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Characterizations of anisotropic Lipschitz functions via the commutators of anisotropic maximal function in total anisotropic Morrey spaces

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Abstract. *We shall give necessary and sufficient conditions for the boundedness of the anisotropic maximal commutators M_b^d and the commutators of the anisotropic maximal operator $[b, M^d]$ in total Morrey spaces $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ when b belongs to anisotropic Lipschitz spaces $\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$, whereby some new characterizations for certain subclasses of anisotropic Lipschitz spaces $\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$ are obtained.*

Keywords. Total anisotropic Morrey spaces, anisotropic maximal function, anisotropic fractional maximal function, commutator, anisotropic Lipschitz spaces

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1 Introduction

The aim of this paper is to study anisotropic maximal commutators M_b^d and commutators of the anisotropic maximal operator $[b, M^d]$ in total anisotropic Morrey spaces $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ when b belongs to anisotropic Lipschitz spaces $\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$.

Let $\mathbb{R}^n$ be the n-dimension Euclidean space with the norm $|x|$ for each $x \in \mathbb{R}^n$, S^{n-1} denotes the unit sphere on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ and $r > 0$, let $\mathcal{E}(x, r)$ denote the open ball centered at x of radius r and ${}^{\complement}\mathcal{E}(x, r)$ denote the set $\mathbb{R}^n \backslash \mathcal{E}(x, r)$. Let $d = (d_1, \ldots, d_n)$, $d_i \geq 1$, $i = 1, \ldots, n$, $|d| = \sum_{i=1}^n d_i$ and $t^d x \equiv \left(t^{d_1} x_1, \ldots, t^{d_n} x_n\right)$. By [5,7], the function $F(x, \rho) = \sum_{i=1}^n x_i^2 \rho^{-2d_i}$, considered for any fixed $x \in \mathbb{R}^n$, is a decreasing one with respect to $\rho > 0$ and the equation $F(x, \rho) = 1$ is uniquely solvable. This unique solution will be denoted by $\rho(x)$. It is a simple matter to check that $\rho(x - y)$ defines a distance between any

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two points $x,\ y \in \mathbb{R}^n$. Thus $\mathbb{R}^n$, endowed with the metric ρ, defines a homogeneous metric space ([5–7]). The balls with respect to ρ, centered at x of radius r, are just the ellipsoids

$$\mathcal{E}_d(x,r) = \left\{ y \in \mathbb{R}^n : \frac{(y_1 - x_1)^2}{r^{2d_1}} + \cdots + \frac{(y_n - x_n)^2}{r^{2d_n}} < 1 \right\},$$

with the Lebesgue measure $|\mathcal{E}_d(x,r)| = v_n r^{|d|}$, where v_n is the volume of the unit ball in $\mathbb{R}^n$. Let also $\Pi_d(x,r) = \{y \in \mathbb{R}^n : \max_{1\le i\le n} |x_i - y_i|^{1/d_i} < r\}$ denote the parallelopiped, ${}^{\complement}\mathcal{E}_d(x,r) = \mathbb{R}^n \setminus \mathcal{E}_d(x,r)$ be the complement of $\mathcal{E}_d(0,r)$. If $d = \mathbf{1} \equiv (1,\ldots,1)$, then clearly $\rho(x) = |x|$ and $\mathcal{E}_{\mathbf{1}}(x,r) = \mathcal{E}(x,r)$. Note that in the standard parabolic case $d = (1,\ldots,1,2)$ we have

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + x_n^2}}{2}}, \qquad x = (x', x_n).$$

Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. The anisotropic fractional maximal operator M^d_α is given by

$$M^d_\alpha f(x) = \sup_{t>0} |\mathcal{E}(x,t)|^{-1+\frac{\alpha}{|d|}} \int_{\mathcal{E}(x,t)} |f(y)| dy, \ \ 0 \le \alpha < |d|,$$

where $|\mathcal{E}(x,t)|$ is the Lebesgue measure of the ellipsoid $\mathcal{E}(x,t)$. If $\alpha = 0$, then $M^d \equiv M^d_0$ is the anisotropic Hardy-Littlewood maximal operator. If $d = \mathbf{1}$, then $M_\alpha \equiv M^d_\alpha$ is the fractional maximal operator and $M \equiv M^d$ is the classical Hardy-Littlewood maximal operator.

The anisotropic maximal commutator of M^d with a locally integrable function b is defined by

$$M^d_b f(x) = \sup_{t>0} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} |b(x) - b(y)||f(y)| dy.$$

If $d = \mathbf{1}$, then $M_b \equiv M^d_b$ is the maximal commutator. The operators M^d_α and M^d_b play an important role in real and harmonic analysis (see, for example [28, 29]).

On the other hand, we can define the (nonlinear) commutator of the anisotropic maximal operator M^d with a locally integrable function b by

$$[b, M^d] f(x) f = b(x) M^d f(x) - M^d(bf)(x).$$

Obviously, operators M^d_b and $[b, M^d]$ essentially differ from each other since M^d_b is positive and sublinear and $[b, M^d]$ is neither positive nor sublinear.

The operators M, $[b, M]$ and M_b play an important role in real and harmonic analysis and applications (see, for instance [1–4, 13, 14, 16–18, 21, 22, 24, 25, 30]).

In 1978, Janson [20] gave some characterizations of the Lipschitz space $\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$ via commutator $[b, T]$ and the author proved that $b \in \dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$ if and only if $[b, T]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $1 < p < n/\beta$, $1/p - 1/q = \beta/n$ and T is the classical singular integral operator (see also [26]).

Morrey spaces, introduced by C. B. Morrey [23], play important roles in the regularity theory of PDE, including heat equations and Navier-Stokes equations. In [13] Guliyev introduce a variant of Morrey spaces called total Morrey spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$, $0 < p < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. In [1] the authors was consider the total anisotropic Morrey spaces $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$, give basic properties of the spaces $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ and study some embeddings into the Morrey space $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$. Was also given necessary and sufficient conditions for the boundedness of the anisotropic maximal commutator operator M_b^d and commutator of anisotropic maximal operator $[b, M^d]$ on $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$. Was obtained some new characterizations for certain subclasses of $BMO(\mathbb{R}^n)$.

The aim of this paper is to give necessary and sufficient conditions for the boundedness of the anisotropic maximal commutator operator M_b^d and commutator of anisotropic maximal operator $[b, M^d]$ on $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ when b belongs to anisotropic Lipschitz spaces $\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$. New characterizations of some subclasses of anisotropic Lipschitz spaces $\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$ are obtained.

The structure of the paper is as follows. In Section 2 we give some theorems about the boundedness of anisotropic fractional maximal operator M^d_α on the total anisotropic Morrey spaces $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$. In Section 3 we find necessary and sufficient conditions for the boundedness of the anisotropic maximal commutator M_b^d on $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ spaces. In Section 4 we find necessary and sufficient conditions for the boundedness of the commutator of anisotropic maximal operator $[b, M^d]$ on $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ spaces.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Preliminaries

Definition 2.1 *Let $d = (d_1, \ldots, d_n)$, $d_i \geq 1$, $i = 1, \ldots, n$. Let also $0 < p < \infty$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $[t]_1 = \min\{1, t\}$, $t > 0$. We denote by $L^d_{p,\lambda}(\mathbb{R}^n)$ the anisotropic Morrey space, by $\widetilde{L}^d_{p,\lambda}(\mathbb{R}^n)$ the modified anisotropic Morrey space [10,12], and by $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ the total anisotropic Morrey space [1,13] the set of all classes of locally integrable functions f with the finite norms*

$$\|f\|_{L^d_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} t^{-\frac{\lambda}{p}} \|f\|_{L_p(\mathcal{E}(x,t))},$$

$$\|f\|_{\widetilde{L}^d_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{L_p(\mathcal{E}(x,t))},$$

$$\|f\|_{L^d_{p,\lambda,\mu}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{L_p(\mathcal{E}(x,t))},$$

respectively.

Definition 2.2 *Let $d = (d_1, \ldots, d_n)$, $d_i \geq 1$, $i = 1, \ldots, n$. Let also $0 < p < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. We define the weak anisotropic Morrey space $WL^d_{p,\lambda}(\mathbb{R}^n)$, the weak modified anisotropic Morrey space $W\widetilde{L}^d_{p,\lambda}(\mathbb{R}^n)$ [10,12] and the weak total anisotropic Morrey space $WL^d_{p,\lambda,\mu}(\mathbb{R}^n)$ [1,13] as the set of all locally integrable functions f with finite norms*

$$\|f\|_{WL^d_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} t^{-\frac{\lambda}{p}} \|f\|_{WL_p(\mathcal{E}(x,t))},$$

$$\|f\|_{W\widetilde{L}^d_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{WL_p(\mathcal{E}(x,t))},$$

$$\|f\|_{WL^d_{p,\lambda,\mu}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{WL_p(\mathcal{E}(x,t))},$$

respectively.

Lemma 2.1 *If* $0 < p < \infty$, $0 \le \mu \le \lambda \le \gamma$, *then*

$$L^P_{p,\lambda,\mu}(\mathbb{R}^n) = L^P_{p,\lambda}(\mathbb{R}^n) \cap L^P_{p,\mu}(\mathbb{R}^n)$$

and

$$\|f\|_{L^P_{p,\lambda,\mu}(\mathbb{R}^n)} = \max\left\{\|f\|_{L^P_{p,\lambda}}, \|f\|_{L^P_{p,\mu}}\right\}.$$

Proof. Let $f \in L^P_{p,\lambda,\mu}(\mathbb{R}^n)$ and $0 \le \mu \le \lambda \le \gamma$. Then

$$\begin{aligned}
&\|f\|_{L^P_{p,\lambda}} = \|f\|_{L^{p,\lambda,\lambda}} \\
&= \max\left\{\sup_{x\in\mathbb{R}^n, 0<t\le 1} t^{-\frac{\lambda}{p}} \|f\|_{L^p(\mathcal{E}(x,t))}, \sup_{x\in\mathbb{R}^n, t>1} t^{-\frac{\mu}{p}} t^{\frac{\mu-\lambda}{p}} \|f\|_{L^p(\mathcal{E}(x,t))}\right\} \\
&\le \|f\|_{L^P_{p,\lambda,\mu}(\mathbb{R}^n)}
\end{aligned}$$

and

$$\begin{aligned}
&\|f\|_{L^P_{p,\mu}} = \|f\|_{L^{p,\mu,\mu}} \\
&= \max\left\{\sup_{x\in\mathbb{R}^n, 0<t\le 1} t^{\frac{\lambda-\mu}{p}} t^{-\frac{\lambda}{p}} \|f\|_{L^p(\mathcal{E}(x,t))}, \sup_{x\in\mathbb{R}^n, t>1} t^{-\frac{\mu}{p}} \|f\|_{L^p(\mathcal{E}(x,t))}\right\} \\
&\le \|f\|_{L^P_{p,\lambda,\mu}(\mathbb{R}^n)}.
\end{aligned}$$

Therefore, $f \in L^P_{p,\lambda}(\mathbb{R}^n) \cap L^P_{p,\mu}(\mathbb{R}^n)$ and $\max\left\{\|f\|_{L^P_{p,\lambda}}, \|f\|_{L^P_{p,\mu}}\right\} \le \|f\|_{L^P_{p,\lambda,\mu}(\mathbb{R}^n)}$.

Now let $f \in L^P_{p,\lambda}(\mathbb{R}^n) \cap L^P_{p,\mu}(\mathbb{R}^n)$. Then

$$\begin{aligned}
\|f\|_{L^P_{p,\lambda,\mu}} &= \sup_{x\in\mathbb{R}^n, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{L^p(\mathcal{E}(x,t))} \\
&= \max\left\{\sup_{x\in\mathbb{R}^n, 0<t\le 1} t^{-\frac{\lambda}{p}} \|f\|_{L^p(\mathcal{E}(x,t))}, \sup_{x\in\mathbb{R}^n, t>1} t^{-\frac{\mu}{p}} \|f\|_{L^p(\mathcal{E}(x,t))}\right\} \\
&\le \max\left\{\|f\|_{L^P_{p,\lambda}}, \|f\|_{L^P_{p,\mu}}\right\}.
\end{aligned}$$

Therefore, $f \in L^P_{p,\lambda,\mu}(\mathbb{R}^n)$ and $\|f\|_{L^P_{p,\lambda,\mu}(\mathbb{R}^n)} \le \max\left\{\|f\|_{L^P_{p,\lambda}}, \|f\|_{L^P_{p,\mu}}\right\}$.

The following lemma is a weak version of Lemma 2.1 and is proved similarly.

Lemma 2.2 *If* $0 < p < \infty$, $0 \le \mu \le \lambda \le \gamma$, *then*

$$WL^P_{p,\lambda,\mu}(\mathbb{R}^n) = WL^P_{p,\lambda}(\mathbb{R}^n) \cap WL^P_{p,\mu}(\mathbb{R}^n)$$

and

$$\|f\|_{WL^P_{p,\lambda,\mu}} = \max\left\{\|f\|_{WL^P_{p,\lambda}}, \|f\|_{WL^P_{p,\mu}}\right\}.$$

Remark 2.1 Let $0 < p < \infty$. If $\lambda < 0$ or $\mu > \gamma$, then

$$L^P_{p,\lambda,\mu}(\mathbb{R}^n) = WL^P_{p,\lambda,\mu}(\mathbb{R}^n) = \Theta(\mathbb{R}^n).$$

The following local estimate is valid (see also [11]).

Lemma 2.3 *[11, Lemma 4.1] Let $0 \le \alpha < |d|$, $1 \le p < \frac{|d|}{\alpha}$, and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{|d|}$. Then, for $p > 1$ the inequality*

$$\|M^d_\alpha f\|_{L_q(\mathcal{E}(x,r))} \lesssim r^{\frac{|d|}{q}} \sup_{t>2r} t^{-\frac{|d|}{q}} \|f\|_{L_p(\mathcal{E}(x,t))} \tag{2.1}$$

holds for all $\mathcal{E}(x,r)$ and for all $f \in L^{\rm loc}_p(\mathbb{R}^n)$.

Moreover if $p = 1$, then the inequality

$$\|M^d_\alpha f\|_{WL_q(\mathcal{E}(x,r))} \lesssim r^{\frac{|d|}{q}} \sup_{t>2r} t^{-\frac{|d|}{q}} \|f\|_{L_1(\mathcal{E}(x,t))} \tag{2.2}$$

holds for all $\mathcal{E}(x,r)$ and for all $f \in L^{\rm loc}_1(\mathbb{R}^n)$.

The following is Spanne's type result for fractional anisotropic maximal operators in total anisotropic Morrey spaces (see, for example, [11]).

Theorem 2.1 *(Spanne type result) [18, Theorem 2.1] Let $1 \le p < \infty$, $0 \le \mu \le \lambda < |d|$, $0 \le \alpha < \frac{|d|-\lambda}{p}$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{|d|}$.*

1. If $p > 1$, $f \in L^d_{p,\lambda,\mu}(\mathbb{R}^n)$, then $M^d_\alpha f \in L^d_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}(\mathbb{R}^n)$ and

$$\|M^d_\alpha f\|_{L^d_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}} \le C_{p,q,\lambda,\mu} \|f\|_{L^d_{p,\lambda,\mu}}, \tag{2.3}$$

where $C_{p,q,\lambda,\mu}$ depends only on p,q,λ,μ and n.

2. If $p = 1$, $f \in L^d_{1,\lambda,\mu}(\mathbb{R}^n)$, then $Mf \in WL^d_{q,\lambda q,\mu q}(\mathbb{R}^n)$ and

$$\|M^d_\alpha f\|_{WL^d_{q,\lambda q,\mu q}} \le C_{q,\lambda,\mu} \|f\|_{L^d_{1,\lambda,\mu}}, \tag{2.4}$$

where $C_{q,\lambda,\mu}$ is independent of f.

The following is Adam's type result for fractional anisotropic maximal operators in total anisotropic Morrey spaces (see, for example, [10]).

Theorem 2.2 *(Adams type result) [18, Theorem 2.2] Let $1 \le p < \infty$, $0 \le \mu \le \lambda < |d|$, $0 \le \alpha < \frac{|d|-\lambda}{p}$.*

1) If $1 < p < \frac{|d|-\lambda}{\alpha}$, then condition $\frac{\alpha}{|d|-\mu} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{|d|-\lambda}$ is necessary and sufficient for the boundedness of the operator M^d_α from $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ to $L^d_{q,\lambda,\mu}(\mathbb{R}^n)$.

2) If $p = 1 < \frac{|d|-\lambda}{\alpha}$, then condition $\frac{\alpha}{|d|-\mu} \le 1 - \frac{1}{q} \le \frac{\alpha}{|d|-\lambda}$ is necessary and sufficient for the boundedness of the operator M^d_α from $L^d_{1,\lambda,\mu}(\mathbb{R}^n)$ to $WL^d_{q,\lambda,\mu}(\mathbb{R}^n)$.

3) If $\frac{|d|-\lambda}{\alpha} \le p \le \frac{|d|-\mu}{\alpha}$, then the operator M^d_α is bounded from $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ to $L_\infty(\mathbb{R}^n)$.

3 Anisotropic maximal commutator in total anisotropic Morrey spaces

In this section, as an application of the theorems of the previous section we consider the boundedness of the anisotropic maximal commutator M^d_b on total anisotropic Morrey spaces when b belongs to an anisotropic Lipschitz space, by which some new characterizations of the anisotropic Lipschitz spaces are given. Such a characterization was given in [31] for the boundedness of M_b on Lebesgue and Morrey spaces.

Definition 3.1 *Let $0 < \beta < 1$, we say a function b belongs to the anisotropic Lipschitz space $\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$ if there exists a constant C such that for all $x, y \in \mathbb{R}^n$,*

$$|b(x) - b(y)| \le C\rho(x-y)^{\beta}.$$

The smallest such constant C is called the $\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$ norm of b and is denoted by $\|b\|_{\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)}$.

To prove the theorems, we need auxiliary results. The first one is the following characterizations of Lipschitz space, which is due to DeVore and Sharply [8].

Lemma 3.1 *Let $0 < \beta < 1$, we have*

$$\|f\|_{\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)} \approx \sup_{\mathcal{E}} \frac{1}{|\mathcal{E}|^{1+\beta/n}} \int_{\mathcal{E}} |f(x) - f_{\mathcal{E}}|dx,$$

where $f_{\mathcal{E}} = \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} f(y)dy$.

If $b \in \dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$ and $\lambda > 0$, then the function b_{λ^d} is defined by $b_{\lambda^d}(x) = b(\lambda^d x)$ is also in $\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$ and

$$\|b_{\lambda^d}\|_{\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)} = \|b\|_{\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)}. \tag{3.1}$$

See, for example, [9, Proposition 7.1.2 (6)].

Lemma 3.2 *Let $0 < \beta < 1$ and $b \in \dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$, then the following pointwise estimate holds:*

$$M_b^d f(x) \lesssim \|b\|_{\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)} M_\beta^d f(x).$$

Proof. If $b \in \dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$, then

$$\begin{aligned} M_b^d(f)(x) &\approx \sup_{\mathcal{E} \ni x} |\mathcal{E}|^{-1} \int_{\mathcal{E}} |b(x) - b(y)||f(y)|dy \\ &\lesssim \|b\|_{\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)} \sup_{\mathcal{E} \ni x} |\mathcal{E}|^{-1+\frac{\beta}{|d|}} \int_{\mathcal{E}} |f(y)|dy \\ &\approx \|b\|_{\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)} M_\beta^d f(x). \end{aligned}$$

The following is Spanne's type result for the anisotropic maximal commutator operators in total anisotropic Morrey spaces.

Theorem 3.1 *(Spanne type result) Let $0 < \beta < 1$ and $b \in \dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$. Let also $0 \le \mu \le \lambda < |d|$, $1 < p < \frac{|d|-\lambda}{\beta}$ and $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{|d|}$.*

If $f \in L^d_{p,\lambda,\mu}(\mathbb{R}^n)$, then $M_b^d f \in L^d_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}(\mathbb{R}^n)$ and

$$\|M_b^d f\|_{L^d_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}} \le C_{p,q,\lambda,\mu} \|b\|_{\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)} \|f\|_{L^d_{p,\lambda,\mu}}, \tag{3.2}$$

where $C_{p,q,\lambda,\mu}$ depends only on p, q, λ, μ, d and n.

Proof. Let $1 < p < \infty$. From Theorem 2.1 and Lemma 3.2 we get

$$\begin{aligned} \|M_b^d f\|_{L^d_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}} &\lesssim \|b\|_{\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)} \|M_\beta^d f\|_{L^d_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}} \\ &\lesssim \|b\|_{\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)} \|f\|_{L^d_{p,\lambda,\mu}}, \end{aligned}$$

which implies that the operator M_α^d is bounded from $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ to $L^d_{q,\frac{\lambda q}{p},\frac{\mu q}{p}}(\mathbb{R}^n)$.

The following is an Adams type result for the anisotropic maximal commutator operators in total anisotropic Morrey spaces.

Theorem 3.2 *(Adams type result) Let $0 < \beta < 1$ and $b \in \dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$. Let also $0 \le \mu \le \lambda < |d| - \beta$ and $1 < p < \frac{|d|-\lambda}{\beta}$.*

Then condition $\frac{\beta}{|d|-\mu} \le \frac{1}{p} - \frac{1}{q} \le \frac{\beta}{|d|-\lambda}$ is necessary and sufficient for the boundedness of the operator M_b^d from $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ to $L^d_{q,\lambda,\mu}(\mathbb{R}^n)$.

Proof. Sufficiency follows from Theorem 2.2 and Lemma 3.2.

Now we will prove the necessity.

Let $1 < p < \frac{|d|-\lambda}{\alpha}$, $\frac{\alpha}{|d|-\mu} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{|d|-\lambda}$, $b \in \dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$, $f \in L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ and assume that M_b^d is bounded from $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ to $L^d_{q,\lambda,\mu}(\mathbb{R}^n)$.

Note that

$$M_b^d f_{t^d}(x) = M^d_{b_{\frac{1}{t^d}}} f(tx),$$

$$\begin{aligned}
\left\| M_b^d f_{t^d} \right\|_{L^d_{q,\lambda,\mu}} &= \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\frac{\lambda}{p}} \, [1/r]_1^{\frac{\mu}{p}} \, \|M^d_{b_{\frac{1}{t^d}}} f(t^d \cdot)\|_{L_q(\mathcal{E}(x,r))} \\
&= t^{-\frac{n}{q}} \sup_{r>0} \Big(\frac{[tr]_1}{[r]_1}\Big)^{\lambda/q} \sup_{r>0} \Big(\frac{[1/r]_1}{[1/(tr)]_1}\Big)^{\mu/q} \sup_{x\in\mathbb{R}^n,\, r>0} [tr]_1^{-\frac{\lambda}{p}} \, [1/(tr)]_1^{\frac{\mu}{p}} \, \|M^d_{b_{\frac{1}{t^d}}} f\|_{L_q(\mathcal{E}(t^d x,tr))} \\
&= t^{-\frac{n}{q}} \, [t]_{1,+}^{\frac{\lambda}{q}} \, [1/t]_{1,+}^{-\frac{\mu}{q}} \left\| M^d_{b_{\frac{1}{t^d}}} f \right\|_{L^d_{q,\lambda,\mu}}.
\end{aligned}$$

By the boundedness of M_β^d from $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ to $L^d_{q,\lambda,\mu}(\mathbb{R}^n)$ and from the equality (3.1) we get

$$\begin{aligned}
\left\| M_b^d f \right\|_{L^d_{q,\lambda,\mu}} &= t^{\frac{|d|}{q}} \, [t]_{1,+}^{-\frac{\lambda}{q}} \, [1/t]_{1,+}^{\frac{\mu}{q}} \, \|M^d_{b_{\frac{1}{t^d}}} f_t\|_{L^d_{q,\lambda,\mu}} \\
&\lesssim t^{\frac{|d|}{q}} \, [t]_{1,+}^{-\frac{\lambda}{q}} \, [1/t]_{1,+}^{\frac{\mu}{q}} \, \|f_{t^d}\|_{L^d_{p,\lambda,\mu}} \\
&= t^{\frac{|d|}{q} - \frac{n}{p}} \, [t]_{1,+}^{\frac{\lambda}{p} - \frac{\lambda}{q}} \, [1/t]_{1,+}^{-\frac{\mu}{p} + \frac{\mu}{q}} \, \|f\|_{L^d_{p,\lambda,\mu}} \\
&= [t]_{1,+}^{-\frac{|d|-\lambda}{p} + \frac{|d|-\lambda}{q}} \, [1/t]_{1,+}^{\frac{|d|-\mu}{p} - \frac{|d|-\mu}{q}} \, \|f\|_{L^d_{p,\lambda,\mu}}.
\end{aligned}$$

Since $L^d_{p,\lambda,\mu}(\mathbb{R}^n) = L^d_{p,\mu,\lambda}(\mathbb{R}^n)$, we can assume that $\lambda < \mu$, and then $\mu = \lambda$, $\lambda = \mu$.

If $\frac{1}{p} < \frac{1}{q} + \frac{\alpha}{|d|-\lambda}$, then by letting $t \to 0$ we have $\left\| M^d_{b,\alpha} f \right\|_{L^d_{q,\lambda,\mu}} = 0$ for all $f \in L^d_{p,\lambda,\mu}(\mathbb{R}^n)$.

As well as if $\frac{1}{p} > \frac{1}{q} + \frac{\alpha}{|d|-\mu}$, then at $t \to \infty$ we obtain $\left\| M^d_{b,\alpha} f \right\|_{L^d_{q,\lambda,\mu}} = 0$ for all $f \in L^d_{p,\lambda,\mu}(\mathbb{R}^n)$.

Therefore $\frac{\alpha}{|d|-\mu} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{|d|-\lambda}$.

Theorem 3.3 *Let $0 < \beta < 1$ and $b \in L_1^{\rm loc}(\mathbb{R}^n)$. Let also $0 \le \mu \le \lambda < |d|$, $1 < p < \frac{|d|-\lambda}{\beta}$ and $\frac{\beta}{|d|-\mu} \le \frac{1}{p} - \frac{1}{q} \le \frac{\beta}{|d|-\lambda}$. Then, the following statements are equivalent:*

(i) $b \in \dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$.

(ii) *The operator M_b^d is bounded from $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ to $L^d_{q,\lambda,\mu}(\mathbb{R}^n)$.*

Proof. $(i) \Rightarrow (ii)$. Suppose that $b \in \dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$. Combining Lemma 3.2 and Theorems 2.2, we get

$$\|M_b^d f\|_{L^d_{q,\lambda,\mu}} \lesssim \|b\|_{\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)} \|M_\beta^d f\|_{L_{q,\lambda,\mu}} \lesssim \|b\|_{\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)} \|f\|_{L^d_{p,\lambda,\mu}}.$$

$(ii) \Rightarrow (i)$. Assume that M_b^d is bounded from $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ to $L^d_{q,\lambda,\mu}(\mathbb{R}^n)$. Let $\mathcal{E} = \mathcal{E}(x,r)$ be a fixed ball. We consider $f = \chi_{\mathcal{E}}$. It is easy to compute that

$$\begin{aligned}
\|\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}} &\approx \sup_{y\in\mathbb{R}^n, t>0} \Big([t]_1^{-\lambda}\, [1/t]_1^{\mu} \int_{\mathcal{E}(y,t)} \chi_{\mathcal{E}}(z)dz\Big)^{\frac{1}{p}} \\
&= \sup_{y\in\mathbb{R}^n, t>0} \Big(|\mathcal{E}(y,t)\cap\mathcal{E}|\, [t]_1^{-\lambda}\, [1/t]_1^{\mu}\Big)^{\frac{1}{p}} \\
&= \sup_{\mathcal{E}(y,t)\subseteq\mathcal{E}} \Big(|\mathcal{E}(y,t)|\, [t]_1^{-\lambda}\, [1/t]_1^{\mu}\Big)^{\frac{1}{p}} = r^{\frac{|d|}{p}}\, [r]_1^{-\frac{\lambda}{p}}\, [1/r]_1^{\frac{\mu}{p}}. \qquad (3.3)
\end{aligned}$$

On the other hand, since

$$M_b^d(\chi_{\mathcal{E}})(x) \gtrsim \frac{1}{|\mathcal{E}|} \int_{\mathcal{E}} |b(z) - b_{\mathcal{E}}|dz \quad \text{for all} \quad x \in B,$$

we have

$$\begin{aligned}
\|M_b^d(\chi_{\mathcal{E}})\|_{L^d_{q,\lambda,\mu}} &\approx \sup_{\mathcal{E}(y,t)\subseteq\mathcal{E}} \Big(\,[t]_1^{-\lambda}\, [1/t]_1^{\mu} \int_{\mathcal{E}(y,t)} |M_b(\chi_{\mathcal{E}})(z)|^q dz\Big)^{\frac{1}{q}} \\
&\gtrsim r^{\frac{n}{q}}\, [r]_1^{-\frac{\lambda}{q}}\, [1/r]_1^{\frac{\mu}{q}} \frac{1}{|\mathcal{E}|} \int_B |b(z) - b_B|dz \\
&= r^{\beta+\frac{n}{q}}\, [r]_1^{-\frac{\lambda}{q}}\, [1/r]_1^{\frac{\mu}{q}} \frac{1}{|\mathcal{E}|^{1+\beta/|d|}} \int_{\mathcal{E}} |b(z) - b_{\mathcal{E}}|dz. \qquad (3.4)
\end{aligned}$$

Since $L^d_{p,\lambda,\mu}(\mathbb{R}^n) = L^d_{p,\mu,\lambda}(\mathbb{R}^n)$, we can assume that $\lambda < \mu$, and then $\mu = \lambda$, $\lambda = \mu$. On the other hand, by assumption

$$\|M_b^d(\chi_{\mathcal{E}})\|_{L^d_{q,\lambda,\mu}} \lesssim \|\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}},$$

by (3.3) and (3.4), we get that

$$\begin{aligned}
\frac{1}{|\mathcal{E}|^{1+\beta/|d|}} \int_{\mathcal{E}} |b(z) - b_{\mathcal{E}}|dz &\lesssim r^{-\beta-\frac{|d|}{q}}\, [r]_1^{\frac{\lambda}{q}}\, [1/r]_1^{-\frac{\mu}{q}}\, \|M_b^d(\chi_{\mathcal{E}})\|_{L^d_{q,\lambda,\mu}} \\
&\lesssim r^{-\beta-\frac{|d|}{q}}\, [r]_1^{\frac{\lambda}{q}}\, [1/r]_1^{-\frac{\mu}{q}}\, \|\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}} \\
&\lesssim r^{-\beta-\frac{|d|}{q}}\, [r]_1^{\frac{\lambda}{q}}\, [1/r]_1^{-\frac{\mu}{q}}\, r^{\frac{|d|}{p}}\, [r]_1^{-\frac{\lambda}{p}}\, [1/r]_1^{\frac{\mu}{p}} \\
&\lesssim [r]_1^{-\beta+\frac{|d|-\lambda}{p}-\frac{|d|-\lambda}{q}}\, [1/r]_1^{\beta-\frac{|d|-\mu}{p}+\frac{|d|-\mu}{q}} \\
&\lesssim 1.
\end{aligned}$$

From Theorem 3.3 in the case $\lambda = \mu$ or $\mu = 0$ we get the following corollaries.

Corollary 3.1 *[31] Let* $0 < \beta < 1$ *and* $b \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. *Let also* $0 \le \lambda < |d| - \beta$, $1 < p < \frac{|d|-\lambda}{\beta}$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{|d|-\lambda}$. *Then, the following statements are equivalent:*

(i) $b \in \dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$.

(ii) *The operator* M_b^d *is bounded from* $L^d_{p,\lambda}(\mathbb{R}^n)$ *to* $L^d_{q,\lambda}(\mathbb{R}^n)$.

Corollary 3.2 *Let* $0 < \beta < 1$ *and* $b \in L_1^{\rm loc}(\mathbb{R}^n)$. *Let also* $0 \le \lambda < |d| - \beta$, $1 < p < \frac{|d|-\lambda}{\beta}$ *and* $\frac{\beta}{|d|} \le \frac{1}{p} - \frac{1}{q} \le \frac{\beta}{|d|-\lambda}$. *Then, the following statements are equivalent:*

(i) $b \in \dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$.

(ii) *The operator* M_b^d *is bounded from* $\widetilde{L}^d_{p,\lambda}(\mathbb{R}^n)$ *to* $\widetilde{L}^d_{q,\lambda}(\mathbb{R}^n)$.

Remark 3.1 Note that Corollary 3.2 is new.

4 Commutator of anisotropic maximal operator in total anisotropic Morrey spaces

In this section we find necessary and sufficient conditions for the boundedness of the commutator of anisotropic maximal operator $[b, M^d]$ in the $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ spaces.

For a function b defined on $\mathbb{R}^n$, we denote

$$b^-(x) := \begin{cases} 0\,, & \text{if } b(x) \ge 0 \\ |b(x)|, & \text{if } b(x) < 0 \end{cases}$$

and $b^+(x) := |b(x)| - b^-(x)$. Obviously, $b^+(x) - b^-(x) = b(x)$.

The following relations between $[b, M^d_\alpha]$ and $M^d_{b,\alpha}$ are valid :

Let b be any non-negative locally integrable function. Then for all $f \in L_1^{\rm loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ the following inequality is valid

$$\begin{aligned} \big|[b, M^d]f(x)\big| &= \big|b(x)M^d f(x) - M^d(bf)(x)\big| \\ &= \big|M^d(b(x)f)(x) - M^d(bf)(x)\big| \\ &\le M^d(|b(x) - b|f)(x) = M^d_b f(x). \end{aligned} \tag{4.1}$$

Applying Theorem 3.3, we obtain the following result.

Theorem 4.1 *Let* $0 < \beta < 1$ *and* $b \in L_1^{\rm loc}(\mathbb{R}^n)$. *Let also* $0 \le \mu \le \lambda < |d|$, $1 < p < \frac{|d|-\lambda}{\beta}$ *and* $\frac{\beta}{|d|-\mu} \le \frac{1}{p} - \frac{1}{q} \le \frac{\beta}{|d|-\lambda}$. *Then, the following statements are equivalent:*

(i) $b \in \dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$ *and* $b \ge 0$.

(ii) *The operator* $[b, M^d]$ *is bounded from* $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ *to* $L^d_{q,\lambda,\mu}(\mathbb{R}^n)$.

(iii) *There exists a constant* $C > 0$ *such that*

$$\sup_{\mathcal{E}} |\mathcal{E}|^{-\frac{\beta}{|d|}} \frac{\|\big(b - M^d_{\mathcal{E}}(b)\big)\,\chi_{\mathcal{E}}\|_{L^d_{q,\lambda,\mu}}}{\|\chi_{\mathcal{E}}\|_{L^d_{q,\lambda,\mu}}} \le C. \tag{4.2}$$

Proof. $(i) \Rightarrow (ii)$. Suppose that $b \in \dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$. Combining Theorems 2.2 and 3.3, and inequality (4.1), we get

$$\|[b, M^d]f\|_{L^d_{q,\lambda,\mu}} \le \|M^d_b f\|_{L^d_{q,\lambda,\mu}} \lesssim \|b\|_{\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)}\, \|M^d_\beta f\|_{L^d_{q,\lambda,\mu}} \lesssim \|f\|_{L^d_{p,\lambda,\mu}}.$$

$(ii) \Rightarrow (iii)$. Assume that $[b, M^d]$ is bounded from $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$ to $L^d_{q,\lambda,\mu}(\mathbb{R}^n)$. For a given ellipsoid $\mathcal{E}$, we define the following local maximal function:

$$M^d_{\mathcal{E}} f(x) := \sup_{\mathcal{E} \supseteq \mathcal{E}' \ni x} |\mathcal{E}'|^{-1} \int_{\mathcal{E}'} |f(y)|\, dy,$$

where the supremum is taken over all ellipsoid $\mathcal{E}'$ such that $x \in \mathcal{E}' \subseteq \mathcal{E}$.

Since

$$M^d(b\chi_{\mathcal{E}})\chi_{\mathcal{E}} = M^d_{\mathcal{E}}(b)\,\chi_{\mathcal{E}} \quad \text{and} \quad M^d(\chi_{\mathcal{E}})\chi_{\mathcal{E}} = M^d_{\mathcal{E}}\chi_{\mathcal{E}} = \chi_{\mathcal{E}},$$

we have

$$\begin{aligned}\big(b - M^d_{\mathcal{E}}(b)\big)\,\chi_{\mathcal{E}} &= \big(b - M^d_{\mathcal{E}}(b)\big)\,\chi_{\mathcal{E}} \\ &= \big(bM^d(\chi_{\mathcal{E}}) - M^d(b\chi_{\mathcal{E}})\big)\,\chi_{\mathcal{E}} = [b, M^d](\chi_{\mathcal{E}}).\end{aligned} \tag{4.3}$$

By this and $[b, M^d_\alpha] : L^d_{p,\lambda,\mu}(\mathbb{R}^n) \to L^d_{q,\lambda,\mu}(\mathbb{R}^n)$, we obtain

$$\|\big(b - M^d_{\mathcal{E}}(b)\big)\,\chi_{\mathcal{E}}\|_{L_{q,\lambda,\mu}} \le \|[b, M^d]\chi_{\mathcal{E}}\|_{L^d_{q,\lambda,\mu}(\mathbb{R}^n)} \lesssim \|\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}(\mathbb{R}^n)}.$$

Thus from (3.3) we get

$$\begin{aligned}|\mathcal{E}|^{-\frac{\beta}{|d|}}\,\frac{\|\big(b - M^d_{\mathcal{E}}(b)\big)\,\chi_{\mathcal{E}}\|_{L_{q,\lambda,\mu}}}{\|\chi_{\mathcal{E}}\|_{L_{q,\lambda,\mu}}} &\le |\mathcal{E}|^{-\frac{\beta}{|d|}}\,\frac{\|\chi_{\mathcal{E}}\|_{L^d_{p,\lambda,\mu}(\mathbb{R}^n)}}{\|\chi_{\mathcal{E}}\|_{L^d_{q,\lambda,\mu}(\mathbb{R}^n)}} \\ &\approx r^{-\beta + \frac{|d|}{p} - \frac{n}{q}}\,[r]_1^{\frac{\lambda}{q} - \frac{\lambda}{p}}\,[1/r]_1^{\frac{\mu}{p} - \frac{\mu}{q}} \\ &\approx [r]_1^{-\beta + \frac{|d|-\lambda}{p} - \frac{|d|-\lambda}{q}}\,[1/r]_1^{\beta - \frac{|d|-\mu}{p} + \frac{|d|-\mu}{q}} \\ &\lesssim 1.\end{aligned}$$

$(iii) \Rightarrow (i)$. Assume that (4.2) is valid.

Now, let us prove $b \in \dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$ and $b \ge 0$. For any ellipsoid $\mathcal{E}$, let $E = \{y \in \mathcal{E} : b(y) \le b_B\}$ and $F = \{y \in \mathcal{E} : b(y) > b_B\}$. The following equality is true (see [4, page 3331]):

$$\int_E |b(y) - b_{\mathcal{E}}|dy = \int_F |b(y) - b_{\mathcal{E}}|dy.$$

Since $b(y) \le b_{\mathcal{E}} \le |b_{\mathcal{E}}| \le M^d_{\mathcal{E}}(b)(y)$ for any $y \in E$, we obtain

$$|b(y) - b_{\mathcal{E}}| \le \big|b(y) - M^d_{\mathcal{E}}(b)(y)\big|,\ y \in E.$$

Then from Hölder's inequality and (4.2) we have

$$\begin{aligned}&\frac{1}{|\mathcal{E}|^{1+\beta/|d|}}\int_{\mathcal{E}} |b(y) - b_{\mathcal{E}}|dy = \frac{2}{|\mathcal{E}|^{1+\beta/|d|}}\int_E |b(y) - b_{\mathcal{E}}|dy \\ &\le \frac{2}{|\mathcal{E}|^{1+\beta/|d|}}\int_E \big|b(y) - M^d_{\mathcal{E}}(b)(y)\big|dy \le \frac{2}{|\mathcal{E}|^{1+\beta/|d|}}\int_{\mathcal{E}} \big|b(y) - M^d_{\mathcal{E}}(b)(y)\big|dy \\ &\lesssim \frac{2}{|\mathcal{E}|^{1+\beta/|d|}}\,\|b - M^d_{\mathcal{E}}(b)\|_{L_q(\mathcal{E})}\,|\mathcal{E}|^{\frac{1}{q'}} \\ &\lesssim |\mathcal{E}|^{-\frac{1}{q} - \frac{\beta}{|d|}}\,[r]_1^{\frac{\lambda}{q}}\,[1/r]_1^{-\frac{\mu}{q}}\,\|\big(b - M^d_{\mathcal{E}}(b)\big)\,\chi_{\mathcal{E}}\|_{L^d_{q,\lambda,\mu}(\mathbb{R}^n)} \\ &\lesssim r^{-\frac{|d|}{q}}\,[r]_1^{\frac{\lambda}{q}}\,[1/r]_1^{-\frac{\mu}{q}}\,\|\chi_{\mathcal{E}}\|_{L^d_{q,\lambda,\mu}} \\ &\lesssim r^{-\frac{|d|}{q}}\,[r]_1^{\frac{\lambda}{q}}\,[1/r]_1^{-\frac{\mu}{q}}\,r^{\frac{|d|}{q}}\,[r]_1^{-\frac{\lambda}{q}}\,[1/r]_1^{\frac{\mu}{q}} \approx 1.\end{aligned}$$

From Theorem 4.1 in the case $\lambda = \mu$ or $\mu = 0$ we get the following corollaries.

Corollary 4.1 *[31] Let* $0 < \beta < 1$ *and* $b \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$*. Let also* $0 \le \lambda < |d| - \beta$, $1 < p < \frac{|d|-\lambda}{\beta}$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{|d|-\lambda}$*. Then, the following statements are equivalent:*

(i) $b \in \dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$ *and* $b \ge 0$*.*

(ii) *The operator* $[b, M^d]$ *is bounded from* $L^d_{p,\lambda}(\mathbb{R}^n)$ *to* $L^d_{q,\lambda}(\mathbb{R}^n)$*.*

(iii) *There exists a constant* $C > 0$ *such that*

$$\sup_{\mathcal{E}} \frac{\|\left(b - M^d_{\mathcal{E}}(b)\right)\chi_{\mathcal{E}}\|_{L^d_{q,\lambda}}}{\|\chi_{\mathcal{E}}\|_{L^d_{q,\lambda}}} \le C.$$

Corollary 4.2 *Let* $0 < \beta < 1$ *and* $b \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$*. Let also* $0 \le \lambda < |d| - \beta$, $1 < p < \frac{|d|-\lambda}{\beta}$ *and* $\frac{\beta}{|d|} \le \frac{1}{p} - \frac{1}{q} \le \frac{\beta}{|d|-\lambda}$*. Then, the following statements are equivalent:*

(i) $b \in \dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$ *and* $b \ge 0$*.*

(ii) *The operator* $[b, M^d]$ *is bounded from* $\widetilde{L}^d_{p,\lambda}(\mathbb{R}^n)$ *to* $\widetilde{L}^d_{q,\lambda}(\mathbb{R}^n)$*.*

(iii) *There exists a constant* $C > 0$ *such that*

$$\sup_{\mathcal{E}} \frac{\|\left(b - M^d_{\mathcal{E}}(b)\right)\chi_{\mathcal{E}}\|_{\widetilde{L}^d_{q,\lambda}}}{\|\chi_{\mathcal{E}}\|_{\widetilde{L}^d_{q,\lambda}}} \le C.$$

Remark 4.1 Note that Corollaries 4.1 and 4.2 are new. In the case of $d = \mathbf{1} \equiv (1, \ldots, 1)$ the Theorem 4.1 were proven in [15], see also [16,19,24].

5 Conclusion

The paper gives necessary and sufficient conditions for the boundedness of anisotropic maximal commutators M_b^d and commutators of the anisotropic maximal operator $[b, M^d]$ in total anisotropic Morrey spaces $L^d_{p,\lambda,\mu}(\mathbb{R}^n)$, when b belongs to the anisotropic Lipschitz spaces $\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$. As an application, new characterizations of some subclasses of anisotropic Lipschitz spaces $\dot{\Lambda}_{\beta,d}(\mathbb{R}^n)$ are obtained.

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