as the set of locally integrable functions  $f(x)$ ,  $x \in \mathbb{R}_+$  with finite norms

$$\|f\|_{WL_{p,\lambda}(\mathbb{R}_+, dm_\nu)} := \sup_{r>0} r \sup_{x \in \mathbb{R}_+, t>0} \left( t^{-\lambda} m_\nu \{y \in B(x, t) : |f(y)| > r\} \right)^{1/p},$$

$$\|f\|_{W\tilde{L}_{p,\lambda}(\mathbb{R}_+, dm_\nu)} := \sup_{r>0} r \sup_{x \in \mathbb{R}_+, t>0} \left( [t]_1^{-\lambda} m_\nu \{y \in B(x, t) : |f(y)| > r\} \right)^{1/p},$$

$$\|f\|_{WL_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)} := \sup_{r>0} r \sup_{x \in \mathbb{R}_+, t>0} \left( [t]_1^{-\lambda} [1/t]_1^\mu m_\nu \{y \in B(x, t) : |f(y)| > r\} \right)^{1/p},$$

respectively.

We note that

$$L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu) \subset WL_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu) \quad \text{and} \quad \|f\|_{WL_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)} \leq \|f\|_{L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)}.$$

**Remark 3.1** We mention that  $(L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu), \|f\|_{L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)})$  is a Banach space. Moreover, since  $(\mathbb{R}_+, |\cdot|, dm_\nu)$  is a space of homogeneous type,

$$(L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu), \|f\|_{L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)})$$

is a special case of Morrey spaces on spaces of homogeneous type in [10].

**Lemma 3.1** If  $0 < p < \infty$ ,  $0 \leq \lambda \leq 2\nu + 2$  and  $0 \leq \mu \leq 2\nu + 2$ , then

$$L_{p,2\nu+2,\mu}(\mathbb{R}_+, dm_\nu) \subset_{\succ} L_\infty(\mathbb{R}_+, dm_\nu) \subset_{\succ} L_{p,\lambda,2\nu+2}(\mathbb{R}_+, dm_\nu)$$

and

$$\|f\|_{L_{p,\lambda,2\nu+2}(\mathbb{R}_+, dm_\nu)} \leq b_\nu^{1/p} \|f\|_{L_\infty(\mathbb{R}_+, dm_\nu)} \leq \|f\|_{L_{p,2\nu+2,\mu}(\mathbb{R}_+, dm_\nu)}.$$

**Proof.** Let  $f \in L_\infty(\mathbb{R}_+, dm_\nu)$ . Then for all  $x \in \mathbb{R}$  and  $0 < t \leq 1$

$$\left( t^{-\lambda} \int_{B(x,t)} |f(y)|^p dm_\nu(y) \right)^{1/p} \leq b_\nu^{1/p} \|f\|_{L_\infty}, \quad 0 \leq \lambda \leq 2\nu + 2$$

and for all  $x \in \mathbb{R}_+$  and  $t \geq 1$

$$\left( t^{-2\nu-2} \int_{B(x,t)} |f(y)|^p dm_\nu(y) \right)^{1/p} \leq b_\nu^{1/p} \|f\|_{L_\infty}.$$

Therefore  $f \in L_{p,\lambda,2\nu+2}(\mathbb{R}_+, dm_\nu)$  and

$$\|f\|_{L_{p,\lambda,2\nu+2}(\mathbb{R}_+, dm_\nu)} \leq b_\nu^{1/p} \|f\|_{L_\infty}.$$

Let  $f \in L_{p,2\nu+2,\mu}(\mathbb{R}_+, dm_\nu)$ . By the Lebesgue's differentiation theorem we have

$$\lim_{t \rightarrow 0} m_\nu(B(x,t))^{-1} \int_{B(x,t)} |f(y)|^p dm_\nu(y) = |f(x)|^p \quad a.e. \ x \in \mathbb{R}_+.$$

Then

$$\begin{aligned} |f(x)| &= \left( \lim_{t \rightarrow 0} m_\nu(B(x,t))^{-1} \int_{B(x,t)} |f(y)|^p dm_\nu(y) \right)^{1/p} \\ &\leq b_\nu^{1/p} \left( t^{-2\nu-2} \int_{B(x,t)} |f(y)|^p dm_\nu(y) \right)^{1/p} \leq b_\nu^{1/p} \|f\|_{L_{p,2\nu+2,\mu}(\mathbb{R}_+, dm_\nu)}. \end{aligned}$$

Therefore  $f \in L_\infty(\mathbb{R}_+, dm_\nu)$  and

$$\|f\|_{L_\infty(\mathbb{R}_+, dm_\nu)} \leq b_\nu^{1/p} \|f\|_{L_{p,2\nu+2,\mu}(\mathbb{R}_+, dm_\nu)}.$$

**Corollary 3.1** *If  $0 < p < \infty$ , then*

$$L_{p,2\nu+2}(\mathbb{R}_+, dm_\nu) = \tilde{L}_{p,2\nu+2}(\mathbb{R}_+, dm_\nu) = L_\infty(\mathbb{R}_+, dm_\nu)$$

and

$$\|f\|_{L_{p,2\nu+2}(\mathbb{R}_+, dm_\nu)} = \|f\|_{\tilde{L}_{p,2\nu+2}(\mathbb{R}_+, dm_\nu)} = b_\nu^{1/p} \|f\|_{L_\infty}.$$

**Lemma 3.2** *Let  $1 \leq p < \infty$ ,  $0 \leq \lambda \leq 2\nu + 2$  and  $0 \leq \mu \leq 2\nu + 2$ . Then*

$$L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu) = L_{p,\lambda}(\mathbb{R}_+, dm_\nu) \cap L_{p,\mu}(\mathbb{R}_+, dm_\nu)$$

and

$$\|f\|_{L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)} = \max \left\{ \|f\|_{L_{p,\lambda}(\mathbb{R}_+, dm_\nu)}, \|f\|_{L_{p,\mu}(\mathbb{R}_+, dm_\nu)} \right\}.$$

**Proof.** Let  $f \in L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)$ . Then by (3.1) and (3.2) we have

$$L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu) \subset \supset L_{p,\lambda}(\mathbb{R}_+, dm_\nu) \cap L_{p,\mu}(\mathbb{R}_+, dm_\nu)$$

and

$$\max \left\{ \|f\|_{L_{p,\lambda}(\mathbb{R}_+, dm_\nu)}, \|f\|_{L_{p,\mu}(\mathbb{R}_+, dm_\nu)} \right\} \leq \|f\|_{L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)}.$$

Let  $f \in L_{p,\lambda}(\mathbb{R}_+, dm_\nu) \cap L_{p,\mu}(\mathbb{R}_+, dm_\nu)$ . Then we have

$$\begin{aligned} \|f\|_{L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)} &= \sup_{x \in \mathbb{R}_+, t > 0} \left( [t]_1^{-\lambda} [1/t]_1^\mu \int_{B(x,t)} |f(y)|^p dm_\nu(y) \right)^{1/p} \\ &= \max \left\{ \sup_{x \in \mathbb{R}_+, 0 < t \leq 1} \left( t^{-\lambda} \int_{B(x,t)} |f(y)|^p dm_\nu(y) \right)^{1/p}, \right. \\ &\quad \left. \sup_{x \in \mathbb{R}_+, t > 1} \left( t^{-\mu} \int_{B(x,t)} |f(y)|^p dm_\nu(y) \right)^{1/p} \right\} \\ &\leq \max \left\{ \|f\|_{L_{p,\lambda}(\mathbb{R}_+, dm_\nu)}, \|f\|_{L_{p,\mu}(\mathbb{R}_+, dm_\nu)} \right\}. \end{aligned}$$

Therefore,  $f \in L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)$  and the embedding  $L_{p,\lambda}(\mathbb{R}_+, dm_\nu) \cap L_{p,\mu}(\mathbb{R}_+, dm_\nu) \subset_{\succ} L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)$  is valid.

Thus  $L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu) = L_{p,\lambda}(\mathbb{R}_+, dm_\nu) \cap L_{p,\mu}(\mathbb{R}_+, dm_\nu)$  and

$$\|f\|_{L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)} = \max \left\{ \|f\|_{L_{p,\lambda}(\mathbb{R}_+, dm_\nu)}, \|f\|_{L_{p,\mu}(\mathbb{R}_+, dm_\nu)} \right\}.$$

**Corollary 3.2** *If  $0 < p < \infty$ ,  $0 \leq \lambda \leq 2\nu + 2$ , then*

$$\tilde{L}_{p,\lambda}(\mathbb{R}_+, dm_\nu) = L_{p,\lambda}(\mathbb{R}_+, dm_\nu) \cap L_p(\mathbb{R}_+, dm_\nu)$$

and

$$\|f\|_{\tilde{L}_{p,\lambda}(\mathbb{R}_+, dm_\nu)} = \max \left\{ \|f\|_{L_{p,\lambda}(\mathbb{R}_+, dm_\nu)}, \|f\|_{L_p(\mathbb{R}_+, dm_\nu)} \right\}.$$

From Lemmas 3.1 and 3.2 for  $1 \leq p < \infty$  we have

$$\tilde{L}_{p,2\nu+2}(\mathbb{R}_+, dm_\nu) = L_\infty(\mathbb{R}_+, dm_\nu) \cap L_p(\mathbb{R}_+, dm_\nu). \quad (3.3)$$

**Lemma 3.3** *Let  $0 < p < \infty$ ,  $0 \leq \lambda \leq 2\nu + 2$  and  $0 \leq \mu \leq 2\nu + 2$ . Then*

$$WL_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu) = WL_{p,\lambda}(\mathbb{R}_+, dm_\nu) \cap WL_{p,\mu}(\mathbb{R}_+, dm_\nu)$$

and

$$\|f\|_{WL_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)} = \max \left\{ \|f\|_{WL_{p,\lambda}(\mathbb{R}_+, dm_\nu)}, \|f\|_{WL_{p,\mu}(\mathbb{R}_+, dm_\nu)} \right\}.$$

**Corollary 3.3** *If  $0 < p < \infty$ ,  $0 \leq \lambda \leq 2\nu + 2$ , then*

$$W\tilde{L}_{p,\lambda}(\mathbb{R}_+, dm_\nu) = WL_{p,\lambda}(\mathbb{R}_+, dm_\nu) \cap WL_p(\mathbb{R}_+, dm_\nu)$$

and

$$\|f\|_{W\tilde{L}_{p,\lambda}(\mathbb{R}_+, dm_\nu)} = \max \left\{ \|f\|_{WL_{p,\lambda}(\mathbb{R}_+, dm_\nu)}, \|f\|_{WL_p(\mathbb{R}_+, dm_\nu)} \right\}.$$

**Remark 3.2** *If  $0 < p < \infty$ , and  $\min\{\lambda, \mu\} < 0$  or  $\max\{\lambda, \mu\} > 2\nu + 2$ , then*

$$L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu) = WL_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu) = \Theta(\mathbb{R}_+).$$

**Lemma 3.4** *If  $0 < p < \infty$ ,  $0 \leq \lambda_2 \leq \lambda_1 \leq 2\nu + 2$  and  $0 \leq \mu_1 \leq \mu_2 \leq 2\nu + 2$ , then*

$$L_{p,\lambda_1,\mu_1}(\mathbb{R}_+, dm_\nu) \subset_{\succ} L_{p,\lambda_2,\mu_2}(\mathbb{R}_+, dm_\nu)$$

and

$$\|f\|_{L_{p,\lambda_2,\mu_2}(\mathbb{R}_+, dm_\nu)} \leq \|f\|_{L_{p,\lambda_1,\mu_1}(\mathbb{R}_+, dm_\nu)}.$$

**Proof.** Let  $f \in L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)$ ,  $0 < p < \infty$ ,  $0 \leq \lambda_2 \leq \lambda_1 \leq 2\nu + 2$ ,  $0 \leq \mu_1 \leq \mu_2 \leq 2\nu + 2$ . Then

$$\|f\|_{L_{p,\lambda_2,\mu_2}(\mathbb{R}_+, dm_\nu)} = \max \left\{ \sup_{x \in \mathbb{R}_+, 0 < t \leq 1} \left( t^{\lambda_1 - \lambda_2} t^{-\lambda_1} \int_{B(x,t)} |f(y)|^p dm_\nu(y) \right)^{1/p}, \right. \\ \left. \sup_{x \in \mathbb{R}_+, t \geq 1} \left( t^{\mu_1 - \mu_2} t^{-\mu_1} \int_{B(x,t)} |f(y)|^p dm_\nu(y) \right)^{1/p} \right\} \leq \|f\|_{L_{p,\lambda_1,\mu_1}^d}.$$

On the total Morrey spaces associated with the Bessel operator the following embedding is valid.

**Lemma 3.5** Let  $0 \leq \lambda < 2\nu + 2$ ,  $0 \leq \mu < 2\nu + 2$ ,  $0 \leq \beta_1 < 2\nu + 2 - \lambda$  and  $0 \leq \beta_2 < 2\nu + 2 - \mu$ . Then for  $\frac{2\nu+2-\lambda}{\beta_1} \leq p \leq \frac{2\nu+2-\mu}{\beta_2}$

$$L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu) \subset_{\succ} L_{1,2\nu+2-\beta_1,2\nu+2-\beta_2}(\mathbb{R}_+, dm_\nu)$$

and for  $f \in L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)$  the following inequality

$$\|f\|_{L_{1,2\nu+2-\beta_1,2\nu+2-\beta_2}(\mathbb{R}_+, dm_\nu)} \leq b_\nu^{\frac{1}{p'}} \|f\|_{L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)}$$

is valid.

**Proof.** Let  $0 < \lambda < 2\nu + 2$ ,  $0 \leq \mu < 2\nu + 2$ ,  $0 < \beta_1 < 2\nu + 2 - \lambda$ ,  $0 < \beta_2 < 2\nu + 2 - \mu$ ,  $f \in L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)$  and  $\frac{2\nu+2-\lambda}{\beta_1} \leq p \leq \frac{2\nu+2-\mu}{\beta_2}$ . By the Hölder's inequality we have

$$\begin{aligned} \|f\|_{L_{1,2\nu+2-\beta_1,2\nu+2-\beta_2}(\mathbb{R}_+, dm_\nu)} &= \sup_{x \in \mathbb{R}_+, t > 0} [t]_1^{\beta_1-2\nu-2} [1/t]_1^{-\beta_2+2\nu+2} \int_{B(x,t)} |f(y)| dm_\nu(y) \\ &\leq b_\nu^{\frac{1}{p'}} \sup_{x \in \mathbb{R}_+, t > 0} \left( [t]_1 t^{-1} \right)^{-(2\nu+2)/p'} [t]_1^{\beta_1 - \frac{2\nu+2-\lambda}{p}} \left( [t]_1^{-\lambda} [1/t]_1^\mu \int_{B(x,t)} |f(y)|^p dm_\nu(y) \right)^{1/p} \\ &\leq b_\nu^{\frac{1}{p'}} \|f\|_{L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)} \sup_{t > 0} \left( [t]_1 t^{-1} \right)^{\frac{2\nu+2-\mu}{p} - \beta_2} [t]_1^{\beta_1 - \frac{2\nu+2-\lambda}{p}}. \end{aligned}$$

Note that

$$\sup_{t > 0} \left( [t]_1 t^{-1} \right)^{\frac{2\nu+2-\mu}{p} - \beta_2} [t]_1^{\beta_1 - \frac{2\nu+2-\lambda}{p}} = \max \left\{ \sup_{0 < t \leq 1} t^{\beta_1 - \frac{2\nu+2-\lambda}{p}}, \sup_{t > 1} t^{\beta_2 - \frac{2\nu+2-\mu}{p}} \right\} < \infty$$

$$\text{if and only if } \frac{2\nu + 2 - \lambda}{\beta_1} \leq p \leq \frac{2\nu + 2 - \mu}{\beta_2}.$$

Therefore  $f \in L_{1,2\nu+2-\beta_1,2\nu+2-\beta_2}(\mathbb{R}_+, dm_\nu)$  and

$$\|f\|_{L_{1,2\nu+2-\beta_1,2\nu+2-\beta_2}(\mathbb{R}_+, dm_\nu)} \leq b_\nu^{\frac{1}{p'}} \|f\|_{L_{p,\lambda,\mu}(\mathbb{R}_+, dm_\nu)}.$$

**Corollary 3.4** Let  $0 \leq \lambda < 2\nu + 2$  and  $0 \leq \beta < 2\nu + 2 - \lambda$ . Then for  $p = \frac{2\nu+2-\lambda}{\beta}$

$$L_{p,\lambda}(\mathbb{R}_+, dm_\nu) \subset_{\succ} L_{1,2\nu+2-\beta}(\mathbb{R}_+, dm_\nu)$$

and for  $f \in L_{p,\lambda}(\mathbb{R}_+, dm_\nu)$  the following inequality

$$\|f\|_{L_{1,2\nu+2-\beta}(\mathbb{R}_+, dm_\nu)} \leq b_\nu^{\frac{1}{p'}} \|f\|_{L_{p,\lambda}(\mathbb{R}_+, dm_\nu)}$$

is valid.

**Corollary 3.5** Let  $0 \leq \lambda < 2\nu + 2$  and  $0 \leq \beta < 2\nu + 2 - \lambda$ . Then for  $\frac{2\nu+2-\lambda}{\beta} \leq p \leq \frac{2\nu+2-\mu}{\beta}$

$$\tilde{L}_{p,\lambda}(\mathbb{R}_+, dm_\nu) \subset_{\succ} \tilde{L}_{1,2\nu+2-\beta}(\mathbb{R}_+, dm_\nu)$$

and for  $f \in \tilde{L}_{p,\lambda}(\mathbb{R}_+, dm_\nu)$  the following inequality

$$\|f\|_{\tilde{L}_{1,2\nu+2-\beta}(\mathbb{R}_+, dm_\nu)} \leq b_\nu^{\frac{1}{p'}} \|f\|_{\tilde{L}_{p,\lambda}(\mathbb{R}_+, dm_\nu)}$$

is valid.

**Acknowledgements.** The authors express their thank to the referees for carefully reading, helpful comments and suggestions on the manuscript of this paper.

## References

1. Abasova, G.A., Omarova, M.N.: *Commutator of anisotropic maximal function with BMO functions on anisotropic total Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Mathematics **43**(1), 3-15 (2023).
2. Andersen, K.F., Kerman, R.A.: *Weighted norm inequalities for generalized Hankel conjugate transformations*, Studia Math. **71**(1), 15-26 (1981).
3. Andrews, L.C.: *Special functions of mathematics for engineers*, Oxford University Press, 1998.
4. Betancor, J.J., Chicco Ruiz, A., Farina, J.C., Rodriguez-Mesa, L.: *Maximal operators, Riesz transforms and Littlewood-Paley functions associated with Bessel operators on BMO*, J. Math. Anal. Appl. **363**, 310-326 (2010).
5. Betancor, J.J., Dziubanski, J., Torrea, J.L.: *On Hardy spaces associated with Bessel operators*, J. Anal. Math. **107**, 195-219 (2009).
6. Betancor, J.J., Farina, J.C., Buraczewski, D., Martinez, T., Torrea, J.L.: *Riesz transform related to Bessel operators*, Proc. Roy. Soc. Edinburgh Sect. A **137**, 701-725 (2007).
7. Betancor, J.J., Farina, J.C., Sanabria, A.: *On Littlewood-Paley functions associated with Bessel operators*, Glasg. Math. J. **51**, 55-70 (2009).
8. Betancor, J.J., Harboure, E., Nowak, A., Viviani, B.: *Mapping properties of functional operators in harmonic analysis related to Bessel operators*, Studia Math. **197**, 101-140 (2010).
9. Duong, X.T., Li, Ji, Wick, B.D., Yang, D.: *Hardy space via factorization, and BMO space via commutators in the Bessel setting*, Indiana Univ. Math. J. **66**(4), 1081-1106 (2017).
10. Fan, D., Lu, S., Yang, D.: *Boundedness of operators in Morrey spaces on homogeneous spaces and its applications*, Acta Math. Sinica (Chin. Ser.) **42**, 583-590 (1999).
11. Guliyev, V.S.: *Maximal commutator and commutator of maximal function on total Morrey spaces*, J. Math. Inequal. **16**(4), 1509-1524 (2022).
12. Guliyev, V.S.: *Characterizations of Lipschitz functions via the commutators of maximal function in total Morrey spaces*. Math. Meth. Appl. Sci. **47**(11), 8669-8682 (2024).
13. Guliyev, V.S.: *Characterizations for the fractional maximal operator and its commutators on total Morrey spaces*. Positivity **28**, Paper No. 51, 20 pp. (2024).
14. Guliyev, V.S.: *Corrigendum to : "Characterizations of Lipschitz functions via the commutators of maximal function in total Morrey spaces"*. Math. Meth. Appl. Sci. **48**(4), 1-8 (2025).
15. Kerman, R.A.: *Boundedness criteria for generalized Hankel conjugate transformations*, Canad. J. Math. **30**, 147-153 (1978).
16. Krantz, S.G., Li, S.Y.: *Boundedness and compactness of integral operators on spaces of homogeneous type and applications I*, J. Math. Anal. Appl. **258**, 629-641 (2001).
17. S.G. Krantz, S. Y. Li, *Boundedness and compactness of integral operators on spaces of homogeneous type and applications II*, J. Math. Anal. Appl. **258**, 642-657 (2001).
18. Mammadov, Y.Y., Guliyev, V.S., Muslumova, F.A.: *Commutator of the maximal function in total Morrey spaces for the Dunkl operator on the real line*, Azerb. J. Math. **15**(2), 1-18 (2025).
19. Morrey, C.B.: *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43**, 126-166 (1938).
20. Muckenhoupt, B., Stein, E.M.: *Classical expansions and their relation to conjugate harmonic functions*, Trans. Amer. Math. Soc. **118**, 17-92 (1965).
21. Muslumova, F.A.: *Some embeddings into the total Morrey spaces associated with the Dunkl operator on  $\mathbb{R}^d$* , Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Mathematics **43**(1), 94-102 (2023).
22. Muslumova, F.A.: *Corrigendum to : "Some embeddings into the total Morrey spaces associated with the Dunkl operator on  $\mathbb{R}^d$ "*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-



- 
- Tech. Math. Sci., Mathematics **44**(1), 132 (2024).
23. Rösler, M.: *Bessel-type signed hypergroups on  $R$* , in: H. Heyer, A. Mukherjea (Eds.), *Probability Measures on Groups and Related Structures XI*, Proceedings of Oberwolfach, 292-304 (1994).
  24. Trimèche, K.: *Transformation integrale de Weyl et theoreme de Paley-Wiener associes a un operateur differentiel singulier sur  $(0, \infty)$* , J. Math. Pures Appl. **60**, 51-98 (1981).
  25. Trimèche, K.: *Generalized Harmonic Analysis and Wavelet Packets*, Gordon and Breach Science Publishers, 2001.
  26. Villani, M.: *Riesz transforms associated to Bessel operators*, Illinois J. Math. **52**, 77-89 (2008).
  27. Yang, Da., Yang, Do.: *Real-variable characterizations of Hardy spaces associated with Bessel operators*, Anal. Appl. (Singap.) **9**, 345-368 (2011).
  28. Watson, G.N.: *A treatise on theory of Bessel functions*, Cambridge University Press, Cambridge, 1966.