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On the strongly damped wave equation of Kirchhoff type containing a finite number of unknown values and a viscoelastic term

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Abstract. *In this paper, we consider a Robin-Dirichlet problem* (P) *for the strongly damped wave equation of Kirchhoff type containing a nonlinear term with a finite number of unknown values* $\mu = \mu(t, u_x(0,t), u_x(\eta_1,t), \cdots, u_x(\eta_q,t), \|u_x(t)\|^2)$*, and a viscoelastic term in form of a integral with respect to time variable. In Part I, using the linearization method together with Faedo-Galerkin method and weak compact method, we prove the existence and uniqueness of a weak solution of Prob.* (P)*. A lemma of energy which is useful to do proofs is also given here. In Part II, we consider a specific case* (P_q) *of* (P) *in which* $\mu = \mu\left(t, \frac{1}{q}\sum_{i=0}^{q-1} u_x^2(\frac{i}{q}, t)\right)$*. Under suitable conditions, we prove that the solution of* (P_q) *converges to that of the coressponding problem* (P_∞)*, as* $q \to \infty$ *(in a certain sense), where* (P_∞) *is defined by* (P_q) *in which* $\mu = \mu(t, \|u_x(t)\|^2)$*. The main tools in this part are the lemma of Aubin-Lions and the Ascoli-Arzela theorem with a priori estimates.*

Keywords. Kirchhoff equation; Robin-Dirichlet problem; Nonlocal terms; Faedo-Galerkin method; Linearization method; A lemma of energy.

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1 Introduction

In this paper, we investigate the Robin-Dirichlet problem for the strongly damped wave equation of Kirchhoff type as follows

$$\begin{cases} u_{tt} - \lambda u_{txx} - \mu(t, u_x(0,t), u_x(\eta_1,t), \cdots, u_x(\eta_q,t), \|u_x(t)\|^2)u_{xx} \\ \quad + \displaystyle\int_0^t g(t-s)u_{xx}(x,s)ds \\ \quad = f(x,t,u,u_t,u_x,u_x(0,t),u_x(\eta_1,t),\cdots,u_x(\eta_q,t),\|u_x(t)\|^2), \\ \qquad\qquad\qquad\qquad 0 < x < 1, 0 < t < T, \\ u_x(0,t) - \zeta u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{1.1}$$

where μ, f, g, $\tilde{u}_0$, $\tilde{u}_1$ were given functions and $\lambda > 0$, $\zeta \geq 0$, $\eta_1, \eta_2, \cdots, \eta_q$ were given constants with $0 \leq \eta_1 < \eta_2 < \cdots < \eta_q < 1$, and $\|u_x(t)\|^2 = \int_0^1 u_x^2(x,t)\,dx$.

The initial-boundary value problems for wave equations have been studied extensively and obtained numerous interesting results. Among the works of the Kirchhoff-Carrier type we can cite, for example, M.M. Cavalcanti et. al. [3] - [5], N.A. Larkin [11], N.T. Long et. al. [13], N.T. Long [14], L.A. Medeiros [15], J.Y. Park and J.J. Bae [23], M.L. Santos [26] and the references given therein. A survey of the results about the mathematical aspects of Kirchhoff model can be found in L.A. Medeiros et. al. [16], [17]. By using different methods together with various techniques in functional analysis, several results concerning the existence/global existence and the properties of solutions such as blow-up, decay, stability have been established. Especially, in case of the presence of viscoelastic terms, the reciprocal effects between viscoelastic terms and the source term can cause the decayed property or the blow-up phenomena of solutions in some cases, see for example, [8]-[10], [18]-[20], [24], [28]-[30].

Eq. $(1.1)_1$ involves nonlocal terms taking the form of integrals in space, or in time, combined with the presence of the unkown values $u_x(\eta_i,t)$, $i = \overline{1,q}$ in the nonlinear terms, so Prob. (1.1) is related to nonlocal problems arised naturally in some applied sciences such as fluid mechanics, heat transfer theory and control theory, see [1], [2], [6] and the references cited therein.

The nonlocal problems with nonlocal conditions come up when values of the function on the boundary is connected to values inside the domain. The other nonlocal problems have nonlocal nonlinearities in the source terms, for example, we refer to [7], [21], [22], [25], and the references given therein. In [25], M. Pellicer studied the following problem for a nonlocal nonlinear strongly damped wave equation with dynamical boundary conditions

$$\begin{cases} u_{tt} - u_{xx} - \alpha u_{txx} + \varepsilon f\left(u(1,t), \frac{u_t(1,t)}{\sqrt{\varepsilon}}\right) = 0, \\ u(0,t) = 0, \\ u_{tt}(1,t) = -\varepsilon\left[u_x(1,t) + \alpha u_{tx}(1,t) + r u_t(1,t)\right] - \varepsilon f\left(u(1,t), \frac{u_t(1,t)}{\sqrt{\varepsilon}}\right), \end{cases} \tag{1.2}$$

where the term $\varepsilon f\left(u(1,t), \frac{u_t(1,t)}{\sqrt{\varepsilon}}\right)$ represents a control acceleration at $x = 1$ and α, r, ε are positive constants. By using the invariant manifold theory, the authors proved that for small values of the parameter ε, the solution of (1.2) attracted to a two-dimensional invariant manifold. In [21], N.H. Nhan et. al. considered the Robin problem for a wave equation with source containing nonlocal terms as follows

$$\begin{cases} u_{tt} - u_{xx} = f\left(x, t, u(x,t), u(\eta_1, t), \cdots, u(\eta_q, t), u_t(x,t)\right), \\ \qquad\qquad\qquad\qquad 0 < x < 1,\ 0 < t < T, \\ u_x(0,t) - h_0 u(0,t) = u_x(1,t) + h_1 u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{1.3}$$

where $f,\ \tilde{u}_0,\ \tilde{u}_1$ are given functions and $h_0,\ h_1 \geq 0,\ \eta_1,\ \eta_2, \cdots, \eta_q$ are given constants with $h_0 + h_1 > 0$, $0 \leq \eta_1 < \eta_2 < \cdots < \eta_q \leq 1$. Here, the authors proved the existence and uniqueness of a weak solution and established an asymptotic expansion of high order in a small parameter of a weak solution. However, to the best of our knowledge, there are relatively few results about such a nonlocal problem with source containing multi-point nonlocal terms.

In this paper, we investigate Prob. (1.1) and study a specific case (P_q) of Prob. (1.1) with $\mu = \mu\left(t, \frac{1}{q}\sum_{i=0}^{q-1} u_x^2(\frac{i}{q}, t)\right)$ and $f = f\left(x, t, u(x,t), u_t(x,t), u_x(x,t), \frac{1}{q}\sum_{i=0}^{q-1} u_x^2(\frac{i}{q}, t)\right)$.

In this special case, the sum $\frac{1}{q}\sum_{i=0}^{q-1} u_x^2(\frac{i}{q}, t)$ can be considered as a special combination of discrete family $\{u_x(\eta_i, t)\}_{i=1}^{q}$ in Eq. $(1.1)_1$. We note more that, the functions $y \longmapsto u_x(y,t)$ is continuous on $[0,1]$, a.e. $t \in [0,T]$, then we have

$$\frac{1}{q}\sum_{i=0}^{q-1} u_x^2(\frac{i}{q}, t) \to \|u_x(t)\|^2, \text{ as } q \to \infty, \tag{1.4}$$

hence Prob. (P_q) might have a close relationship (in a certain sense) with Prob. (P_∞) for the following equation

$$\begin{aligned} & u_{tt} - \lambda u_{txx} - \mu\left(t, \|u_x(t)\|^2\right) u_{xx} + \int_0^t g(t-s) u_{xx}(x,s)\, ds \\ & \quad = f\left(x, t, u, u_t, u_x, \|u_x(t)\|^2\right), \end{aligned} \tag{1.5}$$

$0 < x < 1$, $0 < t < T$, associated with the Robin-Dirichlet condition and initial condition $(1.1)_{2,3}$.

This paper consists of four sections. In Section 2, we present some preliminaries. In Section 3, by applying the linearization method together with Faedo-Galerkin method and the weak compact method, we prove the existence and the uniqueness of a weak solution. A lemma of energy (Lemma 3.4) which is useful to do proofs is also given here. This lemma is a relative generalization of the inequality and equality of energy given in Lions's book [[12], Lemma 1.6, p. 224]. In Section 4, we consider the couple of problems (P_q), (P_∞) and prove that the solution of (P_q) converges to the solution of (P_∞), as $q \to \infty$ (in a certain sense). The proof of this section is done by the lemma of Aubin-Lions and the Ascoli-Arzela theorem with a priori estimates.

2 Preliminaries

In this paper, with $\Omega = (0,1)$, we will use the usual function spaces $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$. Let $\langle \cdot, \cdot \rangle$ denote either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. We denote by $\|\cdot\|$ the norm in L^2 and by $\|\cdot\|_X$ the norm in a Banach space X. We call X' the dual space of X.

We denote $L^p(0,T;X)$, $1 \le p \le \infty$ the Banach space of real functions $u : (0,T) \to X$ measurable, such that $\|u\|_{L^p(0,T;X)} < +\infty$, with

$$\|u\|_{L^p(0,T;X)} = \begin{cases} \left(\int_0^T \|u(t)\|_X^p \, dt\right)^{1/p}, & \text{if} \quad 1 \le p < \infty, \\ ess\sup\limits_{0<t<T} \|u(t)\|_X \, , & \text{if} \quad p = \infty. \end{cases}$$

Let $u(t)$, $u'(t) = u_t(t) = \dot{u}(t)$, $u''(t) = u_{tt}(t) = \ddot{u}(t)$, $u_x(t) = \nabla u(t)$, $u_{xx}(t) = \Delta u(t)$, denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

Let $T^* > 0$, with $f \in C^k([0,1] \times [0,T^*] \times \mathbb{R}^{q+5})$, $f = f(x,t,y_1,\cdots,y_{q+5})$, we put $D_1 f = \frac{\partial f}{\partial x}$, $D_2 f = \frac{\partial f}{\partial t}$, $D_{i+2} f = \frac{\partial f}{\partial y_i}$ with $i = 1,\cdots,q+5$, and $D^\alpha f = D_1^{\alpha_1} \cdots D_{q+7}^{\alpha_{q+7}} f$, $\alpha = (\alpha_1,\cdots,\alpha_{q+7}) \in \mathbb{Z}_+^{q+7}$, $|\alpha| = \alpha_1 + \cdots + \alpha_{q+7} \le k$, $D^{(0,\cdots,0)} f = f$.

With $\mu \in C^k([0,T^*] \times \mathbb{R}^{q+2})$, $\mu = \mu(t, z_1, \cdots, z_{q+2})$, we put $D_1 \mu = \frac{\partial \mu}{\partial t}$, $D_{i+1}\mu = \frac{\partial \mu}{\partial z_i}$ with $i = 1,\cdots,q+2$, and $D^\alpha \mu = D_1^{\alpha_1} \cdots D_{q+3}^{\alpha_{q+3}} \mu$, $\alpha = (\alpha_1,\cdots,\alpha_{q+3}) \in \mathbb{Z}_+^{q+3}$, $|\alpha| = \alpha_1 + \cdots + \alpha_{q+3} \le k$, $D^{(0,\cdots,0)}\mu = \mu$.

On H^1, we shall use the following norm

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2\right)^{1/2}. \tag{2.1}$$

We put

$$V = \{v \in H^1 : v(1) = 0\}, \tag{2.2}$$

$$a(u,v) = \int_0^1 u_x(x) v_x(x) dx + \zeta u(0) v(0), \; u, v \in V. \tag{2.3}$$

Then, V is a closed subspace of H^1 and on V, three norms $v \longmapsto \|v\|_{H^1}$, $v \longmapsto \|v_x\|$ and $v \longmapsto \|v\|_a = \sqrt{a(v,v)}$ are equivalent norms. We have the following lemmas, the proofs of which are straightforward hence we omit the details.

Lemma 2.1. *The imbedding $H^1 \hookrightarrow C(\overline{\Omega})$ is compact and*

$$\|v\|_{C(\overline{\Omega})} \le \sqrt{2} \, \|v\|_{H^1} \text{ for all } v \in H^1. \tag{2.4}$$

Lemma 2.2. *Let $\zeta \ge 0$. Then the imbedding $V \hookrightarrow C(\overline{\Omega})$ is compact and*

$$\begin{cases} \|v\|_{C(\overline{\Omega})} \le \|v_x\| \le \|v\|_a \, , \\ \frac{1}{\sqrt{2}} \|v\|_{H^1} \le \|v_x\| \le \|v\|_a \le \sqrt{1+\zeta} \, \|v_x\| \le \sqrt{1+\zeta} \, \|v\|_{H^1} \, , \end{cases} \tag{2.5}$$

for all $v \in V$.

Lemma 2.3. *Let $\zeta \ge 0$. Then the symmetric bilinear form $a(\cdot,\cdot)$ defined by* (2.3) *is continuous on $V \times V$ and coercive on V.*

Lemma 2.4. *Let $\zeta \ge 0$. Then there exists the Hilbert orthonormal base $\{w_j\}$ of L^2 consisting of the eigenfunctions w_j corresponding to the eigenvalue λ_j such that*

$$\begin{cases} 0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \cdots, \; \lim\limits_{j \to +\infty} \lambda_j = +\infty, \\ a(w_j, v) = \lambda_j \langle w_j, v \rangle \text{ for all } v \in V, \, j = 1, 2, \cdots. \end{cases} \tag{2.6}$$

Furthermore, the sequence $\{w_j/\sqrt{\lambda_j}\}$ is also a Hilbert orthonormal base of V with respect to the scalar product $a(\cdot,\cdot)$.

On the other hand, we also have w_j satisfying the following boundary value problem

$$\begin{cases} -\Delta w_j = \lambda_j w_j, \text{ in } (0,1), \\ w_{jx}(0) - \zeta w_j(0) = w_j(1) = 0, \, w_j \in C^\infty(\overline{\Omega}). \end{cases} \tag{2.7}$$

The proof of Lemma 2.4 can be found in ([27], p.87, Theorem 7.7), with $H = L^2$ and V, $a(\cdot,\cdot)$ as defined by (2.2), (2.3).

Remark 2.1. The weak formulation of Prob. (1.1) can be given in the following manner: Find $u \in \tilde{V}_T = \{v \in L^\infty(0,T;H^2 \cap V) : v' \in L^\infty(0,T;H^2 \cap V),\, v'' \in L^\infty(0,T;L^2) \cap L^2(0,T;V)\}$, such that u satisfies the following variational equation

$$\begin{aligned} &\langle u''(t), w\rangle + \lambda\, a(u'(t),w) + \mu\,[u](t)\, a(u(t),w) \\ &= \int_0^t g(t-s)\, a(u(s),w)\, ds + \langle f[u](t), w\rangle, \end{aligned} \tag{2.8}$$

for all $w \in V$, a.e., $t \in (0,T)$, together with the initial conditions

$$u(0) = \tilde{u}_0, \; u'(0) = \tilde{u}_1, \tag{2.9}$$

where

$$\begin{aligned} f[u](x,t) &= f\Big(x,t,u(x,t),u'(x,t),u_x(x,t),u_x(0,t), \\ &\qquad u_x(\eta_1,t),\cdots,u_x(\eta_q,t),\|u_x(t)\|^2\Big), \\ \mu[u](t) &= \mu\Big(t,u_x(0,t),u_x(\eta_1,t),\cdots,u_x(\eta_q,t),\|u_x(t)\|^2\Big). \end{aligned} \tag{2.10}$$

3 Existence and uniqueness

We make the following assumptions:

(H_1) : $\tilde{u}_0, \tilde{u}_1 \in V \cap H^2,\; \tilde{u}_{0x}(0) - \zeta\tilde{u}_0(0) = 0$;

(H_2) : $g \in H^1(0,T^*)$;

(H_3) : $\mu \in C^1([0,T^*] \times \mathbb{R}^{q+1} \times \mathbb{R}_+)$ such that
$\mu\,(t,z_1,\cdots,z_{q+2}) \geq \mu_* > 0, \forall t \in [0,T^*], \forall\,(z_1,\cdots,z_{q+1}) \in \mathbb{R}^{q+1}, \forall z_{q+2} \in \mathbb{R}_+$;

(H_4) : $f \in C^1([0,1] \times [0,T^*] \times \mathbb{R}^{q+4} \times \mathbb{R}_+)$.

For each $M > 0$ given, we set the constants $\bar{K}_M(f)$, $\tilde{K}_M(\mu)$ as follows

$$\begin{aligned} \bar{K}_M &= \bar{K}_M(f) = \|f\|_{C^1(\bar{A}_M)} = \|f\|_{C(\bar{A}_M)} + \sum_{i=1}^{q+7} \|D_i f\|_{C(\bar{A}_M)}, \\ \tilde{K}_M &= \tilde{K}_M(\mu) = \|\mu\|_{C^1(\tilde{A}_M)} = \|\mu\|_{C(\tilde{A}_M)} + \sum_{i=1}^{q+3} \|D_i \mu\|_{C(\tilde{A}_M)}, \\ \|f\|_{C(\bar{A}_M)} &= \sup_{(x,t,y_1,\ldots,y_{q+5}) \in \bar{A}_M} |f(x,t,y_1,\ldots,y_{q+5})|, \\ \|\mu\|_{C(\tilde{A}_M)} &= \sup_{(t,z_1,\ldots,z_{q+2}) \in \tilde{A}_M} |\mu(t,z_1,\ldots,z_{q+2})|. \end{aligned} \tag{3.1}$$

where

$$\bar{A}_M = [0,1] \times [0,T^*] \times [-M,M]^2 \times [-\sqrt{2}M, \sqrt{2}M]^{q+2} \times [0,M^2], \tag{3.2}$$

$$\tilde{A}_M = [0,T^*] \times [-M,M]^{q+1} \times [0,M^2].$$

For every $T \in (0,T^*]$, we put

$$V_T = \{v \in L^\infty(0,T;H^2 \cap V) : v' \in L^\infty(0,T;H^2 \cap V),\ v'' \in L^2(0,T;V)\}, \tag{3.3}$$

then V_T is a Banach space with respect to the following norm (see Lions [12])

$$\|v\|_{V_T} = \max\left\{\|v\|_{L^\infty(0,T;H^2\cap V)},\ \|v'\|_{L^\infty(0,T;H^2\cap V)},\ \|v''\|_{L^2(0,T;V)}\right\}. \tag{3.4}$$

For every $M > 0$, we put

$$W(M,T) = \left\{v \in V_T : \|v\|_{V_T} \le M\right\}, \tag{3.5}$$
$$W_1(M,T) = \left\{v \in W(M,T) : v'' \in L^\infty(0,T;L^2)\right\}.$$

We note that

$$H_T = \left\{v \in C\left([0,T];H^2 \cap V\right) \cap C^1(0,T;V) : v' \in L^2(0,T;H^2 \cap V)\right\} \tag{3.6}$$

is a Banach space with respect to the norm

$$\|v\|_{H_T} = \|v\|_{C([0,T];H^2\cap V)} + \|v'\|_{C([0,T];V)} + \|v'\|_{L^2(0,T;H^2\cap V)}. \tag{3.7}$$

Now, we establish the recurrent sequence $\{u_m\}$. The first term is chosen as $u_0 \equiv \tilde{u}_0$, suppose that

$$u_{m-1} \in W_1(M,T), \tag{3.8}$$

we associate Prob. (1.1) with the following problem.

Find $u_m \in W(M,T)$ $(m \ge 1)$ satisfying the linear variational problem

$$\begin{cases} \langle u_m''(t), w\rangle + \lambda a(u_m'(t), w) + \mu_m(t) a(u_m(t), w) \\ \qquad = \int_0^t g(t-s) a(u_m(s), w) ds + \langle F_m(t), w\rangle, \forall w \in V, \\ u_m(0) = \tilde{u}_0,\ u_m'(0) = \tilde{u}_1, \end{cases} \tag{3.9}$$

where

$$\begin{aligned} F_m(x,t) &= f[u_{m-1}](x,t) \\ &= f\Big(x,t,u_{m-1}(x,t),u_{m-1}'(x,t),\nabla u_{m-1}(x,t),\nabla u_{m-1}(0,t), \\ &\qquad \nabla u_{m-1}(\eta_1,t),\cdots,\nabla u_{m-1}(\eta_q,t),\|\nabla u_{m-1}(t)\|^2\Big), \\ \mu_m(t) &= \mu[u_{m-1}](t) \\ &= \mu\Big(t,\nabla u_{m-1}(0,t),\nabla u_{m-1}(\eta_1,t),\ldots,\nabla u_{m-1}(\eta_q,t),\|\nabla u_{m-1}(t)\|^2\Big). \end{aligned} \tag{3.10}$$

Then, we have the following theorem.

Theorem 3.1. *Let $(H_1)-(H_4)$ hold. Then, there exist positive constants M, T such that, for $u_0 \equiv \tilde{u}_0$, there exists a recurrent sequence $\{u_m\} \subset W(M,T)$ defined by* (3.8)-(3.10).

Proof of Theorem 3.1. The proof consists of several steps.

Step 1. The Faedo-Galerkin approximation (introduced by Lions [12]). Consider the basis $\{w_j\}$ for L^2 as in Lemma 2.4. Put

$$u_m^{(k)}(t) = \sum_{j=1}^k c_{mj}^{(k)}(t) w_j, \tag{3.11}$$

where the coefficients $c_{mj}^{(k)}$, $j = 1, \cdots, k$ satisfy the system of linear differential equations

$$\begin{cases} \langle \ddot{u}_m^{(k)}(t), w_j \rangle + \lambda a(\dot{u}_m^{(k)}(t), w_j) + \mu_m(t) a(u_m^{(k)}(t), w_j) \\ \qquad = \int_0^t g(t-s) a(u_m^{(k)}(s), w_j) ds + \langle F_m(t), w_j \rangle, \, 1 \leq j \leq k, \\ u_m^{(k)}(0) = \tilde{u}_{0k}, \, \dot{u}_m^{(k)}(0) = \tilde{u}_{1k}, \end{cases} \tag{3.12}$$

where

$$\begin{cases} \tilde{u}_{0k} = \sum_{j=1}^k \alpha_j^{(k)} w_j \to \tilde{u}_0 \text{ strongly in } H^2 \cap V, \\ \tilde{u}_{1k} = \sum_{j=1}^k \beta_j^{(k)} w_j \to \tilde{u}_1 \text{ strongly in } H^2 \cap V. \end{cases} \tag{3.13}$$

Using contraction mapping principle, it is not difficult to show that the system (3.12) has a unique solution $u_m^{(k)}(t)$ in $[0, T]$.

Step 2. A priori estimates. Put

$$\begin{aligned} S_m^{(k)}(t) &= \left\| \dot{u}_m^{(k)}(t) \right\|^2 + \left\| \dot{u}_m^{(k)}(t) \right\|_a^2 + \lambda \left\| \Delta \dot{u}_m^{(k)}(t) \right\|^2 \\ &+ \mu_m(t) \left(\left\| u_m^{(k)}(t) \right\|_a^2 + \left\| \Delta u_m^{(k)}(t) \right\|^2 \right) \\ &+ 2 \int_0^t \left[\lambda \left(\left\| \dot{u}_m^{(k)}(s) \right\|_a^2 + \left\| \Delta \dot{u}_m^{(k)}(s) \right\|^2 \right) + \left\| \ddot{u}_m^{(k)}(s) \right\|_a^2 \right] ds, \end{aligned} \tag{3.14}$$

then we deduce from (3.12) that

$$\begin{aligned} \bar{\mu}_* \bar{S}_m^{(k)}(t) &\leq S_m^{(k)}(t) \\ &= S_m^{(k)}(0) + 2\langle \Delta \tilde{u}_{0k}, \Delta \tilde{u}_{1k} \rangle + 2\langle F_m(0), \Delta \tilde{u}_{1k} \rangle + g(0) \left\| \Delta \tilde{u}_{0k} \right\|^2 \\ &+ \int_0^t \left(\mu_m'(s) - 2g(0) \right) \left[\left\| u_m^{(k)}(s) \right\|_a^2 + \left\| \Delta u_m^{(k)}(s) \right\|^2 \right] ds \\ &+ 2 \int_0^t g(t-s) \left[a(u_m^{(k)}(s), u_m^{(k)}(t)) + \langle \Delta u_m^{(k)}(s), \Delta u_m^{(k)}(t) + \Delta \dot{u}_m^{(k)}(t) \rangle \right] ds \\ &- 2 \int_0^t \int_0^\tau g'(\tau - s) \left[a(u_m^{(k)}(s), u_m^{(k)}(\tau)) + \langle \Delta u_m^{(k)}(s), \Delta u_m^{(k)}(\tau) + \Delta \dot{u}_m^{(k)}(\tau) \rangle \right] ds d\tau \\ &+ 2 \int_0^t \langle F_m(s), \dot{u}_m^{(k)}(s) - \Delta \dot{u}_m^{(k)}(s) \rangle ds + 2 \int_0^t \langle F_m'(s), \Delta \dot{u}_m^{(k)}(s) \rangle ds \\ &+ 2 \int_0^t \left\| \Delta \dot{u}_m^{(k)}(s) \right\|^2 ds - g(0) \left\| \Delta u_m^{(k)}(t) \right\|^2 - 2\langle \Delta u_m^{(k)}(t), \Delta \dot{u}_m^{(k)}(t) \rangle \\ &- 2\langle F_m(t), \Delta \dot{u}_m^{(k)}(t) \rangle \\ &= S_m^{(k)}(0) + 2\langle \Delta \tilde{u}_{0k}, \Delta \tilde{u}_{1k} \rangle + 2\langle F_m(0), \Delta \tilde{u}_{1k} \rangle + g(0) \left\| \Delta \tilde{u}_{0k} \right\|^2 + \sum_{j=2}^9 I_j, \end{aligned} \tag{3.15}$$

where $\bar{\mu}_* = \min\{1, \mu_*, \lambda\}$ and

$$\begin{aligned} \bar{S}_m^{(k)}(t) &= \left\| \dot{u}_m^{(k)}(t) \right\|^2 + \left\| \dot{u}_m^{(k)}(t) \right\|_a^2 + \left\| \Delta \dot{u}_m^{(k)}(t) \right\|^2 + \left\| u_m^{(k)}(t) \right\|_a^2 + \left\| \Delta u_m^{(k)}(t) \right\|^2 \\ &+ \int_0^t \left[\left\| \dot{u}_m^{(k)}(s) \right\|_a^2 + \left\| \Delta \dot{u}_m^{(k)}(s) \right\|^2 + \left\| \ddot{u}_m^{(k)}(s) \right\|_a^2 \right] ds. \end{aligned} \tag{3.16}$$

First, we need the following lemma, the proof of this lemma is easy so we omit the details.

Lemma 3.2. *Every* $M > 0$, *put*

$$\begin{cases} \hat{\eta}_M = \left(1 + \sqrt{2}(q+1)M + 2M^2\right) \tilde{K}_M(\mu), \\ \hat{\rho}_m(M,t) = \left[1 + \left(2 + \sqrt{2}(q+1)\right) M + 2M^2 + \|u''_{m-1}(t)\|\right] \bar{K}_M(f), \\ \bar{\rho}_M = \sqrt{2}\bar{K}_M(f) \left[T^* \left(1 + \left(2 + \sqrt{2}(q+1)\right) M + 2M^2\right)^2 + M^2\right]^{1/2}. \end{cases}$$

Then, we have

(i) $\|\hat{\rho}_m(M,\cdot)\|_{L^2(0,T)} \le \bar{\rho}_M$,

(ii) $\|\hat{\rho}_m(M,\cdot)\|_{L^1(0,T)} \le \sqrt{T}\bar{\rho}_M$,

(iii) $0 < \mu_* \le \mu_m(t) \le \tilde{K}_M(\mu)$,

(iv) $|\mu'_m(t)| \le \hat{\eta}_M$,

(v) $\|F_m(t)\| \le \bar{K}_M(f)$,

(vi) $\|F'_m(t)\| \le \rho_m(M,t)$,

(vii) $\|F_m(t)\| \le \|F_m(0)\| + \sqrt{T}\bar{\rho}_M$.

By using Lemma 3.2, we shall estimate the terms $I_1, \cdots, I_9$ on the right-hand side of (3.15) as follows.

$$\begin{aligned} I_1 &= \int_0^t \left(\mu'_m(s) - 2g(0)\right) \left(\left\|u_m^{(k)}(s)\right\|_a^2 + \left\|\Delta u_m^{(k)}(s)\right\|^2\right) ds \qquad (3.17) \\ &\le (\hat{\eta}_M + 2|g(0)|) \int_0^t \bar{S}_m^{(k)}(s)ds; \\ I_2 &= 2\int_0^t g(t-s) \left[a(u_m^{(k)}(s), u_m^{(k)}(t)) + \left\langle \Delta u_m^{(k)}(s), \Delta u_m^{(k)}(t) + \Delta \dot{u}_m^{(k)}(t)\right\rangle\right] ds \\ &\le 2\int_0^t |g(t-s)| \Big(\left\|u_m^{(k)}(s)\right\|_a \left\|u_m^{(k)}(t)\right\|_a + \left\|\Delta u_m^{(k)}(s)\right\| \left\|\Delta u_m^{(k)}(t)\right\| \\ &\quad + \left\|\Delta u_m^{(k)}(s)\right\| \left\|\Delta \dot{u}_m^{(k)}(t)\right\| \Big) ds \\ &\le 2\int_0^t |g(t-s)| \left(\left\|u_m^{(k)}(s)\right\|_a^2 + 2\left\|\Delta u_m^{(k)}(s)\right\|^2\right)^{1/2} \\ &\quad \times \left(\left\|u_m^{(k)}(t)\right\|_a^2 + \left\|\Delta u_m^{(k)}(t)\right\|^2 + \left\|\Delta \dot{u}_m^{(k)}(t)\right\|^2\right)^{1/2} ds \\ &\le \frac{\bar{\mu}_*}{6}\bar{S}_m^{(k)}(t) + \frac{24}{\bar{\mu}_*}\left(\int_0^t |g(t-s)|\sqrt{\bar{S}_m^{(k)}(s)}ds\right)^2 \\ &\le \frac{\bar{\mu}_*}{6}\bar{S}_m^{(k)}(t) + \frac{24}{\bar{\mu}_*}\|g\|_{L^2(0,T^*)}^2 \int_0^t \bar{S}_m^{(k)}(s)ds; \\ I_3 &= -2\int_0^t d\tau \int_0^\tau g'(\tau - s) \left[a(u_m^{(k)}(s), u_m^{(k)}(\tau)) + \left\langle \Delta u_m^{(k)}(s), \Delta u_m^{(k)}(\tau) + \Delta \dot{u}_m^{(k)}(\tau)\right\rangle\right] ds \\ &\le 4\int_0^t d\tau \int_0^\tau |g'(\tau - s)| \sqrt{\bar{S}_m^{(k)}(s)}\sqrt{\bar{S}_m^{(k)}(\tau)}ds \\ &\le 4\sqrt{T^*}\left\|g'\right\|_{L^2(0,T^*)} \int_0^t \bar{S}_m^{(k)}(s)ds; \end{aligned}$$

$$
\begin{aligned}
I_4 &= 2\int_0^t \left\langle F_m(s), \dot{u}_m^{(k)}(s) - \Delta\dot{u}_m^{(k)}(s)\right\rangle ds \\
&\le 2\int_0^t \|F_m(s)\| \left(\left\|\dot{u}_m^{(k)}(s)\right\| + \left\|\Delta\dot{u}_m^{(k)}(s)\right\|\right) ds \\
&\le 2\bar{K}_M(f)\int_0^t \sqrt{\bar{S}_m^{(k)}(s)}ds \le T\bar{K}_M^2(f) + \int_0^t \bar{S}_m^{(k)}(s)ds; \\
I_5 &= 2\int_0^t \langle F_m'(s), \Delta\dot{u}_m^{(k)}(s)\rangle ds \\
&\le \int_0^t \|F_m'(s)\|ds + \int_0^t \|F_m'(s)\| \cdot \|\Delta\dot{u}_m^{(k)}(s)\|^2 ds \\
&\le \int_0^T \hat{\rho}_m(M,s)ds + \int_0^t \hat{\rho}_m(M,s) \cdot \|\Delta\dot{u}_m^{(k)}(s)\|^2 ds \\
&\le \sqrt{T}\,\bar{\rho}_M + \int_0^t \hat{\rho}_m(M,s) \cdot \bar{S}_m^{(k)}(s)ds; \\
I_6 &= 2\int_0^t \|\Delta\dot{u}_m^{(k)}(s)\|^2 ds \le 2\int_0^t \bar{S}_m^{(k)}(s)ds; \\
I_7 &= -g(0)\|\Delta u_m^{(k)}(t)\|^2 \\
&\le 2|g(0)| \left[\|\Delta\tilde{u}_{0k}\|^2 + T^*\int_0^t \|\Delta\dot{u}_m^{(k)}(s)\|^2 ds\right] \\
&\le 2|g(0)|\|\Delta\tilde{u}_{0k}\|^2 + 2T^*|g(0)|\int_0^t \bar{S}_m^{(k)}(s)ds; \\
I_8 &= -2\langle \Delta u_m^{(k)}(t), \Delta\dot{u}_m^{(k)}(t)\rangle \\
&\le \frac{\bar{\mu}_*}{6}\bar{S}_m^{(k)}(t) + \frac{6}{\bar{\mu}_*}\|\Delta u_m^{(k)}(t)\|^2 \\
&\le \frac{\bar{\mu}_*}{6}\bar{S}_m^{(k)}(t) + \frac{12}{\bar{\mu}_*}\left(\|\Delta\tilde{u}_{0k}\|^2 + T^*\int_0^t \bar{S}_m^{(k)}(s)ds;\right) \\
I_9 &= -2\langle F_m(t), \Delta\dot{u}_m^{(k)}(t)\rangle \\
&\le \frac{\bar{\mu}_*}{6}\bar{S}_m^{(k)}(t) + \frac{6}{\bar{\mu}_*}\|F_m(t)\|^2 \\
&\le \frac{\bar{\mu}_*}{6}\bar{S}_m^{(k)}(t) + \frac{12}{\bar{\mu}_*}\left(\|F_m(0)\|^2 + T\bar{\rho}_M^2\right).
\end{aligned}
$$

It follows from (3.15) and (3.17) that

$$
\bar{S}_m^{(k)}(t) \le \bar{S}_{0m}^{(k)} + \bar{D}_1(M,T) + \int_0^t \left[\bar{D}_2(M) + \frac{2}{\bar{\mu}_*}\hat{\rho}_m(M,s)\right]\bar{S}_m^{(k)}(s)ds, \tag{3.18}
$$

where

$$
\begin{aligned}
\bar{S}_{0m}^{(k)} &= \frac{2}{\bar{\mu}_*}S_m^{(k)}(0) + \frac{4}{\bar{\mu}_*}\left[\langle \Delta\tilde{u}_{0k}, \Delta\tilde{u}_{1k}\rangle + \langle F_m(0), \Delta\tilde{u}_{1k}\rangle\right] \\
&\quad + \frac{6}{\bar{\mu}_*}\left[\frac{4}{\bar{\mu}_*}\|F_m(0)\|^2 + \left(|g(0)| + \frac{4}{\bar{\mu}_*}\right)\|\Delta\tilde{u}_{0k}\|^2\right],
\end{aligned} \tag{3.19}
$$

$$
\begin{aligned}
\bar{D}_1(M,T) &= \frac{2}{\bar{\mu}_*}\left[T\left(\bar{K}_M^2(f)+\frac{12}{\bar{\mu}_*}\bar{\rho}_M^2\right)+\sqrt{T}\bar{\rho}_M\right],\\
\bar{D}_2(M) &= \frac{2}{\bar{\mu}_*}\Bigg(3+\hat{\eta}_M+\frac{12}{\bar{\mu}_*}T^*+2(1+T^*)|g(0)|\\
&\quad +\frac{24}{\bar{\mu}_*}\left\|g\right\|_{L^2(0,T^*)}^2+4\sqrt{T^*}\left\|g'\right\|_{L^2(0,T^*)}\Bigg).
\end{aligned}
$$

By (3.13), (3.14), it follows from $(3.19)_1$ that

$$
\bar{S}_{0m}^{(k)} \leq \frac{1}{2}M^2, \text{ for all } m,\ k\in\mathbb{N}, \tag{3.20}
$$

where M is a constant depending only on $\mu,\, f,\, g,\, \tilde{u}_0,\, \tilde{u}_1,\, \lambda,\, \zeta,\, \eta_1,\, \eta_2, \cdots, \eta_q$.

We choose $T\in(0,T^*]$, such that

$$
\left(\frac{1}{2}M^2+\bar{D}_1(M,T)\right)\exp\left(T\bar{D}_2(M)+\frac{2}{\bar{\mu}_*}\sqrt{T}\bar{\rho}_M\right)\leq M^2, \tag{3.21}
$$

and

$$
k_T = 3\sqrt{T\tilde{D}_1(M)}\exp\left(T\tilde{D}_2(M)\right)<1, \tag{3.22}
$$

where

$$
\begin{aligned}
\tilde{D}_1(M) &= \frac{24}{\bar{\mu}_*^2}\left(M^2\tilde{K}_M^2+\bar{K}_M^2(f)\right)\left(2+\sqrt{2}(q+1)+2M\right)^2, \qquad (3.23)\\
\tilde{D}_2(M) &= \frac{1}{\bar{\mu}_*}\left(\hat{\eta}_M+2|g(0)|+2\sqrt{T^*}\left\|g'\right\|_{L^2(0,T^*)}+\frac{6}{\bar{\mu}_*}\left\|g\right\|_{L^2(0,T^*)}^2\right).
\end{aligned}
$$

Then, we deduce from (3.18), (3.20) and (3.21) that

$$
\begin{aligned}
\bar{S}_m^{(k)}(t) &\leq \exp\left(-T\bar{D}_2(M)-\frac{2}{\bar{\mu}_*}\sqrt{T}\bar{\rho}_M\right)M^2 \qquad (3.24)\\
&\quad +\int_0^t\left(\bar{D}_2(M)+\frac{2}{\bar{\mu}_*}\hat{\rho}_m(M,s)\right)\bar{S}_m^{(k)}(s)\,ds.
\end{aligned}
$$

Finally, by using Gronwall's Lemma, we deduce from (3.24) that

$$
\begin{aligned}
\bar{S}_m^{(k)}(t) &\leq \exp\left(-T\bar{D}_2(M)-\frac{2}{\bar{\mu}_*}\sqrt{T}\bar{\rho}_M\right)M^2 \qquad (3.25)\\
&\quad\times\exp\left[\int_0^t\left(\bar{D}_2(M)+\frac{2}{\bar{\mu}_*}\hat{\rho}_m(M,s)\right)ds\right]\\
&\leq \exp\left(-T\bar{D}_2(M)-\frac{2}{\bar{\mu}_*}\sqrt{T}\bar{\rho}_M\right)M^2\\
&\quad\times\exp\left(T\bar{D}_2(M)+\frac{2}{\bar{\mu}_*}\left\|\hat{\rho}_m(M,\cdot)\right\|_{L^1(0,T)}\right)\\
&\leq \exp\left(-T\bar{D}_2(M)-\frac{2}{\bar{\mu}_*}\sqrt{T}\bar{\rho}_M\right)M^2\\
&\quad\times\exp\left(T\bar{D}_2(M)+\frac{2}{\bar{\mu}_*}\sqrt{T}\bar{\rho}_M\right)
\end{aligned}
$$

$$\leq M^2.$$

for all $t \in [0,T]$, for all m and $k \in \mathbb{N}$.

Therefore, we have

$$u_m^{(k)} \in W(M,T), \text{ for all } m \text{ and } k \in \mathbb{N}. \tag{3.26}$$

Step 3. Limit procedure. From (3.26), there exists a subsequence of the sequence of $\{u_m^{(k)}\}$, with the same notation, such that

$$\begin{cases} u_m^{(k)} \to u_m & \text{in} \quad L^\infty(0,T;H^2 \cap V) \text{ weak*}, \\ \dot{u}_m^{(k)} \to u_m' & \text{in} \quad L^\infty(0,T;H^2 \cap V) \text{ weak*}, \\ \ddot{u}_m^{(k)} \to u_m'' & \text{in} \quad L^2(0,T;V) \text{ weak}, \\ u_m \in W(M,T). \end{cases} \tag{3.27}$$

Passing to limit in (3.12), we have u_m satisfying (3.9), (3.10) in $L^2(0,T)$ weak. Furthermore, $(3.9)_1$ and $(3.27)_4$ imply that

$$u_m'' = \lambda \Delta u_m' + \mu_m(t)\Delta u_m - \int_0^t g(t-s)\Delta u_m(s)ds + F_m \in L^\infty(0,T;L^2),$$

so we obtain $u_m \in W_1(M,T)$, Theorem 3.1 is proved. □

We will use the result given in Theorem 3.1 and the compact imbedding theorems to prove the existence and uniqueness of a weak solution of Prob. (1.1). Hence, we get the main result in this section as follows.

Theorem 3.3. *Let* $(H_1)-(H_4)$ *hold. Then, there exist positive constants* M, T *such that*

(i) *Prob.* (1.1) *has an unique weak solution* $u \in W_1(M,T)$.

(ii) *The recurrent sequence* $\{u_m\}$ *defined by* (3.8)-(3.10) *converges to the solution* u *of Prob.* (1.1) *strongly in* H_T. *Furthermore, we have the estimation*

$$\|u_m - u\|_{H_T} \leq C_T k_T^m, \text{ for all } m \in \mathbb{N}, \tag{3.28}$$

where $k_T \in [0,1)$ *and* C_T *are the constants depending only on* T, μ, f, g, $\tilde{u}_0$, $\tilde{u}_1$, λ, ζ, η_1, $\eta_2, \cdots, \eta_q$.

Proof of Theorem 3.3.

(a) *Existence of the solution*. We shall prove that $\{u_m\}$ is a Cauchy sequence in H_T. Let $w_m = u_{m+1} - u_m$. Then w_m satisfies the variational problem

$$\begin{cases} \langle w_m''(t), w\rangle + \lambda a(w_m'(t), w) + \mu_{m+1}(t) a(w_m(t), w) \\ = \displaystyle\int_0^t g(t-s) a(w_m(s), w)ds + \langle F_{m+1}(t) - F_m(t), w\rangle \\ \qquad + [\mu_{m+1}(t) - \mu_m(t)] \langle \Delta u_m(t), w\rangle, \forall w \in V, \\ w_m(0) = w_m'(0) = 0. \end{cases} \tag{3.29}$$

Taking $w = w_m'$ in $(3.29)_1$ and integrating in t, we get

$$\begin{aligned} \bar{\mu}_* \bar{X}_m(t) \leq & \int_0^t \left(\mu_{m+1}'(s) - 2g(0)\right) \|w_m(s)\|_a^2 ds \\ & + 2\int_0^t g(t-s)\, a(w_m(s), w_m(t))ds \\ & - 2\int_0^t d\tau \int_0^\tau g'(\tau - s)\, a(w_m(s), w_m(\tau))ds \end{aligned} \tag{3.30}$$

$$+ 2\int_0^t [\mu_{m+1}(s) - \mu_m(s)] \langle \Delta u_m(s), w'_m(s)\rangle ds$$
$$+ 2\int_0^t \langle F_{m+1}(s) - F_m(s), w'_m(s)\rangle ds.$$

where $\bar{\mu}_* = \min\{1, \mu_*, \lambda\}$ and

$$\bar{X}_m(t) = \left\|w'_m(t)\right\|^2 + \|w_m(t)\|_a^2 + \int_0^t \left\|w'_m(s)\right\|_a^2 ds. \tag{3.31}$$

Now, we require the following lemma, which is a relative generalization of the inequality and equality of energy given in Lions's book [[12], Lemma 1.6, p. 224].

Lemma 3.4. *Let $u \in \tilde{V}_T$ be the weak solution of the following problem*

$$\begin{cases} u'' - \lambda u'_{xx} - \mu(t)u_{xx} + \int_0^t g(t-s)u_{xx}(x,s)ds \\ \qquad\qquad = F(x,t), 0 < x < 1, 0 < t < T, \\ u_x(0,t) - \zeta u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \\ \tilde{u}_0, \tilde{u}_1 \in V \cap H^2,\ \tilde{u}_{0x}(0) - \zeta\tilde{u}_0(0) = 0, \\ F \in L^2(0,T;V), \mu \in H^1(0,T), \mu(t) \geq \mu_* > 0. \end{cases} \tag{3.32}$$

Then we have

$$\begin{aligned} &\frac{1}{2}\left\|u'(t)\right\|_a^2 + \frac{1}{2}\mu(t)\|\Delta u(t)\|^2 + \lambda\int_0^t \left\|\Delta u'(s)\right\|^2 ds \\ &\geq \frac{1}{2}\|\tilde{u}_1\|_a^2 + \frac{1}{2}\mu(0)\|\Delta\tilde{u}_0\|^2 + \frac{1}{2}\int_0^t \mu'(s)\|\Delta u(s)\|^2 ds \\ &\quad + \int_0^t\int_0^\tau g(\tau - s)\langle \Delta u(s), \Delta u'(\tau)\rangle ds\, d\tau \\ &\quad + \int_0^t \langle F(s), -\Delta u'(s)\rangle ds, \quad \text{a.e., } t \in (0,T). \end{aligned} \tag{3.33}$$

Furthermore, if $\tilde{u}_0 = \tilde{u}_1 = 0$ then there is an equality in (3.33).

Proof of Lemma 3.4. This proof is similar to the argument of Lions to obtain the inequality and equality of Energy in [12]. The details are follows.

Fix $t_1, t_2, 0 < t_1 < t_2 < T$ and let $w_{km}(x,t)$ be the function defined as follows

$$w_{km}(x,t) = \left[\left(\theta_m(t)\Delta u'(x,t)\right) * \rho_k(t) * \rho_k(t)\right]\theta_m(t), \tag{3.34}$$

where

(i) θ_m is a continuous, piecewise linear function on $[0,T]$ defined as follows

$$\theta_m(t) = \begin{cases} 0, & t \notin [t_1 + 1/m, t_2 - 1/m], \\ 1, & t \in [t_1 + 2/m, t_2 - 2/m], \\ m(t - t_1 - 1/m), & t \in [t_1 + 1/m, t_1 + 2/m], \\ -m(t - t_2 + 1/m), & t \in [t_2 - 2/m, t_2 - 1/m], \end{cases} \tag{3.35}$$

(ii) $\{\rho_k\}$ is a regularizing sequence in $C_c^\infty(\mathbb{R})$, i.e.,

$$\rho_k \in C_c^\infty(\mathbb{R}),\ \text{supp}\,\rho_k \subset [-1/k, 1/k],\ \rho_k(-t) = \rho_k(t),\ \int_{-\infty}^{\infty}\rho_k(t)dt = 1. \tag{3.36}$$

(iii) $(*)$ is the convolution product in the time variable, i.e.,

$$(u*\rho_k)(x,t)=\int_{-\infty}^{\infty}u(x,t-s)\rho_k(s)ds. \tag{3.37}$$

Taking the scalar product of the function $w_{km}(x,t)$ in (3.34) with Eq. $(3.32)_1$, then integrating with respect to the time variable from 0 to T, we have

$$A_{km}+B_{km}+C_{km}+D_{km}=E_{km}, \tag{3.38}$$

where

$$\begin{aligned}
A_{km}&=\int_0^T\langle u''(t),w_{km}(t)\rangle\,dt, \qquad (3.39)\\
B_{km}&=\lambda\int_0^T a(u'(t),w_{km}(t))\,dt,\\
C_{km}&=\int_0^T\mu(t)\,a(u(t),w_{km}(t))\,dt,\\
D_{km}&=\int_0^T\int_0^t g(t-s)\langle\Delta u(s),w_{km}(t)\rangle\,ds\,dt,\\
E_{km}&=\int_0^T\langle F(t),w_{km}(t)\rangle\,dt.
\end{aligned}$$

By using the properties of the functions $\theta_m(t)$ and $\rho_k(t)$, we can show after some lengthy calculations

$$\begin{aligned}
\lim_{k\to\infty}A_{km}&=\int_0^T\theta_m(t)\theta_m'(t)\left\|u'(t)\right\|_a^2\,dt, \qquad (3.40)\\
\lim_{k\to\infty}B_{km}&=-\lambda\int_0^T\theta_m^2(t)\left\|\Delta u'(t)\right\|^2\,dt,\\
\lim_{k\to\infty}C_{km}&=\frac{1}{2}\int_0^T\theta_m^2(t)\mu'(t)\left\|\Delta u(t)\right\|^2\,dt+\int_0^T\theta_m(t)\theta_m'(t)\mu(t)\left\|\Delta u(t)\right\|^2\,dt,\\
\lim_{k\to\infty}D_{km}&=\int_0^T\theta_m^2(t)\,dt\int_0^t g(t-s)\langle\Delta u(s),\Delta u'(t)\rangle\,ds,\\
\lim_{k\to\infty}E_{km}&=\int_0^T\theta_m^2(t)\langle F(t),\Delta u'(t)\rangle\,dt.
\end{aligned}$$

Letting $m\to\infty$, from (3.38)-(3.40) we obtain

$$\begin{aligned}
&\frac{1}{2}\left\|u'(t_1)\right\|_a^2-\frac{1}{2}\left\|u'(t_2)\right\|_a^2-\lambda\int_{t_1}^{t_2}\left\|\Delta u'(t)\right\|^2dt\\
&\quad+\frac{1}{2}\mu(t_1)\left\|\Delta u(t_1)\right\|^2-\frac{1}{2}\mu(t_2)\left\|\Delta u(t_2)\right\|^2+\frac{1}{2}\int_{t_1}^{t_2}\mu'(t)\left\|\Delta u(t)\right\|^2dt\\
&\quad+\int_{t_1}^{t_2}d\tau\int_0^\tau g(\tau-s)\langle\Delta u(s),\Delta u'(\tau)\rangle\,ds\\
&=\int_{t_1}^{t_2}\langle F(t),\Delta u'(t)\rangle\,dt,\quad \text{a.e., } t_1,t_2\in(0,T),\ t_1<t_2<T.
\end{aligned}$$

or

$$\frac{1}{2}\left\|u'(t_2)\right\|_a^2+\frac{1}{2}\mu(t_2)\left\|\Delta u(t_2)\right\|^2+\lambda\int_0^{t_2}\left\|\Delta u'(t)\right\|^2dt-\frac{1}{2}\int_0^{t_2}\mu'(t)\left\|\Delta u(t)\right\|^2dt$$
$$-\int_0^{t_2}\int_0^{\tau}g(\tau-s)\langle\Delta u(s),\Delta u'(\tau)\rangle ds\,d\tau+\int_0^{t_2}\langle F(t),\Delta u'(t)\rangle dt$$
$$=\frac{1}{2}\left\|u'(t_1)\right\|_a^2+\frac{1}{2}\mu(t_1)\left\|\Delta u(t_1)\right\|^2+\lambda\int_0^{t_1}\left\|\Delta u'(t)\right\|^2dt-\frac{1}{2}\int_0^{t_1}\mu'(t)\left\|\Delta u(t)\right\|^2dt$$
$$-\int_0^{t_1}\int_0^{\tau}g(\tau-s)\langle\Delta u(s),\Delta u'(\tau)\rangle ds\,d\tau+\int_0^{t_1}\langle F(t),\Delta u'(t)\rangle dt, \tag{3.41}$$

a.e., $t_1,t_2\in(0,T)$, $t_1<t_2<T$.

From (3.41) we obtain (3.33), by taking $t_2=t$ and passing to the limit as $t_1\to 0_+$, and using the property of weak lower semicontinuity of the functional $v\longmapsto\left\|v\right\|^2$.

In the case of $\tilde{u}_0=\tilde{u}_1=0$, we prolong u, F by 0 and μ by $\mu(0)$, respectively as $t<0$ and we deduce that equality (3.41) is true for almost $t_1<t_2<T$. Taking $t_1<0$ in (3.41), its right-hand side is 0, we take $t_1\to 0_-$ and we have equality (3.33) when $\tilde{u}_0=\tilde{u}_1=0$.

The proof of Lemma 3.4 is completed. □

We note more that $w_m=u_{m+1}-u_m\in\tilde{V}_T$ is the weak solution of the problem (3.32) coresponding to $\tilde{u}_0=\tilde{u}_1=0$, $\mu(t)=\mu_{m+1}(t)$, $F(t)=\left[\mu_{m+1}(t)-\mu_m(t)\right]\Delta u_m+F_{m+1}(t)-F_m(t)$.

By using Lemma 3.4 with $\tilde{u}_0=\tilde{u}_1=0$, we get

$$\frac{1}{2}\left\|w_m'(t)\right\|_a^2+\frac{1}{2}\mu_{m+1}(t)\left\|\Delta w_m(t)\right\|^2+\lambda\int_0^t\left\|\Delta w_m'(s)\right\|^2ds \tag{3.42}$$
$$=\frac{1}{2}\int_0^t\mu_{m+1}'(s)\left\|\Delta w_m(s)\right\|^2ds+\int_0^t\int_0^{\tau}g(\tau-s)\langle\Delta w_m(s),\Delta w_m'(\tau)\rangle ds\,d\tau$$
$$+\int_0^t\left\langle\left[\mu_{m+1}(s)-\mu_m(s)\right]\Delta u_m(s)+F_{m+1}(s)-F_m(s),\,-\Delta w_m'(s)\right\rangle ds.$$

Put

$$Y_m(t)=\left\|w_m'(t)\right\|_a^2+\mu_{m+1}(t)\left\|\Delta w_m(t)\right\|^2+2\lambda\int_0^t\left\|\Delta w_m'(s)\right\|^2ds, \tag{3.43}$$

we obtain

$$\bar{\mu}_*\bar{Y}_m(t)\le Y_m(t)=\int_0^t\mu_{m+1}'(s)\left\|\Delta w_m(s)\right\|^2ds \tag{3.44}$$
$$+2\int_0^t\int_0^{\tau}g(\tau-s)\langle\Delta w_m(s),\Delta w_m'(\tau)\rangle ds\,d\tau$$
$$+2\int_0^t\left[\mu_{m+1}(s)-\mu_m(s)\right]\langle\Delta u_m(s),-\Delta w_m'(s)\rangle ds$$
$$+2\int_0^t\langle F_{m+1}(s)-F_m(s),-\Delta w_m'(s)\rangle ds,$$

where $\bar{\mu}_*=\min\{1,\mu_*,\lambda\}$ and

$$\bar{Y}_m(t)=\left\|w_m'(t)\right\|_a^2+\left\|\Delta w_m(t)\right\|^2+\int_0^t\left\|\Delta w_m'(s)\right\|^2ds. \tag{3.45}$$

Note that

$$\int_0^t d\tau \int_0^\tau g(\tau-s)\langle \Delta w_m(s), \Delta w_m'(\tau)\rangle\, ds = \int_0^t g(t-s)\langle \Delta w_m(s), \Delta w_m(t)\rangle\, ds - g(0)\int_0^t \|\Delta w_m(s)\|^2\, ds - \int_0^t d\tau \int_0^\tau g'(\tau-s)\langle \Delta w_m(s), \Delta w_m(\tau)\rangle ds,$$

it follows from (3.44) that

$$\begin{aligned} \bar{\mu}_* \bar{Y}_m(t) \le & \int_0^t \left(\mu_{m+1}'(s) - 2g(0)\right) \|\Delta w_m(s)\|^2\, ds \\ & + 2\int_0^t [\mu_{m+1}(s) - \mu_m(s)] \left\langle \Delta u_m(s), -\Delta w_m'(s)\right\rangle ds \\ & + 2\int_0^t \left\langle F_{m+1}(s) - F_m(s), -\Delta w_m'(s)\right\rangle ds \\ & + 2\int_0^t g(t-s)\left\langle \Delta w_m(s), \Delta w_m(t)\right\rangle ds \\ & - 2\int_0^t d\tau \int_0^\tau g'(\tau-s)\left\langle \Delta w_m(s), \Delta w_m(\tau)\right\rangle ds. \end{aligned} \tag{3.46}$$

It follows from (3.30), (3.31), (3.45) and (3.46) that

$$\begin{aligned} \bar{\mu}_* \bar{S}_m(t) \le & \int_0^t \left(\mu_{m+1}'(s) - 2g(0)\right)\left(\|w_m(s)\|_a^2 + \|\Delta w_m(s)\|^2\right) ds \\ & + 2\int_0^t g(t-s)\left[a(w_m(s), w_m(t)) + \langle \Delta w_m(s), \Delta w_m(t)\rangle\right] ds \\ & - 2\int_0^t d\tau \int_0^\tau g'(\tau-s)\left[a(w_m(s), w_m(\tau)) + \langle \Delta w_m(s), \Delta w_m(\tau)\rangle\right] ds \\ & + 2\int_0^t [\mu_{m+1}(s) - \mu_m(s)]\left\langle \Delta u_m(s), w_m'(s) - \Delta w_m'(s)\right\rangle ds \\ & + 2\int_0^t \left\langle F_{m+1}(s) - F_m(s), w_m'(s) - \Delta w_m'(s)\right\rangle ds \\ = & \, J_1 + J_2 + J_3 + J_4 + J_5, \end{aligned} \tag{3.47}$$

where

$$\begin{aligned} \bar{S}_m(t) &= \bar{X}_m(t) + \bar{Y}_m(t) \\ &= \left\|w_m'(t)\right\|_a^2 + \left\|w_m'(t)\right\|^2 + \|w_m(t)\|_a^2 + \|\Delta w_m(t)\|^2 \\ &\quad + \int_0^t \left(\left\|w_m'(s)\right\|_a^2 + \left\|\Delta w_m'(s)\right\|^2\right) ds. \end{aligned} \tag{3.48}$$

We shall estimate the terms J_1, J_2, J_3 on the right-hand side of (3.47) as follows

$$J_1 = \int_0^t \left(\mu_{m+1}'(s) - 2g(0)\right)\left(\|w_m(s)\|_a^2 + \|\Delta w_m(s)\|^2\right) ds \tag{3.49}$$

$$\le (\hat{\eta}_M + 2|g(0)|) \int_0^t \bar{S}_m(s)\, ds;$$

$$\begin{aligned} J_2 &= 2\int_0^t g(t-s)\left[a(w_m(s), w_m(t)) + \langle \Delta w_m(s), \Delta w_m(t)\rangle\right] ds \\ &\le 2\int_0^t |g(t-s)| \sqrt{\bar{S}_m(s)}\sqrt{\bar{S}_m(t)}\, ds \\ &\le \frac{\bar{\mu}_*}{6}\bar{S}_m(t) + \frac{6}{\bar{\mu}_*}\|g\|^2_{L^2(0,T^*)} \int_0^t \bar{S}_m(s)\, ds; \\ J_3 &= -2\int_0^t d\tau \int_0^\tau g'(\tau-s)\left[a(w_m(s), w_m(\tau)) + \langle \Delta w_m(s), \Delta w_m(\tau)\rangle\right] ds \\ &\le 2\int_0^t |g'(t-s)| \sqrt{\bar{S}_m(s)}\sqrt{\bar{S}_m(\tau)}\, ds \\ &\le 2\sqrt{T^*}\left\|g'\right\|_{L^2(0,T^*)} \int_0^t \bar{S}_m(s) ds. \end{aligned}$$

With two last temrs J_4, J_5 on the right-hand side of (3.47), we note that

$$\begin{aligned} |\nabla w_{m-1}(\eta_i, t)| &\le \|\nabla w_{m-1}(t)\|_{C^0([0,1])} \le \sqrt{2}\,\|\nabla w_{m-1}(t)\|_{H^1} \\ &= \sqrt{2}\sqrt{\|\nabla w_{m-1}(t)\|^2 + \|\Delta w_{m-1}(t)\|^2} \\ &\le \sqrt{2}\,\|w_{m-1}(t)\|_{H^2\cap V} \le \sqrt{2}\,\|w_{m-1}\|_{C^0([0,T];H^2\cap V)} \\ &\le \sqrt{2}\,\|w_{m-1}\|_{H_T}, \end{aligned} \tag{3.50}$$

it implies that

$$\begin{aligned} |\mu_{m+1}(t) - \mu_m(t)| &\le \tilde{K}_M \left[\sum_{i=0}^q |\nabla w_{m-1}(\eta_i, t)| + \left|\|\nabla u_m(t)\|^2 - \|\nabla u_{m-1}(t)\|^2\right|\right] \\ &\le \tilde{K}_M \left[\sum_{i=0}^q |\nabla w_{m-1}(\eta_i, t)| + 2M\,\|\nabla w_{m-1}(t)\|\right] \\ &\le \tilde{K}_M \left[\sqrt{2}(q+1)\,\|w_{m-1}\|_{H_T} + 2M\,\|w_{m-1}\|_{H_T}\right] \\ &= \tilde{K}_M \left(\sqrt{2}(q+1) + 2M\right)\|w_{m-1}\|_{H_T}, \end{aligned} \tag{3.51}$$

and

$$\begin{aligned} \|F_{m+1}(t) - F_m(t)\| \le \bar{K}_M(f)\Big[&\|w_{m-1}(t)\| + \left\|w'_{m-1}(t)\right\| + \|\nabla w_{m-1}(t)\| \\ &+ \sum_{i=0}^q |\nabla w_{m-1}(\eta_i, t)| + \left|\|\nabla u_m(t)\|^2 - \|\nabla u_{m-1}(t)\|^2\right|\Big] \\ \le \bar{K}_M(f)\Big[&2\,\|w_{m-1}\|_{H_T} + \sqrt{2}(q+1)\,\|w_{m-1}\|_{H_T} + 2M\,\|w_{m-1}\|_{H_T}\Big] \\ = \bar{K}_M(f)&\left(2 + \sqrt{2}(q+1) + 2M\right)\|w_{m-1}\|_{H_T}. \end{aligned} \tag{3.52}$$

Hence, two terms J_4, J_5 are estimated as follows

$$J_4 = 2\int_0^t \left[\mu_{m+1}(s) - \mu_m(s)\right] \langle \Delta u_m(s), w'_m(s) - \Delta w'_m(s)\rangle ds \tag{3.53}$$

$$\leq 2\sqrt{2}\tilde{K}_M\left(\sqrt{2}(q+1)+2M\right)M\left\|w_{m-1}\right\|_{H_T}\int_0^t\left(\left\|w_m'(s)\right\|_a^2+\left\|\Delta w_m'(s)\right\|^2\right)^{1/2}ds$$
$$\leq \frac{\bar{\mu}_*}{6}\bar{S}_m(t)+\frac{12}{\bar{\mu}_*}TM^2\tilde{K}_M^2\left(\sqrt{2}(q+1)+2M\right)^2\left\|w_{m-1}\right\|_{H_T}^2;$$

$$J_5=2\int_0^t\left\langle F_{m+1}(s)-F_m(s),w_m'(s)-\Delta w_m'(s)\right\rangle ds$$
$$\leq 2\sqrt{2}\bar{K}_M(f)\left(2+\sqrt{2}(q+1)+2M\right)\left\|w_{m-1}\right\|_{H_T}\sqrt{T}\sqrt{\bar{S}_m(t)}$$
$$\leq \frac{\bar{\mu}_*}{6}\bar{S}_m(t)+\frac{12}{\bar{\mu}_*}T\bar{K}_M^2(f)\left(2+\sqrt{2}(q+1)+2M\right)^2\left\|w_{m-1}\right\|_{H_T}^2.$$

It follows from (3.47), (3.49) and (3.53) that

$$\bar{S}_m(t)\leq T\tilde{D}_1(M)\left\|w_{m-1}\right\|_{H_T}^2+2\tilde{D}_2(M)\int_0^t\bar{S}_m(s)ds, \tag{3.54}$$

where $\tilde{D}_1(M)$ and $\tilde{D}_2(M)$ are defined as in (3.23).

Using Gronwall's Lemma, we deduce from (3.54) that

$$\left\|w_m\right\|_{H_T}\leq k_T\left\|w_{m-1}\right\|_{H_T}\quad\forall m\in\mathbb{N}, \tag{3.55}$$

where $k_T=3\sqrt{T\tilde{D}_1(M)}\exp\left(T\tilde{D}_2(M)\right)\in(0,1)$ is defined as in (3.22), it leads to

$$\left\|u_m-u_{m+p}\right\|_{H_T}\leq\left\|u_0-u_1\right\|_{H_T}(1-k_T)^{-1}k_T^m,\ \forall m,\ p\in\mathbb{N}. \tag{3.56}$$

It follows that $\{u_m\}$ is a Cauchy sequence in H_T. Then there exists $u\in H_T$ such that

$$u_m\to u \text{ strongly in } H_T. \tag{3.57}$$

Note that $u_m\in W(M,T)$, then there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\begin{cases} u_{m_j}\to u & \text{in} \quad L^\infty(0,T;H^2\cap V) \text{ weak*},\\ u_{m_j}'\to u' & \text{in} \quad L^\infty(0,T;H^2\cap V) \text{ weak*},\\ u_{m_j}''\to u'' & \text{in} \quad L^2(0,T;V) \text{ weak},\\ u\in W(M,T). \end{cases} \tag{3.58}$$

We also note that

$$\left\|F_m-f[u]\right\|_{L^\infty(0,T;L^2)}\leq\bar{K}_M(f)\left(2+\sqrt{2}(q+1)+2M\right)\left\|u_{m-1}-u\right\|_{H_T}, \tag{3.59}$$
$$\left\|\mu_m-\mu[u]\right\|_{L^\infty(0,T)}\leq\tilde{K}_M(\mu)\left(\sqrt{2}(q+1)+2M\right)\left\|u_{m-1}-u\right\|_{H_T}.$$

Combining (3.57) and (3.59), we obtain

$$\begin{aligned} F_m&\to f[u] \quad \text{strongly in } L^\infty(0,T;L^2),\\ \mu_m&\to\mu[u] \quad \text{strongly in } L^\infty(0,T). \end{aligned} \tag{3.60}$$

Finally, passing to limit in (3.9), (3.10) as $m=m_j\to\infty$, it implies from (3.57), (3.58) and (3.60) that there exists $u\in W(M,T)$ satisfying (2.8)-(2.9).

Furthermore, (2.8) and $(3.58)_4$ imply that

$$u''=\lambda\Delta u'+\mu[u](t)\Delta u-\int_0^t g(t-s)\Delta u(s)ds+f[u]\in L^\infty(0,T;L^2),$$

so we obtain $u \in W_1(M,T)$. The existence proof is completed.

(b) *Uniqueness of the solution.* Let $u_1,\, u_2 \in W_1(M,T)$ be two weak solution of Prob. (1.1). Then $u = u_1 - u_2$ satisfies the variational problem

$$\begin{cases} \langle u''(t), w\rangle + \lambda a(u'(t), w) + \mu_1(t) a(u(t), w) = \bar{\mu}(t)\langle \Delta u_2(t), w\rangle \\ \qquad + \displaystyle\int_0^t g(t-s) a(u(s), w) ds + \left\langle \bar{F}(t), w \right\rangle, \forall w \in V, \\ u(0) = u'(0) = 0, \end{cases} \tag{3.61}$$

where

$$\begin{aligned} \bar{\mu}(t) &= \mu_1(t) - \mu_2(t) = \mu[u_1](t) - \mu[u_2](t), \\ \bar{F}(x,t) &= F_1(x,t) - F_2(x,t), \\ F_i(x,t) &= f[u_i](x,t), \quad \mu_i(t) = \mu[u_i](t), \quad i = 1, 2. \end{aligned} \tag{3.62}$$

Using Lemma 3.4 with $\tilde{u}_0 = \tilde{u}_1 = 0$, we have

$$\begin{aligned} \bar{\mu}_* \bar{Z}(t) \leq & \int_0^t \left(\mu_1'(s) - 2g(0)\right) \left(\|u(s)\|_a^2 + \|\Delta u(s)\|^2\right) ds \\ & + 2\int_0^t g(t-s) \left(a(u(s), u(t)) + \langle \Delta u(s), \Delta u(t)\rangle\right) ds \\ & - 2\int_0^t d\tau \int_0^\tau g'(\tau - s) \left(a(u(s), u(\tau)) + \langle \Delta u(s), \Delta u(\tau)\rangle\right) ds \\ & + 2\int_0^t \bar{\mu}(s)\langle \Delta u_2(s), u'(s) - \Delta u'(s)\rangle ds \\ & + 2\int_0^t \langle \bar{F}(s), u'(s) - \Delta u'(s)\rangle ds. \end{aligned} \tag{3.63}$$

where $\bar{\mu}_* = \min\{1, \mu_*, \lambda\}$ and $\bar{Z}(t) = \|u'(t)\|^2 + \|u'(t)\|_a^2 + \|u(t)\|_a^2 + \|\Delta u(t)\|^2 + \int_0^t \left(\left\|u'(s)\right\|_a^2 + \left\|\Delta u'(s)\right\|^2\right) ds.$

With the estimations similar to that for the terms of $\bar{S}_m(t)$ as above, we also obtain the following estimate

$$\bar{Z}(t) \leq K_M^* \int_0^t \bar{Z}(s) ds, \tag{3.64}$$

where

$$\begin{aligned} K_M^* = & \frac{2}{\bar{\mu}_*}\left[\hat{\eta}_M + 2\left(|g(0)| + \frac{1}{\bar{\mu}_*}\|g\|_{L^2(0,T^*)}^2 + \sqrt{T^*}\left\|g'\right\|_{L^2(0,T^*)}\right)\right] \\ & + \frac{8}{\bar{\mu}_*}\left[M\tilde{K}_M(\mu) + \bar{K}_M(f)\right]\left(3 + \sqrt{2}(q+1) + 2M\right). \end{aligned}$$

Using Gronwall's Lemma, it follows from (3.64) that $Z(t) \equiv 0$, ie., $u_1 \equiv u_2$.

Theorem 3.3 is proved completely. □

4 A special case

In this section, we shall consider the following problem

$$(P_q)\begin{cases} u_{tt}-\lambda u_{txx}-\mu_q[u](t)u_{xx}+\int_0^t g(t-s)u_{xx}(x,s)ds \\ \qquad = f_q[u](x,t),\, 0<x<1,\, 0<t<T, \\ u_x(0,t)-\zeta u(0,t)=u(1,t)=0, \\ u(x,0)=\tilde{u}_0(x),\ u_t(x,0)=\tilde{u}_1(x), \end{cases}$$

which is a specific case of Prob. (1.1). We also consider the Robin-Dirichlet problem defined as follows

$$(P_\infty)\begin{cases} u_{tt}-\lambda u_{txx}-\mu[u](t)u_{xx}+\int_0^t g(t-s)u_{xx}(x,s)ds \\ \qquad = f[u](x,t),\, 0<x<1,\, 0<t<T, \\ u_x(0,t)-\zeta u(0,t)=u(1,t)=0, \\ u(x,0)=\tilde{u}_0(x),\ u_t(x,0)=\tilde{u}_1(x), \end{cases}$$

where $\zeta\geq 0$, $\lambda>0$ are constants and μ, f, g, $\tilde{u}_0$, $\tilde{u}_1$ are given functions, in which

$$\begin{aligned} \mu[u](t) &= \mu\left(t,\|u_x(t)\|^2\right), \\ \mu_q[u](t) &= \mu\left(t,S_q[u](t)\right), \\ f[u](x,t) &= f\left(x,t,u(x,t),u'(x,t),u_x(x,t),\|u_x(t)\|^2\right), \\ f_q[u](x,t) &= f\left(x,t,u(x,t),u'(x,t),u_x(x,t),S_q[u](t)\right), \\ S_q[u](t) &= \frac{1}{q}\sum_{i=0}^{q-1}u_x^2\left(\frac{i}{q},t\right). \end{aligned} \tag{4.1}$$

We first note that, for a.e. $t\in[0,T]$, because the function $y\longmapsto u_x(y,t)$ is continuous on $[0,1]$, we get

$$S_q[u](t)=\frac{1}{q}\sum\nolimits_{i=0}^{q-1}u_x^2(\frac{i}{q},t)\to\|u_x(t)\|^2,\ \text{khi } q\to\infty.$$

Therefore, in a sense defined below (Lemma 4.1), we can verify that $\mu_q[u](t)$, $f_q[u](x,t)$ converge to $\mu[u](t)$, $f[u](x,t)$ as $q\to\infty$, respectively. In this case, we shall prove that the solution of (P_q) converges to that of (P_∞) as $q\to+\infty$. For this goal, we make the following assumptions.

(H_1) : $\tilde{u}_0,\tilde{u}_1\in V\cap H^2,\ \tilde{u}_{0x}(0)-\zeta\tilde{u}_0(0)=0$;
(H_2) : $g\in H^1(0,T^*)$;
$(\bar{H}_3)$: $\mu\in C^1([0,T^*]\times\mathbb{R}_+^2)$ such that $\mu(t,z)\geq\mu_*>0,\ \forall(t,z)\in[0,T^*]\times\mathbb{R}_+$;
$(\bar{H}_4)$: $f\in C^1([0,1]\times[0,T^*]\times\mathbb{R}^3\times\mathbb{R}_+)$.

For each $M>0$ given, we set the constants $\bar{K}_M(f)$, $\tilde{K}_M(\mu)$ as follows

$$\bar{K}_M=\bar{K}_M(f)=\|f\|_{C^1(\bar{A}_M)}=\|f\|_{C^0(\bar{A}_M)}+\sum_{i=1}^{6}\|D_if\|_{C^0(\bar{A}_M)},$$

$$\tilde{K}_M=\tilde{K}_M(\mu)=\|\mu\|_{C^1(\tilde{A}_M)}=\|\mu\|_{C^0(\tilde{A}_M)}+\sum_{i=1}^{2}\|D_i\mu\|_{C^0(\tilde{A}_M)},$$

$$\|f\|_{C^0(\bar{A}_M)}=\sup_{(x,t,y_1,\dots,y_4)\in\bar{A}_M}|f(x,t,y_1,\dots,y_4)|,$$

$$
\begin{aligned}
\|\mu\|_{C^0(\tilde{A}_M)} &= \sup_{(t,z)\in\tilde{A}_M} |\mu(t,z)|\,,\\
\bar{A}_M &= [0,1]\times[0,T^*]\times[-M,M]^2\times[-\sqrt{2}M,\sqrt{2}M]\times[0,M^2],\\
\tilde{A}_M &= [0,T^*]\times[0,M^2].
\end{aligned}
$$

Using the assumptions (H_1), (H_2), $(\bar{H}_3)$, $(\bar{H}_4)$ and the results of Theorem 3.3, there exist positive constants M, T independent of q such that the problems (P_q), (P_∞), respectively, have the unique weak solutions u_q, u_∞, satisfying

$$u_q,\ u_\infty \in W_1(M,T),\ \text{for all } q\in\mathbb{N}. \tag{4.2}$$

From (4.2), we deduce that there exists a subsequence of $\{u_q\}$, with the same notation, such that

$$
\begin{cases}
u_q \to u & \text{in} \quad L^\infty(0,T;H^2\cap V) \text{ weak*},\\
u_q' \to u' & \text{in} \quad L^\infty(0,T;H^2\cap V) \text{ weak*},\\
u_q'' \to u'' & \text{in} \quad L^2(0,T;V) \text{ weak},\\
u\in W_1(M,T).
\end{cases} \tag{4.3}
$$

Applying the lemma of Aubin-Lions, a classical compactness result in the spaces $L^p(0,T;X)$, there exists a subsequence $\{u_q\}$, with the same symbol, such that

$$
\begin{cases}
u_q \to u & \text{in} \quad C([0,T];V) \text{ strongly},\\
u_q' \to u' & \text{in} \quad C([0,T];V) \text{ strongly}.
\end{cases} \tag{4.4}
$$

Because u_q is the unique weak solution of (P_q), we have

$$
\begin{aligned}
&\int_0^T \langle u_q''(t),w\rangle\varphi(t)\,dt + \lambda\int_0^T a(u_q'(t),w)\varphi(t)\,dt + \int_0^T \mu_q[u_q](t)a(u_q(t),w)\varphi(t)\,dt\\
&= \int_0^T \left(\int_0^t g(t-s)a(u_q(s),w)\,ds\right)\varphi(t)\,dt + \int_0^T \langle f_q[u_q](t),w\rangle\,\varphi(t)\,dt.
\end{aligned} \tag{4.5}
$$

By (4.3) and (4.4) we get

$$
\begin{aligned}
\int_0^T \langle u_q''(t),w\rangle\varphi(t)\,dt &\to \int_0^T \langle u''(t),w\rangle\varphi(t)\,dt,\\
\int_0^T a(u_q(t),w)\varphi(t)\,dt &\to \int_0^T a(u(t),w)\varphi(t)\,dt,\\
\lambda\int_0^T a(u_q'(t),w)\varphi(t)\,dt &\to \lambda\int_0^T a(u'(t),w)\varphi(t)\,dt,\\
\int_0^T \varphi(t)\,dt\int_0^t g(t-s)a(u_q(s),w)\,ds &\to \int_0^T \varphi(t)\,dt\int_0^t g(t-s)a(u(s),w)\,ds.
\end{aligned} \tag{4.6}
$$

We need to check

$$
\begin{aligned}
&\text{(i)} \quad \int_0^T \mu_q[u_q](t)a(u_q(t),w)\varphi(t)dt \to \int_0^T \mu[u](t)a(u(t),w)\varphi(t)dt,\\
&\text{(ii)} \quad \int_0^T \langle f_q[u_q]\,(t)\,,w\rangle\,\varphi(t)dt \to \int_0^T \langle f[u]\,(t)\,,w\rangle\,\varphi(t)dt,
\end{aligned} \tag{4.7}
$$

so we need prove the following lemma.

Lemma 4.1: *There exists a subsequence of* $\{u_q\}$, *which is also denoted by* $\{u_q\}$, *such that*

$$\begin{aligned}
&\text{(i)}\ \left\| S_q[u] - \|u_x(\cdot)\|^2 \right\|^2_{L^2(0,T)} = \int_0^T \left| S_q[u](t) - \int_0^1 u_x^2(y,t)dy \right|^2 dt \to 0,\ \text{as } q \to \infty,\\
&\text{(ii)}\ \|S_q[u_q] - S_q[u]\|_{C([0,T])} \to 0,\ \text{as } q \to \infty,\\
&\text{(iii)}\ \left\| S_q[u_q] - \|u_x(\cdot)\|^2 \right\|^2_{L^2(0,T)} = \int_0^T \left| S_q[u_q](t) - \int_0^1 u_x^2(y,t)dy \right|^2 dt \to 0,\ \text{as } q \to \infty.
\end{aligned} \tag{4.8}$$

*Proof of Lemma 4.1***.**
Proof (i). We note that

$$\frac{1}{q}\sum\nolimits_{i=0}^{q-1} h(\frac{i}{q}) \to \int_0^1 h(y)dy,\ \forall h \in C([0,1]). \tag{4.9}$$

Since $u_x \in L^\infty(0,T;V) \hookrightarrow L^\infty(0,T;C(\bar{\Omega}))$, so the function $y \longmapsto u_x^2(y,t)$, a.e. $t \in [0,T]$ belongs to $C(\bar{\Omega})$, then,

$$S_q[u](t) = \frac{1}{q}\sum\nolimits_{i=0}^{q-1} u_x^2(\frac{i}{q},t) \to \int_0^1 u_x^2(y,t)dy = \|u_x(t)\|^2,\ \text{as } q \to \infty. \tag{4.10}$$

Note that

$$\begin{aligned}
|S_q[u](t)| &\le \frac{1}{q}\sum_{i=0}^{q-1} u_x^2\left(\frac{i}{q},t\right) \le \frac{1}{q}\sum_{i=0}^{q-1} \|u_x(t)\|^2_{C^0(\bar{\Omega})} \\
&= \|u_x(t)\|^2_{C^0(\bar{\Omega})} \le 2\,\|u_x(t)\|^2_{H^1} = 2\left(\|u_x(t)\|^2 + \|\Delta u(t)\|^2\right) \le 2M^2.
\end{aligned} \tag{4.11}$$

so

$$\left| S_q[u](t) - \|u_x(t)\|^2 \right| \le 3M^2,\ \text{for all } q \in \mathbb{N} \text{ and a.e. } t \in [0,T]. \tag{4.12}$$

Applying the dominated convergence theorem, we deduce that (i) is true.
Proof (ii). By $u_q \in W(M,T)$, we deduce that

$$\begin{aligned}
&u_q \in C\left([0,T];H^2 \cap V\right) \cap C^1([0,T];V) \cap L^\infty(0,T;H^2 \cap V),\\
&u_q' \in C([0,T];V) \cap L^\infty(0,T;H^2 \cap V).
\end{aligned} \tag{4.13}$$

Consider the sequence $\{h_q\}$ defined by $h_q = u_{qx}$, by $H^1 \hookrightarrow C([0,1]) \equiv E$, we have $\{h_q\} \subset C\left([0,T];H^1\right) \subset C([0,T];E)$. We shall show that there exists a subsequence of $\{h_q\}$, it is also denoted by $\{h_q\}$, such that

$$h_q \to u_x \text{ strongly in } C([0,T];E). \tag{4.14}$$

Using Ascoli-Arzela theorem in $C([0,T];E)$, we shall prove that

$$\begin{aligned}
&\text{(j)}\quad \{h_q\} \text{ is equicontinuous in } C([0,T];E),\\
&\text{(jj)}\quad \text{for every } t \in [0,T],\ \{h_q(t) : q \in \mathbb{N}\} \text{ is relatively compact in } E.
\end{aligned} \tag{4.15}$$

Proof $(4.15)_{(j)}$. For all $t_1, t_2 \in [0,T]$, $t_1 \le t_2$, $\forall q \in \mathbb{N}$, by $(4.13)_{(ii)}$, we have

$$\|h_q(t_2) - h_q(t_1)\|_E = \left\| \int_{t_1}^{t_2} h_q'(t)\,dt \right\|_E \le \int_{t_1}^{t_2} \left\| h_q'(t) \right\|_E\,dt \tag{4.16}$$

$$= \int_{t_1}^{t_2} \left\| u'_{qx}(t) \right\|_E dt \le \sqrt{2} \int_{t_1}^{t_2} \left\| u'_{qx}(t) \right\|_{H^1} dt$$
$$\le \sqrt{2} \left| t_2 - t_1 \right| \left\| u'_q \right\|_{L^\infty(0,T;H^2 \cap V)} \le \sqrt{2} M \left| t_2 - t_1 \right| .$$

This implies that $(4.15)_{(j)}$ holds.
Proof $(4.15)_{(jj)}$. By $(4.13)_{(i)}$, we have

$$\left\| h_q(t) \right\|_{H^1} = \left\| u_{qx}(t) \right\|_{H^1} = \left\| u_q(t) \right\|_{H^2 \cap V} \le \left\| u_q \right\|_{L^\infty(0,T;H^2 \cap V)} \le M. \tag{4.17}$$

Because the imbedding $H^1 \hookrightarrow C([0,1]) = E$ is compact, there exists a convergent subsequence of $\{h_q(t)\}$ (in E). This implies $(4.15)_{(jj)}$ holds.

From (4.15), we deduce that there exists a subsequence of $\{h_q\}$, also denoted by $\{h_q\}$, such that

$$h_q \to h \text{ strongly in } C([0,T];E) . \tag{4.18}$$

By $C([0,T];E) \hookrightarrow L^2(Q_T)$, we deduce that

$$h_q \to h \text{ strongly in } L^2(Q_T) . \tag{4.19}$$

On the other hand, from $(4.4)_{(i)}$ we obtain

$$h_q = u_{qx} \to u_x \text{ strongly in } L^2(Q_T) . \tag{4.20}$$

It follows from (4.19) and (4.20) that $h = u_x$, thus (4.14) is proved.
On the other hand, from (4.2) we obtain the following estimation

$$\begin{aligned} \left| S_q[u_q](t) - S_q[u](t) \right| &\le \frac{1}{q} \sum_{i=0}^{q-1} \left| u_{qx}^2 \left(\frac{i}{q}, t \right) - u_x^2 \left(\frac{i}{q}, t \right) \right| \\ &\le \left(\left\| u_{qx}(t) \right\|_E + \left\| u_x(t) \right\|_E \right) \left\| u_{qx}(t) - u_x(t) \right\|_E \\ &\le \sqrt{2} \left(\left\| u_{qx}(t) \right\|_{H^1} + \left\| u_x(t) \right\|_{H^1} \right) \left\| u_{qx} - u_x \right\|_{C([0,T];E)} \\ &\le 2\sqrt{2} M \left\| u_{qx} - u_x \right\|_{C([0,T];E)} . \end{aligned} \tag{4.21}$$

Hence

$$\left\| S_q[u_q] - S_q[u] \right\|_{C([0,T])} \le 2\sqrt{2} M \left\| u_{qx} - u_x \right\|_{C([0,T];E)} . \tag{4.22}$$

From (4.14) and (4.22), we obtain $(4.8)_{(ii)}$ holds. We also deduce that (ii) is true.
Proof (iii). By $(4.8)_{(i)}$ and $(4.8)_{(ii)}$, we obtain

$$\begin{aligned} \left\| S_q[u_q] - \left\| u_x(\cdot) \right\|^2 \right\|_{L^2(0,T)} &\le \left\| S_q[u_q] - S_q[u] \right\|_{L^2(0,T)} + \left\| S_q[u] - \left\| u_x(\cdot) \right\|^2 \right\|_{L^2(0,T)} \\ &\le \sqrt{T} \left\| S_q[u_q] - S_q[u] \right\|_{C([0,T])} + \left\| S_q[u] - \left\| u_x(\cdot) \right\|^2 \right\|_{L^2(0,T)} \\ &\to 0. \end{aligned}$$

as $q \to \infty$. Thus, $(4.8)_{(iii)}$ holds. Lemma 4.1 is proved. □
Now, we continue with the proof of (4.7).
Proof $(4.7)_{(i)}$.
By the following inequality

$$\left| \mu_q[u_q](t) - \mu[u](t) \right| = \left| \mu\left(t, S_q[u_q](t)\right) - \mu\left(t, \left\| u_x(t) \right\|^2\right) \right|$$

$$\leq \tilde{K}_M(\mu) \left| S_q[u_q](t) - \|u_x(t)\|^2 \right|,$$

we deduce from $(4.8)_{(iii)}$ that

$$\|\mu_q[u_q] - \mu[u]\|_{L^2(0,T)} \leq \tilde{K}_M(\mu) \left\| S_q[u_q] - \|u_x(\cdot)\|^2 \right\|_{L^2(0,T)} \to 0, \quad \text{as } q \to \infty. \tag{4.23}$$

Note more that $|\mu_q[u_q](t)| \leq \tilde{K}_M(\mu)$, we obtain

$$\begin{aligned}
&\left| \int_0^T \mu_q[u_q](t)\, a(u_q(t), w)\, \varphi(t)\, dt - \int_0^T \mu[u](t)\, a(u(t), w)\, \varphi(t)\, dt \right| \qquad (4.24)\\
&\leq \int_0^T \mu_q[u_q](t)\, |a(u_q(t) - u(t), w)\varphi(t)|\, dt \\
&\quad + \int_0^T |\mu_q[u_q](t) - \mu[u](t)|\, |a(u(t), w)\varphi(t)|\, dt \\
&\leq \int_0^T \mu_q[u_q](t)\, \|u_q(t) - u(t)\|_a\, \|w\|_a\, |\varphi(t)|\, dt \\
&\quad + \int_0^T |\mu_q[u_q](t) - \mu[u](t)|\, \|u(t)\|_a\, \|w\|_a\, |\varphi(t)|\, dt \\
&\leq \|w\|_a\, \|\varphi\|_{L^2(0,T)} \left[\sqrt{T}\tilde{K}_M(\mu)\, \|u_q - u\|_{C([0,T];V)} + \|u\|_{C([0,T];V)}\, \|\mu_q[u_q] - \mu[u]\|_{L^2(0,T)} \right] \\
&\to 0, \quad \text{as } q \to \infty.
\end{aligned}$$

It follows from $(4.4)_1$, (4.23) and (4.24) that $(4.7)_{(i)}$ holds.
Proof $(4.7)_{(ii)}$. We have

$$\begin{aligned}
&\|f_q[u_q](t) - f[u](t)\| \qquad (4.25)\\
&\leq \bar{K}_M(f) \Big[\|u_q(t) - u(t)\| + \left\|u_q'(t) - u'(t)\right\| + \|u_{qx}(t) - u_x(t)\| \\
&\qquad + \left| S_q[u_q](t) - \|u_x(t)\|^2 \right| \Big] \\
&\leq \bar{K}_M(f) \Big[2\, \|u_q - u\|_{C([0,T];V)} + \left\|u_q' - u'\right\|_{C([0,T];V)} \\
&\qquad + \left| S_q[u_q](t) - \|u_x(t)\|^2 \right| \Big],
\end{aligned}$$

so

$$\begin{aligned}
&\|f_q[u_q] - f[u]\|_{L^2(Q_T)} \qquad (4.26)\\
&\leq \bar{K}_M(f) \left[\sqrt{T} \left(2\, \|u_q - u\|_{C([0,T];V)} + \left\|u_q' - u'\right\|_{C([0,T];V)} \right) \right. \\
&\qquad \left. + \left\| S_q[u_q] - \|u_x(\cdot)\|^2 \right\|_{L^2(0,T)} \right].
\end{aligned}$$

Therefore

$$\left| \int_0^T \langle f_q[u_q](t), w \rangle\, \varphi(t)\, dt - \int_0^T \langle f[u](t), w \rangle\, \varphi(t)\, dt \right| \tag{4.27}$$

$$\leq \int_0^T |\langle f_q[u_q](t) - f[u](t), w\rangle \, \varphi(t)| \, dt$$
$$\leq \|f_q[u_q] - f[u]\|_{L^2(Q_T)} \|w\| \, \|\varphi\|_{L^2(0,T)} \, .$$

Thus, it follows from (4.4), $(4.8)_{(iii)}$, (4.26), (4.27), that $(4.7)_{(ii)}$ holds. □

Finally, letting $q \to \infty$, in (4.5), it follows from (4.6) and (4.7) that $u \in W(M,T)$ satisfying the equation

$$\int_0^T \langle u''(t), w\rangle \varphi(t) \, dt + \lambda \int_0^T a(u'(t), w)\varphi(t) \, dt + \int_0^T \mu[u](t) \, a(u(t), w) \, \varphi(t) \, dt$$
$$= \int_0^T \left(\int_0^t g(t-s) \, a(u(s), w) \, ds \right) \varphi(t) \, dt + \int_0^T \langle f[u](t), w\rangle \, \varphi(t) \, dt. \tag{4.28}$$

for all $w \in V$, $\varphi \in C_c^\infty(0,T)$, together with the initial conditions

$$u(0) = \tilde{u}_0, \; u'(0) = \tilde{u}_1. \tag{4.29}$$

Furthermore, $(4.28)_1$ and $(4.3)_4$ lead to

$$u'' = \lambda \Delta u' + \mu[u]\Delta u - \int_0^t g(t-s)\Delta u(s) ds + f[u] \in L^\infty(0,T;L^2),$$

so we obtain $u \in W_1(M,T)$. The existence proof is completed.

Next, it is not difficult to prove the uniqueness of a weak solution of (P) and so $u = u_\infty$. Then, we have the following theorem.

Theorem 4.2. *Let (H_1), (H_2), $(\bar{H}_3)$, $(\bar{H}_4)$ hold. Then, there exist positive constants M, $T > 0$ such that*

(i) (P_∞) has an unique weak solution $u \in W_1(M,T)$.

(ii) The sequence $\{u_q\}$ converges to the solution u of (P_∞) strongly in H_T.

Furthermore, we also have the estimation

$$\|u_q - u\|_{H_T} \leq C_T \left\| S_q[u_q] - \|u_x(\cdot)\|^2 \right\|_{L^2(0,T)}, \text{ for all } q \in \mathbb{N}, \tag{4.30}$$

where C_T is the constant depending only on T, μ, f, g, $\tilde{u}_0$, $\tilde{u}_1$, λ, ζ.

Proof of Theorem 4.2. We only need prove that (ii).

Put

$$\begin{aligned} v_q &= u_q - u, \\ \bar{\mu}_q(t) &= \mu_q[u_q](t) - \mu[u](t), \\ \bar{f}_q(x,t) &= f_q[u_q](x,t) - f[u](x,t), \end{aligned} \tag{4.31}$$

then $v_q \in \tilde{V}_T$ be the weak solution of the following problem (3.32) coresponding to $\tilde{u}_0 = \tilde{u}_1 = 0$, $\mu(t) = \mu_q[u_q](t)$, $F(t) = \bar{\mu}_q(t)\Delta u + \bar{f}_q(x,t)$.

Similarly, by using Lemma 3.4 with $\tilde{u}_0 = \tilde{u}_1 = 0$, we have

$$\bar{\mu}_* \bar{Z}_q(t) \leq \int_0^t \left(\frac{d}{ds}\mu_q[u_q](s) - 2g(0) \right) \left(\|v_q(s)\|_a^2 + \|\Delta v_q(s)\|^2 \right) ds \tag{4.32}$$
$$+ 2\int_0^t g(t-s) \left(a(v_q(s), v_q(t)) + \langle \Delta v_q(s), \Delta v_q(t)\rangle \right) ds$$

$$
\begin{aligned}
&- 2\int_0^t d\tau \int_0^\tau g'(\tau - s)\left(a(v_q(s), v_q(\tau)) + \langle \Delta v_q(s), \Delta v_q(\tau)\rangle\right) ds \\
&+ 2\int_0^t \bar{\mu}_q(s)\langle \Delta u(s), v_q'(s) - \Delta v_q'(s)\rangle ds \\
&+ 2\int_0^t \langle \bar{f}_q(s), v_q'(s) - \Delta v_q'(s)\rangle ds \\
&= J_1 + \cdots + J_5,
\end{aligned}
$$

where $\bar{\mu}_* = \min\{1, \mu_*, \lambda\}$ and

$$
\begin{aligned}
\bar{Z}_q(t) &= \left\|v_q'(t)\right\|^2 + \left\|v_q'(t)\right\|_a^2 + \|v_q(t)\|_a^2 \\
&\quad + \|\Delta v_q(t)\|^2 + \int_0^t \left(\left\|v_q'(s)\right\|_a^2 + \left\|\Delta v_q'(s)\right\|^2\right) ds.
\end{aligned} \tag{4.33}
$$

With the following estimations

$$
\begin{aligned}
\left|\frac{d}{dt}\left(\mu_q[u_q](t)\right)\right| &= \left|D_1\mu\left(t, S_q[u_q](t)\right) + D_2\mu\left(t, S_q[u_q](t)\right)\cdot \frac{2}{q}\sum_{i=0}^{q-1} u_{qx}\left(\tfrac{i}{q}, t\right) u_{qx}'\left(\tfrac{i}{q}, t\right)\right| \\
&\leq \tilde{K}_M(\mu)\left[1 + \frac{4}{q}\sum_{i=0}^{q-1}\|u_{qx}(t)\|_{H^1}\left\|u_{qx}'(t)\right\|_{H^1}\right] \\
&= \tilde{K}_M(\mu)\left[1 + 4\|u_{qx}(t)\|_{H^1}\left\|u_{qx}'(t)\right\|_{H^1}\right] \\
&\leq \tilde{K}_M(\mu)(1 + 4M^2) \equiv \eta_M^*, \qquad (4.34) \\
|\bar{\mu}_q(t)| &= |\mu_q[u_q](t) - \mu[u](t)| \\
&\leq \tilde{K}_M(\mu)\left|S_q[u_q](t) - \|u_x(t)\|^2\right|, \\
\left\|\bar{f}_q(t)\right\| &= \|f_q[u_q](t) - f[u](t)\| \\
&\leq \bar{K}_M(f)\left[\left\|v_q'(t)\right\| + 2\|v_{qx}(t)\| + \left|S_q[u_q](t) - \|u_x(t)\|^2\right|\right] \\
&\leq \bar{K}_M(f)\left[3\sqrt{\bar{Z}_q(t)} + \left|S_q[u_q](t) - \|u_x(t)\|^2\right|\right],
\end{aligned}
$$

$$
\int_0^t \left\|v_q'(s) - \Delta v_q'(s)\right\|^2 ds \leq 2\int_0^t \left(\left\|v_q'(s)\right\|^2 + \left\|\Delta v_q'(s)\right\|^2\right) ds \leq 2\bar{Z}_q(t),
$$

we obtain the estimations for the terms in the right-hand side of (4.32) as follows

$$
\begin{aligned}
J_1 &= \int_0^t \left(\frac{d}{ds}\left(\mu_q[u_q](s)\right) - 2g(0)\right)\left(\|v_q(s)\|_a^2 + \|\Delta v_q(s)\|^2\right) ds \\
&\leq \left(\eta_M^* + 2|g(0)|\right)\int_0^t \bar{Z}_q(s) ds; \qquad (4.35) \\
J_2 &= 2\int_0^t g(t - s)\left(a(v_q(s), v_q(t)) + \langle \Delta v_q(s), \Delta v_q(t)\rangle\right) ds \\
&\leq \frac{\bar{\mu}_*}{6}\bar{Z}_q(t) + \frac{6}{\bar{\mu}_*}\|g\|_{L^2(0,T^*)}^2 \int_0^t \bar{Z}_q(s) ds; \\
J_3 &= -2\int_0^t d\tau \int_0^\tau g'(\tau - s)\left(a(v_q(s), v_q(\tau)) + \langle \Delta v_q(s), \Delta v_q(\tau)\rangle\right) ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq 2\sqrt{T^*}\left\|g'\right\|_{L^2(0,T^*)}\int_0^t \bar{Z}_q(s)ds;\\
J_4 &= 2\int_0^t \bar{\mu}_q(s)\left\langle \Delta u(s), v_q'(s)-\Delta v_q'(s)\right\rangle ds\\
&\leq 2\int_0^t |\bar{\mu}_q(s)|\left\|\Delta u(s)\right\|\left\|v_q'(s)-\Delta v_q'(s)\right\| ds\\
&\leq 2M\left(\int_0^t |\bar{\mu}_q(s)|^2 ds\right)^{1/2}\left(\int_0^t \left\|v_q'(s)-\Delta v_q'(s)\right\|^2 ds\right)^{1/2}\\
&\leq 2\sqrt{2}M\left(\int_0^t |\bar{\mu}_q(s)|^2 ds\right)^{1/2}\sqrt{\bar{Z}_q(t)}\\
&\leq \frac{\bar{\mu}_*}{6}\bar{Z}_q(t)+\frac{12}{\bar{\mu}_*}M^2\tilde{K}_M^2(\mu)\left\|S_q[u_q]-\|u_x(\cdot)\|^2\right\|^2_{L^2(0,T)};\\
J_5 &= 2\int_0^t \left\langle \bar{f}_q(s), v_q'(s)-\Delta v_q'(s)\right\rangle ds\\
&\leq 2\int_0^t \left\|\bar{f}_q(s)\right\|\left\|v_q'(s)-\Delta v_q'(s)\right\| ds\\
&\leq \frac{\bar{\mu}_*}{6}\bar{Z}_q(t)+\frac{6}{\bar{\mu}_*}\bar{K}_M^2(f)\int_0^t\left[3\sqrt{\bar{Z}_q(s)}+\left|S_q[u_q](s)-\|u_x(s)\|^2\right|\right]^2 ds\\
&\leq \frac{\bar{\mu}_*}{6}\bar{Z}_q(t)+\frac{108}{\bar{\mu}_*}\bar{K}_M^2(f)\int_0^t \bar{Z}_q(s)ds+\frac{12}{\bar{\mu}_*}\bar{K}_M^2(f)\left\|S_q[u_q]-\|u_x(\cdot)\|^2\right\|^2_{L^2(0,T)}.
\end{aligned}
$$

It follows from (4.32) and (4.35) that

$$
\bar{Z}_q(t)\leq \bar{d}_1(M)\left\|S_q[u_q]-\|u_x(\cdot)\|^2\right\|^2_{L^2(0,T)}+\bar{d}_2(M)\int_0^t \bar{Z}_q(s)ds, \tag{4.36}
$$

where

$$
\begin{aligned}
\bar{d}_1(M) &= \frac{24}{\bar{\mu}_*^2}\left(M^2\tilde{K}_M^2(\mu)+\bar{K}_M^2(f)\right), \qquad (4.37)\\
\bar{d}_2(M) &= \frac{2}{\bar{\mu}_*}\left(\eta_M^*+2|g(0)|+2\sqrt{T^*}\left\|g'\right\|_{L^2(0,T^*)}\right)\\
&\quad+\frac{12}{\bar{\mu}_*^2}\left(\|g\|^2_{L^2(0,T^*)}+18\bar{K}_M^2(f)\right).
\end{aligned}
$$

Using Gronwall's Lemma, it follows from (4.36) that

$$
\bar{Z}_q(t)\leq \bar{d}_1(M)\exp(T\bar{d}_2(M))\left\|S_q[u_q]-\|u_x(\cdot)\|^2\right\|^2_{L^2(0,T)}. \tag{4.38}
$$

Combining (3.7), (4.33) and (4.36), we get

$$
\|v_q\|_{H_T}\leq 3\sqrt{\bar{d}_1(M)\exp(T\bar{d}_2(M))}\left\|S_q[u_q]-\|u_x(\cdot)\|^2\right\|_{L^2(0,T)}. \tag{4.39}
$$

Theorem 4.2 is proved. □

Remark 4.1.

(i) The arguments and methods used for proving the unique existence of solutions of (P_q) can be applied to the problem $(\bar{P}_q)$ in which $S_q[u](t)$ is replaced by the following arithmetic-mean terms

$$\bar{S}_q[u_x](t) = \frac{1}{q}\sum_{i=0}^{q-1} u_x^2\left(\frac{i+\theta_i}{q}, t\right),$$

respectively, where $\theta_i \in [0,1)$, $i = \overline{0, q-1}$, are given constants.

(ii) The methods used in the above sections can be applied again to obtain the same results for the following problem

$$(\bar{P}_q)\begin{cases} u_{tt} - \lambda u_{txx} - \mu\left(t, \bar{S}_q[u](t), \bar{S}_q[u_x](t)\right) u_{xx} \\ \quad = f\left(x, t, u(x,t), u'(x,t), u_x(x,t), \bar{S}_q[u](t), \bar{S}_q[u_x](t)\right),\ 0 < x < 1,\ 0 < t < T, \\ u_x(0,t) - \zeta u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \end{cases}$$

where $\lambda > 0$, $\zeta \geq 0$ are given constants, μ, f, $\tilde{u}_0$, $\tilde{u}_1$ are given functions and

$$\bar{S}_q[u](t) = \frac{1}{q}\sum_{i=0}^{q-1} u^2\left(\frac{i+\theta_i}{q}, t\right), \bar{S}_q[u_x](t) = \frac{1}{q}\sum_{i=0}^{q-1} u_x^2\left(\frac{i+\theta_i}{q}, t\right),$$

with $\theta_i \in [0,1)$, $i = 0, \cdots, q-1$ are given constants.

Moreover, the weak solution of this problem converges strongly in appropriate spaces to the weak solution of the following problem

$$(\bar{P}_\infty)\begin{cases} u_{tt} - \lambda u_{txx} - \mu\left(t, \|u(t)\|^2, \|u_x(t)\|^2\right) u_{xx} \\ \quad = f\left(x, t, u(x,t), u'(x,t), u_x(x,t), \|u(t)\|^2, \|u_x(t)\|^2\right),\ 0 < x < 1,\ 0 < t < T, \\ u_x(0,t) - \zeta u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \end{cases}$$

where $\|u(t)\|^2 = \int_0^1 u^2(y,t)dy$, $\|u_x(t)\|^2 = \int_0^1 u_x^2(y,t)dy$.

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