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New methodology for a system of nonlinear time-fractional partial differential equations based on the Khalouta-Caputo-Katugampola fractional derivative

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Abstract. *This paper focuses on a new methodology of the ρ-Khalouta decomposition method (ρ-KHDM) to investigate the approximate analytical solutions for a system of time-fractional nonlinear partial differential equations based on the Caputo-Katugampola fractional derivative. The proposed method is a mixture of the ρ-Khalouta transform method and the new decomposition method. The uniqueness and convergence of the solution for the proposed system are proven. The effectiveness of the method is demonstrated through three numerical applications. The proposed method is efficient and reliable compared to other methods, and it produces accurate results based on the obtained results.*

Keywords. System of nonlinear time-fractional partial differential equations, Caputo-Katugampola fractional derivative, Khalouta transform method, new decomposition method.

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1 Introduction

Fractional partial differential equations (FPDEs) have recently proven to be valuable tools for modeling many real-world problems in different domains [7,9,10,24]. This is because realistic modeling of a physical phenomenon depends not only on instantaneous time, but also on the history of past time, which can also be successfully achieved using fractional calculus. For example, half-order derivatives and integrals have proven to be more useful for formulating some electrochemical problems than classical models [19,21,25, 26]. Recently, a large amount of studies have been developed regarding the application of FPDEs in various applications in fluid mechanics, viscoelasticity, biology, physics, and engineering. An excellent account of the study of FPDEs can be found in,[2,5,17,22]. Now, in this paper, we will be interested in studying the system of m-nonlinear time-fractional partial differential equations which is as follows:

$$\mathbb{D}_{i\varphi}^{\alpha,\rho}\mathcal{X}_i(\varpi,\varphi)+\mathfrak{L}_i\left(\mathcal{X}_i(\varpi,\varphi)\right)+\mathfrak{N}_i\left(\mathcal{X}_i(\varpi,\varphi)\right)=f_i(\varpi,\varphi), i=1,2,...,m$$

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with the initial conditions

$$\mathcal{X}_i(\varpi,0)=\mathcal{X}_{i_0}(\varpi),$$

where $\mathbb{D}_{i\varphi}^{\alpha,\rho}$ are the Caputo-Katugampola fractional derivative operators of order α,ρ with $0<\alpha\leq 1$ and $\rho>0$, $\mathfrak{L}_i$ and $\mathfrak{N}_i$ represents linear and nonlinear operators, respectively, and f_i is the nonhomogeneous term.

Systems of nonlinear time-fractional partial differential equations are of valuable interest in many areas of applied sciences, nonlinear hydrodynamics, mathematical physics, mathematical biology, chemistry, engineering, and finance. In [18], the author efficiently employed the homotopy perturbation trasform method (HPTM) and the variational iteration transform method (VITM) for solving a system of time-fractional nonlinear equations describing the unsteady flow of a polytropic gas under Caputo's fractional derivative. In [23], the approximate solutions of multi-dimensional time-fractional Navier-Stokes system have been provided based on the Laplace residual power series method (LRPSM). In [4], by adopting the natural decomposition method (NDM), the approximate analytical solutions of the Kersten-Krasil'shchik coupled KdV-mKdV systemswere considered in the sense of Atangana-Baleanu derivative and Caputo-Fabrizio derivative. In [1], the numerical solutions of the one and two dimensional fractional coupled Burger's equations was investigated using the iterative Elzaki transform method (IETM).

In general, semi-approximate analysis techniques or traditional analysis techniques cannot provide exact closed-form or approximate solutions for systems of nonlinear time-fractional partial differential equations. Therefore, there is an urgent need for efficient numerical techniques capable of finding exact or accurate approximate solutions for these these systems.

The main motivation of this paper is to propose a new methodology of ρ-KHDM to tackle solutions of a system of nonlinear time-fractional partial differential equations based on the Khalouta-Caputo-Katugampola fractional derivative. The ρ-KHDM approach combines the ρ-Khalouta transform method [15] and new decomposition method [16]. The suggested approach provides a closed-type solution in terms of infinite series and the resultant series converges rapidly to the exact solution. Also, it has been observed that the results obtained give better results than the methods in the literature.

The structure of this paper is as follows. In Section 2, we provide some basic definitions, theorems, and formulas related to the paper. In Section 3, we use the proposed method for a system of nonlinear nonhomogeneous time-fractional partial differential equations and obtain an approximate solution to a general problem. The uniqueness and convergence are demonstrated in Section 4. The approximate solutions are found using ρ-KHDM on several applications presented in Section 5. Finally, the conclusion is given in Section 6.

2 Definitions and mathematical formulas

This section explains some basic formulas, concepts, results for the ρ-Khalouta transform of the Caputo-Katugampola time-fractional derivative and related formulas that will be relevant throughout the work.

Definition 2.1 *[13] Let the function $\mathcal{X}:\mathbb{R}\times\mathbb{R}^{+}\rightarrow\mathbb{R}$, then the Katugampola fractional integral of order α,ρ is defined as*

$$\mathbb{I}_{\varphi}^{\alpha,\rho}\mathcal{X}(\varpi,\varphi)=\frac{1}{\Gamma(\alpha)}\int_0^{\varphi}\left(\frac{\varphi^{\rho}-\tau^{\rho}}{\rho}\right)^{\alpha-1}\frac{\mathcal{X}(\varpi,\tau)}{\tau^{1-\rho}}d\tau,\rho>0. \tag{2.1}$$

where $\Gamma(.)$ is the gamma function and $0<\alpha\leq 1,\rho>0$.

Definition 2.2 *[11] Let the function $\mathcal{X} : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$, then the Caputo-Katugampola time-fractional derivative of order α, ρ is defined as*

$$\mathbb{D}_{\varphi}^{\alpha,\rho}\mathcal{X}(\varpi,\varphi) = \mathbb{I}_{\vartheta}^{n-\alpha,\rho}\xi^n\mathcal{X}(\varpi,\varphi) = \frac{1}{\Gamma(n-\alpha)}\int_0^{\varphi}\left(\frac{\varphi^\rho-\tau^\rho}{\rho}\right)^{n-\alpha-1}\frac{\xi^n\mathcal{X}(\varpi,\tau)}{\tau^{1-\rho}}d\tau, \tag{2.2}$$

where the differential operator ξ is given by $\xi = \varphi^{1-\rho}\frac{d}{d\varphi}$ and $n-1 < \alpha \leq n, \rho > 0$.

For equations (2.1) and (2.2), we have the following relations

$$\mathbb{D}_{\varphi}^{\alpha,\rho}\mathbb{I}_{\varphi}^{\alpha,\rho}\mathcal{X}(\varpi,\varphi) = \mathcal{X}(\varpi,\varphi), \tag{2.3}$$

and

$$\mathbb{I}_{\varphi}^{\alpha,\rho}\mathbb{D}_{\varphi}^{\alpha,\rho}\mathcal{X}(\varpi,\varphi) = \mathcal{X}(\varpi,\varphi) - \sum_{k=0}^{n}\frac{D^{\alpha-k,\rho}\mathcal{X}(\varpi,0)}{\Gamma(\alpha-k+1)}\left(\frac{\varphi^\rho}{\rho}\right)^{\alpha-k}. \tag{2.4}$$

Now, we present our main results related to the ρ-Khalouta transform of the Caputo-Katugampola time-fractional derivative.

Definition 2.3 *[15] The ρ-Khalouta transform of the function $\mathcal{X}(\varpi,\varphi)$ with respect to the variable "φ" is defined as*

$$\mathbb{KH}_\rho\left[\mathcal{X}(\varpi,\varphi)\right] = \mathcal{K}_\rho(\varpi,s,\gamma,\eta) = \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{s}{\gamma\eta}\frac{\varphi^\rho}{\rho}\right)\frac{\mathcal{X}(\varpi,\varphi)}{\varphi^{1-\rho}}d\varphi, \rho > 0, \tag{2.5}$$

where $s > 0, \gamma > 0$ and $\eta > 0$ are the Khalouta transform variables.

Theorem 2.1 *[15]*
1) For all real constants λ and μ , we have

$$\mathbb{KH}_\rho\left[\lambda\mathcal{X}(\varpi,\varphi) \pm \mu\mathcal{Y}(\varpi,\varphi)\right] = \lambda\mathbb{KH}_\rho\left[\mathcal{X}(\varpi,\varphi)\right] \pm \mu\mathbb{KH}_\rho\left[\mathcal{Y}(\varpi,\varphi)\right], \tag{2.6}$$

2) Let $\mathcal{K}_\rho(\varpi,s,\gamma,\eta)$ and $\mathcal{H}_\rho(\varpi,s,\gamma,\eta)$ are the ρ-Khalouta transform of $\mathcal{X}$ and $\mathcal{Y}$ respectively. Then

$$\mathbb{KH}_\rho\left[\left(\mathcal{X} *_\rho \mathcal{Y}\right)(\varpi,\varphi)\right] = \frac{\gamma\eta}{s}\mathcal{K}_\rho(\varpi,s,\gamma,\eta)\mathcal{H}_\rho(\varpi,s,\gamma,\eta), \tag{2.7}$$

where $\mathcal{X} *_\rho \mathcal{Y}$ is the ρ-convolution integral defined by

$$\begin{aligned}\left(\mathcal{X} *_\rho \mathcal{Y}\right)(\varpi,\varphi) &= \textstyle\int_0^\varphi \mathcal{X}(\varkappa,(\varphi^\rho-\tau^\rho)^{\frac{1}{\rho}})\frac{\mathcal{Y}(\varkappa,\tau)}{\tau^{1-\rho}}d\tau \\ &= \textstyle\int_0^t \mathcal{Y}(\varkappa,(\varphi^\rho-\mu^\rho)^{\frac{1}{\rho}})\frac{\mathcal{X}(\varkappa,\mu)}{\mu^{1-\rho}}d\mu \\ &= \left(\mathcal{Y} *_\rho \mathcal{X}\right)(\varpi,\varphi).\end{aligned} \tag{2.8}$$

3) Let a, b and $c \in \mathbb{R}$ and $\rho > 0$, then

$$\begin{aligned}\mathbb{KH}_\rho\left[a\right] &= a, \\ \mathbb{KH}_\rho\left[\varphi^b\right] &= \left(\frac{\rho\gamma\eta}{s}\right)^{\frac{b}{\rho}}\Gamma\left(\frac{b}{\rho}+1\right), \\ \mathbb{KH}_\rho\left[\frac{\varphi^{n\rho}}{\rho^n}\right] &= \left(\frac{\gamma\eta}{s}\right)^n\Gamma\left(n+1\right), \\ \mathbb{KH}_\rho\left[\exp\left(c\frac{\varphi^\alpha}{\alpha}\right)\right] &= \frac{s}{s-c\gamma\eta}.\end{aligned} \tag{2.9}$$

4) Let $\mathcal{X} \in C_{\xi}^{n-1}(\mathbb{R} \times \mathbb{R}^{+})$, then the ρ-Khalouta transform of $\xi^{n}\mathcal{X}(\varpi,\varphi)$ with respect to the variable "φ" is defined by

$$\mathbb{KH}_{\rho}\left[\xi^{n}\mathcal{X}(\varpi,\varphi)\right] = \left(\frac{s}{\gamma\eta}\right)^{n}\mathbb{KH}_{\rho}\left[\mathcal{X}(\varpi,\varphi)\right] - \sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{n-k}\xi^{k}\mathcal{X}(\varpi,0) \tag{2.10}$$

Proof. To prove the Theorem, see. [15].

Theorem 2.2 *The ρ-Khalouta transform of the Katugampola fractional integral of order $\alpha, \rho > 0$ of the function $\mathcal{X}(\varpi,\varphi)$ is expressed as*

$$\mathbb{KH}_{\rho}\left[\mathbb{I}_{\varphi}^{\alpha,\rho}\mathcal{X}(\varpi,\varphi)\right] = \left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}_{\rho}\left[\mathcal{X}(\varpi,\varphi)\right]. \tag{2.11}$$

Proof. By applying the ρ-Khalouta transform to equation (2.1) and using Theorem 2.1, we get

$$\begin{aligned}
\mathbb{KH}_{\rho}\left[\mathbb{I}_{\varphi}^{\alpha,\rho}\mathcal{X}(\varpi,\varphi)\right] &= \mathbb{KH}_{\rho}\left[\frac{1}{\Gamma(\alpha)}\int_{0}^{\varphi}\left(\frac{\varphi^{\rho}-\tau^{\rho}}{\rho}\right)^{\alpha-1}\frac{\mathcal{X}(\varpi,\tau)}{\tau^{1-\rho}}d\tau\right] \\
&= \frac{s}{\gamma\eta}\int_{0}^{\infty}\exp\left(-\frac{s}{\gamma\eta}\frac{\varphi^{\rho}}{\rho}\right)\left[\frac{1}{\Gamma(\alpha)}\int_{0}^{\varphi}\left(\frac{\varphi^{\rho}-\tau^{\rho}}{\rho}\right)^{\alpha-1}\frac{\mathcal{X}(\varpi,\tau)}{\tau^{1-\rho}}d\tau\right]\frac{d\varphi}{\varphi^{1-\rho}} \\
&= \frac{s}{\gamma\eta}\int_{0}^{\infty}\exp\left(-\frac{s}{\gamma\eta}\frac{\varphi^{\rho}}{\rho}\right)\left[\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{0}^{\varphi}\left((\varphi^{\rho}-\tau^{\rho})^{\frac{1}{\rho}}\right)^{\rho(\alpha-1)}\frac{\mathcal{X}(\varpi,\tau)}{\tau^{1-\rho}}d\tau\right]\frac{d\varphi}{\varphi^{1-\rho}} \\
&= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\frac{s}{\gamma\eta}\int_{0}^{\infty}\exp\left(-\frac{s}{\gamma\eta}\frac{\varphi^{\rho}}{\rho}\right)\left(\varphi^{\rho(\alpha-1)} *_{\rho}\mathcal{X}(\varpi,\varphi)\right)\frac{d\varphi}{\varphi^{1-\rho}} \\
&= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\mathbb{KH}_{\rho}\left[\varphi^{\rho(\alpha-1)} *_{\rho}\mathcal{X}(\varpi,\varphi)\right].
\end{aligned} \tag{2.12}$$

By using Theorem 2.2 and properties (2) and (3) of Theorem 2.1, we get

$$\begin{aligned}
\mathbb{KH}_{\rho}\left[\mathbb{I}_{\varphi}^{\alpha,\rho}\mathcal{X}(\varpi,\varphi)\right] &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\frac{\gamma\eta}{s}\mathbb{KH}_{\rho}\left[\varphi^{\rho(\alpha-1)}\right]\mathbb{KH}_{\rho}\left[\mathcal{X}(\varpi,\varphi)\right] \\
&= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\frac{\gamma\eta}{s}\left(\frac{\rho\gamma\eta}{s}\right)^{\frac{\rho(\alpha-1)}{\rho}}\Gamma\left(\frac{\rho(\alpha-1)}{\rho}+1\right)\mathbb{KH}_{\rho}\left[\mathcal{X}(\varpi,\varphi)\right] \\
&= \left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}_{\rho}\left[\mathcal{X}(\varpi,\varphi)\right].
\end{aligned} \tag{2.13}$$

So, the proof is complete.

Theorem 2.3 *The ρ-Khalouta transform of the Caputo-Katugampola time-fractional derivative of order $\alpha, \rho > 0$ of the function $\mathcal{X}(\varpi,\varphi)$ is expressed as*

$$\mathbb{KH}_{\rho}\left[\mathbb{D}_{\varphi}^{\alpha,\rho}\mathcal{X}(\varpi,\varphi)\right] = \left(\frac{s}{\gamma\eta}\right)^{\alpha}\mathbb{KH}_{\rho}\left[\mathcal{X}(\varpi,\varphi)\right] - \sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{\alpha-k}\xi^{k}\mathcal{X}(\varpi,0). \tag{2.14}$$

Proof. By applying the ρ-Khalouta transform to equation (2.2) and using Theorem 2.1, we get

$$\mathbb{KH}_{\rho}\left[\mathbb{D}_{\varphi}^{\alpha,\rho}\mathcal{X}(\varpi,\varphi)\right] = \mathbb{KH}_{\rho}\left[\mathbb{I}_{\varphi}^{\alpha-n,\rho}\xi^{n}\mathcal{X}(\varpi,0)\right]. \tag{2.15}$$

By using Theorem 2.2, we get

$$\begin{aligned}\mathbb{KH}_\rho\left[\mathbb{D}_\varphi^{\alpha,\rho}\mathcal{X}(\varpi,\varphi)\right] &= \mathbb{KH}_\rho\left[\mathbb{I}_\varphi^{\alpha-n,\rho}\xi^n\mathcal{X}(\varpi,0)\right]\\ &= \left(\frac{\gamma\eta}{s}\right)^{n-\alpha}\mathbb{KH}_\rho\left[\xi^n\mathcal{X}(\varpi,0)\right].\end{aligned} \tag{2.16}$$

From property (4) of Theorem 2.1, we get

$$\begin{aligned}\mathbb{KH}_\rho\left[\mathbb{D}_\varphi^{\alpha,\rho}\mathcal{X}(\varpi,\varphi)\right] &= \left(\frac{\gamma\eta}{s}\right)^{n-\alpha}\mathbb{KH}_\rho\left[\xi^n\mathcal{X}(\varpi,0)\right]\\ &= \left(\frac{\gamma\eta}{s}\right)^{n-\alpha}\left(\left(\frac{s}{\gamma\eta}\right)^{n}\mathbb{KH}_\rho\left[\mathcal{X}(\varpi,\varphi)\right]-\sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{n-k}\xi^k\mathcal{X}(\varpi,0)\right)\\ &= \left(\frac{s}{\gamma\eta}\right)^{\alpha}\mathbb{KH}_\rho\left[\mathcal{X}(\varpi,\varphi)\right]-\sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{\alpha-k}\xi^k\mathcal{X}(\varpi,0).\end{aligned} \tag{2.17}$$

So, the proof is complete.

3 Description of ρ-Khalouta decomposition method

This section describes the new methodology of ρ-Khalouta decomposition method to solve a system of nonlinear nonhomogeneous time-fractional partial differential equations.

Theorem 3.1 *Consider the system of m-nonlinear nonhomogeneous time-fractional partial differential equations*

$$\mathbb{D}_{i\varphi}^{\alpha,\rho}\mathcal{X}_i(\varpi,\varphi)+\mathfrak{L}_i\left(\mathcal{X}_i(\varpi,\varphi)\right)+\mathfrak{N}_i\left(\mathcal{X}_i(\varpi,\varphi)\right)=f_i(\varpi,\varphi), i=1,2,...,m, \tag{3.1}$$

with the initial conditions

$$\mathcal{X}_i(\varpi,0)=\mathcal{X}_{i_0}(\varpi), \tag{3.2}$$

where $\mathbb{D}_{i\varphi}^{\alpha,\rho}$ are the Caputo-Katugampola time-fractional derivative operators of order α,ρ with $0<\alpha\leq 1$ and $\rho>0$, $\mathfrak{L}_i$ and $\mathfrak{N}_i$ represents linear and nonlinear operators, respectively, and f_i are the nonhomogeneous terms.

The solution of the system of m-nonlinear nonhomogeneous time- fractional partial differential equations (3.1) with the initial conditions (3.2) can be formulated as

$$\begin{aligned}\mathcal{X}_i(\varpi,\varphi) &= \sum_{n=0}^{\infty}\mathcal{X}_{in}(\varpi,\varphi)\\ &= \mathcal{X}_{i0}(\varpi,\varphi)+\sum_{n=1}^{\infty}\mathbb{KH}_\rho^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}_\rho\left[\begin{array}{c}f_i(\varpi,\varphi)-\mathfrak{L}_i\left(\mathcal{X}_{i(n-1)}(\varpi,\varphi)\right)\\ -K_{i(n-1)}\end{array}\right]\right],\end{aligned} \tag{3.3}$$

where K_{in} are polynomials of $\mathcal{X}_{i0},\mathcal{X}_{i1},...,\mathcal{X}_{in}$ defined in [16].

Proof. The solution $\mathcal{X}_i(\varpi,\varphi)$ of the system (3.1) be assumed, as

$$\mathcal{X}_i(\varpi,\varphi)=\sum_{n=0}^{\infty}\mathcal{X}_{in}(\varpi,\varphi). \tag{3.4}$$

To resolve the system (3.1) with the initial conditions (3.2), we consider for $i = 1, 2, ..., m$ the following system

$$\mathbb{D}_{i\varphi}^{\alpha,\rho}\mathcal{X}_{i\lambda_i}(\varpi,\varphi) = \lambda_i\left[f_{i\lambda_i}(\varpi,\varphi) - \mathfrak{L}_i\left(\mathcal{X}_{i\lambda_i}(\varpi,\varphi)\right) - \mathfrak{N}_i\left(\mathcal{X}_{i\lambda_i}(\varpi,\varphi)\right)\right], \lambda_i \in [0,1], \tag{3.5}$$

with the initial conditions

$$\mathcal{X}_{i\lambda_i}(\varpi,0) = \mathcal{X}_{i_0\lambda_i}(\varpi). \tag{3.6}$$

Now, we assume that the solution of (3.5)-(3.6) can be expressed as follows

$$\mathcal{X}_{i\lambda_i}(\varpi,\varphi) = \sum_{n=0}^{\infty}\lambda_i^n\mathcal{X}_{in}(\varpi,\varphi). \tag{3.7}$$

Applying the ρ-Khalouta transform on equation (3.5) with respect to the variable "φ", we get

$$\mathbb{KH}_\rho\left[\mathbb{D}_{i\varphi}^{\alpha,\rho}\mathcal{X}_{i\lambda_i}(\varpi,\varphi)\right] = \lambda_i\mathbb{KH}_\rho\left[f_{i\lambda_i}(\varpi,\varphi) - \mathfrak{L}_i\left(\mathcal{X}_{i\lambda_i}(\varpi,\varphi)\right) - \mathfrak{N}_i\left(\mathcal{X}_{i\lambda_i}(\varpi,\varphi)\right)\right]. \tag{3.8}$$

Using Theorem 2.3 and the initial conditions (3.2), we have

$$\mathbb{KH}_\rho\left[\mathcal{X}_{i\lambda_i}(\varpi,\varphi)\right] = \mathcal{X}_{i\lambda_i}(\varpi,0) + \left(\frac{\gamma\eta}{s}\right)^\alpha\lambda_i\mathbb{KH}_\rho\left[\begin{array}{c} f_{i\lambda_i}(\varpi,\varphi) - \mathfrak{L}_i\left(\mathcal{X}_{i\lambda_i}(\varpi,\varphi)\right) \\ -\mathfrak{N}_i\left(\mathcal{X}_{i\lambda_i}(\varpi,\varphi)\right)\end{array}\right]. \tag{3.9}$$

Taking the inverse ρ-Khalouta transform of equation (3.9) to get

$$\mathcal{X}_{i\lambda_i}(\varpi,\varphi) = \mathcal{X}_{i_0\lambda_i}(\varpi) + \lambda_i\mathbb{KH}_\rho^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^\alpha\lambda_i\mathbb{KH}_\rho\left[\begin{array}{c} f_{i\lambda_i}(\varpi,\varphi) - \mathfrak{L}_i\left(\mathcal{X}_{i\lambda_i}(\varpi,\varphi)\right) \\ -\mathfrak{N}_i\left(\mathcal{X}_{i\lambda_i}(\varpi,\varphi)\right)\end{array}\right]\right]. \tag{3.10}$$

Replacing (3.7) into (3.10), the following equation is obtained.

$$\sum_{n=0}^{\infty}\lambda_i^n\mathcal{X}_{in}(\varpi,\varphi) = \mathcal{X}_{i_0\lambda_i}(\varpi) + \lambda_i\mathbb{KH}_\rho^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^\alpha\mathbb{KH}_\rho\left[\begin{array}{c} f_{i\lambda_i}(\varpi,\varphi) - \mathfrak{L}_i\left(\sum\limits_{n=0}^{\infty}\lambda_i^n\mathcal{X}_{in}(\varpi,\varphi)\right) \\ -\mathfrak{N}_i\left(\sum\limits_{n=0}^{\infty}\lambda_i^n\mathcal{X}_{in}(\varpi,\varphi)\right)\end{array}\right]\right]. \tag{3.11}$$

Application of the new decomposition method [16] to equation (3.11) implies

$$\sum_{n=0}^{\infty}\lambda_i^n\mathcal{X}_{in}(\varpi,\varphi) = \mathcal{X}_{i_0\lambda_i}(\varpi) + \lambda_i\mathbb{KH}_\rho^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^\alpha\mathbb{KH}_\rho\left[\begin{array}{c} f_{i\lambda_i}(\varpi,\varphi) - \sum\limits_{n=0}^{\infty}\mathfrak{L}_i\left(\lambda_i^n\mathcal{X}_{in}(\varpi,\varphi)\right) \\ -\sum\limits_{n=0}^{\infty}\lambda_i^n K_{in}\end{array}\right]\right], \tag{3.12}$$

where K_{in} are polynomials of $\mathcal{X}_{i0}, \mathcal{X}_{i1}, ..., \mathcal{X}_{in}$ defined by

$$K_{in} = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\mathfrak{N}\left(\sum_{j=0}^{n}\lambda_i^j\mathcal{X}_{ij}\right)\right]_{\lambda=0}, n = 0, 1, 2, ... \tag{3.13}$$

By equating the terms in (3.12) with identical powers of λ_i, the following relation is obtained.

$$\begin{aligned}
&\mathcal{X}_{i0}(\varpi,\varphi) = \mathcal{X}_{i_0}(\varpi), \\
&\mathcal{X}_{i1}(\varpi,\varphi) = \mathbb{KH}_\rho^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^\alpha\mathbb{KH}_\rho\left[f_i(\varpi,\varphi) - \mathfrak{L}_i\left(\mathcal{X}_{i0}(\varpi,\varphi)\right) - K_{i0}\right]\right], \qquad (3.14) \\
&\mathcal{X}_{in}(\varpi,\varphi) = \mathbb{KH}_\rho^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^\alpha\mathbb{KH}_\rho\left[-\mathfrak{L}_i\left(\mathcal{X}_{i(n-1)}(\varpi,\varphi)\right) - K_{i(n-1)}\right]\right], n = 2, 3, 4, ...
\end{aligned}$$

Substituting the components of (3.14) into (3.7) gives the solution of the system (3.5). Now, according to (3.4) and (3.7), we have

$$\mathcal{X}_i(\varpi,\varphi)=\lim_{\lambda_i\to 1}\mathcal{X}_{i\lambda_i}(\varpi,\varphi)=\mathcal{X}_{i0}(\varpi,\varphi)+\sum_{n=1}^{\infty}\mathcal{X}_{in}(\varpi,\varphi). \tag{3.15}$$

Using the first equation of (3.14), we see that

$$\mathcal{X}_{i0}(\varpi,0)=\lim_{\lambda_i\to 1}\mathcal{X}_{i_0\lambda_i}(\varpi). \tag{3.16}$$

Substituting (3.14) into (3.15), we get

$$\begin{gathered}\mathcal{X}_i(\varpi,\varphi)=\textstyle\sum_{n=0}^{\infty}\mathcal{X}_{in}(\varpi,\varphi)\\ =\mathcal{X}_{i0}(\varpi,\varphi)+\sum_{n=1}^{\infty}\mathbb{KH}_\rho^{-1}\left[\left(\tfrac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}_\rho\left[\begin{array}{r}f_i(\varpi,\varphi)-\mathfrak{L}_i\left(\mathcal{X}_{i(n-1)}(\varpi,\varphi)\right)\\ -K_{i(n-1)}\end{array}\right]\right].\end{gathered} \tag{3.17}$$

So, the proof is complete.

4 Convergence analysis

This section states the convergence and uniqueness statements of the ρ-KHDM solutions.

Theorem 4.1 ***(Uniqueness theorem)*** *The solution for the system of m-nonlinear nonhomogeneous time-fractional partial differential equations (3.1) obtained by ρ-KHDM is unique for $0<\epsilon<1$, where $\epsilon=(\psi+\phi)\,MT$.*

Proof. The solution of system of m-nonlinear nonhomogeneous time-fractional partial differential equations is given by

$$\mathcal{X}_i(\varpi,\varphi)=\sum_{n=0}^{\infty}\mathcal{X}_{in}(\varpi,\varphi), \tag{4.1}$$

where

$$\begin{gathered}\mathcal{X}_{in}(\varpi,\varphi)=\mathcal{X}_{i_0}(\varpi)+\mathbb{KH}_\rho^{-1}\times\\ \times\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}_\rho\left[f_i(\varpi,\varphi)-\mathfrak{L}_i\left(\mathcal{X}_{i(n-1)}(\varpi,\varphi)\right)-\mathfrak{N}_i\left(\mathcal{X}_{i(n-1)}(\varpi,\varphi)\right)\right]\right].\end{gathered} \tag{4.2}$$

Suppose $\mathcal{X}_i$ and $\mathcal{Y}_i$ are two different solutions to the system (3.1), $\mathfrak{L}_i$ and $\mathfrak{N}_i$ satisfies the conditions $\left|\mathfrak{L}_i\left(\mathcal{X}_i-\mathcal{Y}_i\right)\right|\leq\psi\left|\mathcal{X}_i-\mathcal{Y}_i\right|$ and $\left|\mathfrak{N}_i\left(\mathcal{X}_i-\mathcal{Y}_i\right)\right|\leq\phi\left|\mathcal{X}_i-\mathcal{Y}_i\right|$, where ψ and ϕ are constants, respectively. Then using the aforementioned system, we get

$$\begin{aligned}\left|\mathcal{X}_i-\mathcal{Y}_i\right|&=\left|\mathbb{KH}_\rho^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}_\rho\left[\mathfrak{L}_i\left(\mathcal{X}_i-\mathcal{Y}_i\right)+\mathfrak{N}_i\left(\mathcal{X}_i-\mathcal{Y}_i\right)\right]\right]\right|\\ &\leq\mathbb{KH}_\rho^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}_\rho\left|\mathfrak{L}_i\left(\mathcal{X}_i-\mathcal{Y}_i\right)\right|+\left|\mathfrak{N}_i\left(\mathcal{X}_i-\mathcal{Y}_i\right)\right|\right]\\ &\leq\left(\psi\left|\mathcal{X}_i-\mathcal{Y}_i\right|+\phi\left|\mathcal{X}_i-\mathcal{Y}_i\right|\right)\mathbb{KH}_\rho^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha}\right]\\ &\leq\left(\psi\left|\mathcal{X}_i-\mathcal{Y}_i\right|+\phi\left|\mathcal{X}_i-\mathcal{Y}_i\right|\right)\frac{\varphi^{\alpha\rho}}{\Gamma\left(\alpha+1\right)\rho^{\alpha}}.\end{aligned} \tag{4.3}$$

Now, using the convolution theorem, we obtain the following formula

$$\left|\mathcal{X}_i-\mathcal{Y}_i\right|\leq\int_0^{\varphi}\left(\psi\left|\mathcal{X}_i-\mathcal{Y}_i\right|+\phi\left|\mathcal{X}_i-\mathcal{Y}_i\right|\right)\frac{(\varphi-\tau)^{\alpha\rho}}{\Gamma\left(\alpha+1\right)\rho^{\alpha}}d\tau. \tag{4.4}$$

Using mean value theorem of integral calculus [6], we obtain

$$\begin{aligned} |\mathcal{X}_i - \mathcal{Y}_i| &\le ((\psi + \phi)\, MT)\, |\mathcal{X}_i - \mathcal{Y}_i| \\ &\le \epsilon\, |\mathcal{X}_i - \mathcal{Y}_i|\,, \end{aligned} \tag{4.5}$$

where $M = \max\limits_{\varphi\in[0,T]} \frac{(\varphi-\tau)^{\alpha\rho}}{\Gamma(\alpha+1)\rho^{\alpha}}$.

Consequently, we have $(1-\epsilon)\,|\mathcal{X}_i - \mathcal{Y}_i| \le 0$. As $0 < \epsilon < 1$, then $|\mathcal{X}_i - \mathcal{Y}_i| = 0$, which implies that $\mathcal{X}_i = \mathcal{Y}_i$. Therefore, the solution is unique.

Theorem 4.2 ***(Convergence theorem)*** *Suppose that $\mathcal{B}$ is a Banach space and that $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ is a nonlinear mapping. If the inequality*

$$\|\mathcal{F}(\mathcal{X}_i) - \mathcal{F}(\mathcal{Y}_i)\|_{\mathcal{B}} \le \theta\, \|\mathcal{X}_i - \mathcal{Y}_i\|_{\mathcal{B}}\,, \forall \mathcal{X}_i, \mathcal{Y}_i \in \mathcal{B}, \tag{4.6}$$

exists, then $\mathcal{F}$ has a fixed point according to Banach's fixed point theorem [12]. Moreover, the sequence generated by ρ-KHDM converges to a fixed point of $\mathcal{F}$ and

$$\|S_{in} - S_{iq}\|_{\mathcal{B}} \le \frac{\theta^q}{1-\theta}\, \|S_{i1} - S_{i0}\|_{\mathcal{B}}\,, \tag{4.7}$$

where $\{S_{in}\}_{n\ge 0}$ is the sequence of partial sums of the series defined as $S_{in} = \sum_{j=0}^{n} \mathcal{X}_{ij}(\varpi,\varphi)$.

Proof. Let us take a Banach space $\mathcal{B} = (C[\Omega], \|.\|_{\mathcal{B}})$ of all continuous functions on $\Omega \subset \mathbb{R}\times[0,T]$ with the norm expressed as $\|\mathcal{X}_i(\varpi,\varphi)\|_{\mathcal{B}} = \max\limits_{(\varpi,\varphi)\in\Omega} |\mathcal{X}_i(\varpi,\varphi)|$.

Now, we demonstrate that the sequence $\{S_{in}\}_{n\ge 0}$ is a Cauchy sequence in the Banach space

$$\begin{aligned} \|S_{in} - S_{iq}\|_{\mathcal{B}} &= \max_{(\varpi,\varphi)\in\Omega} |S_{in} - S_{iq}| \\ &= \max_{(\varpi,\varphi)\in\Omega} \left| \mathbb{KH}_\rho^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha} \mathbb{KH}_\rho \left[\mathfrak{L}_i \left(S_{i(n-1)} - S_{i(q-1)}\right) + \mathfrak{N}_i \left(S_{i(n-1)} - S_{i(q-1)}\right)\right]\right]\right| \\ &\le \max_{(\varpi,\varphi)\in\Omega} \left| \mathbb{KH}_\rho^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha} \mathbb{KH}_\rho \left[\mathfrak{L}_i \left|S_{i(n-1)} - S_{i(q-1)}\right| + \mathfrak{N}_i \left|S_{i(n-1)} - S_{i(q-1)}\right|\right]\right]\right| \\ &\le \max_{(\varpi,\varphi)\in\Omega} \left(\psi \left|S_{i(n-1)} S_{i(q-1)}\right| + \phi \left|S_{i(n-1)} - S_{i(q-1)}\right|\right) \mathbb{KH}_\rho^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha}\right] \\ &\le \max_{(\varpi,\varphi)\in\Omega} \left(\psi \left|S_{i(n-1)} - S_{i(q-1)}\right| + \phi \left|S_{i(n-1)} - S_{i(q-1)}\right|\right) \frac{\varphi^{\alpha\rho}}{\Gamma(\alpha+1)\,\rho^{\alpha}} \\ &\le (\psi+\phi) \left\|S_{i(n-1)} - S_{i(q-1)}\right\|_{\mathcal{B}} \frac{\varphi^{\alpha\rho}}{\Gamma(\alpha+1)\,\rho^{\alpha}}. \end{aligned} \tag{4.8}$$

Now, using the convolution theorem, we obtain the following formula

$$\|S_{in} - S_{iq}\|_{\mathcal{B}} \le \int_0^{\varphi} (\psi+\phi) \left\|S_{i(n-1)} - S_{i(q-1)}\right\|_{\mathcal{B}} \frac{(\varphi-\tau)^{\alpha\rho}}{\Gamma(\alpha+1)\,\rho^{\alpha}} d\tau. \tag{4.9}$$

Using mean value theorem of integral calculus [6], we obtain

$$\begin{aligned} \|S_{in} - S_{iq}\|_{\mathcal{B}} &\le ((\psi+\phi)\, MT) \left\|S_{i(n-1)} - S_{i(q-1)}\right\|_{\mathcal{B}} \\ &\le \epsilon \left\|S_{i(n-1)} - S_{i(q-1)}\right\|_{\mathcal{B}}, \end{aligned} \tag{4.10}$$

where $M = \max\limits_{\varphi\in[0,T]} \frac{(\varphi-\tau)^{\alpha\rho}}{\Gamma(\alpha+1)\rho^{\alpha}}$.

Choosing $n = q + 1$, then we obtain

$$\begin{aligned}\left\|S_{i(q+1)} - S_{iq}\right\|_{\mathcal{B}} &\leq \epsilon \left\|S_{iq} - S_{i(q-1)}\right\|_{\mathcal{B}} \leq \epsilon^2 \left\|S_{i(q-1)} - S_{i(q-2)}\right\|_{\mathcal{B}} \\ &\leq ... \leq \epsilon^q \left\|S_{i1} - S_{i0}\right\|_{\mathcal{B}},\end{aligned} \tag{4.11}$$

Using the triangular inequality, we get

$$\begin{aligned}\left\|S_{in} - S_{iq}\right\|_{\mathcal{B}} &= \left\|S_{i(q+1)} - S_{iq} + S_{i(q+2)} - S_{i(q+1)} + ... + S_{in} - S_{i(n-1)}\right\|_{\mathcal{B}} \\ &\leq \left\|S_{i(q+1)} - S_{iq}\right\|_{B} + \left\|S_{i(q+2)} - S_{i(q+1)}\right\|_{B} + ... + \left\|S_{in} - S_{i(n-1)}\right\|_{B} \\ &\leq \epsilon^q \left\|S_{i1} - S_{i0}\right\|_{\mathcal{B}} + \epsilon^{q+1} \left\|S_{i1} - S_{i0}\right\|_{\mathcal{B}} + ... + \epsilon^{n-1} \left\|S_{i1} - S_{i0}\right\|_{\mathcal{B}} \\ &= \epsilon^q \left(1 + \epsilon + ... + \epsilon^{n-q-1}\right) \left\|S_{i1} - S_{i0}\right\|_{\mathcal{B}} \\ &\leq \epsilon^q \left(\frac{1 - \epsilon^{n-q}}{1 - \epsilon}\right) \left\|S_{i1} - S_{i0}\right\|_{\mathcal{B}}.\end{aligned} \tag{4.12}$$

Now by definition $0 < \epsilon < 1$, we have $1 - \epsilon^{n-q} < 1$, thus we have

$$\left\|S_{in} - S_{iq}\right\|_{\mathcal{B}} \leq \frac{\epsilon^q}{1 - \epsilon} \left\|S_{i1} - S_{i0}\right\|_{\mathcal{B}}. \tag{4.13}$$

For $\left\|S_{i1} - S_{i0}\right\|_{\mathcal{B}} < +\infty$, so as $q \to \infty$ then $\left\|S_{in} - S_{iq}\right\|_{\mathcal{B}} \to 0$. Thus, the sequence $\{S_{in}\}_{n\geq 0}$ is a Cauchy sequence in $\mathcal{B} = (C[\Omega], \|.\|_{\mathcal{B}})$, and so the sequence is convergent.

5 Applications

This section presents various examples of system of nonlinear time-fractional partial differential equations involving the Caputo-Katugampola fractional derivative to demonstrate the effectiveness of our new methodology.

Example 1 Consider the system of nonlinear nonhomogeneous time-fractional partial differential equations of the form

$$\begin{cases} \mathbb{D}_{\varphi}^{\alpha,\rho}\mathcal{X}_1 + \mathcal{X}_2\mathcal{X}_{1\varpi} + \mathcal{X}_1 = 1 \\ \mathbb{D}_{\varphi}^{\alpha,\rho}\mathcal{X}_2 + \mathcal{X}_1\mathcal{X}_{2\varpi} - \mathcal{X}_2 = -1 \end{cases}, \tag{5.1}$$

with the initial conditions

$$\begin{cases} \mathcal{X}_1(\varpi, 0) = e^{\varpi} \\ \mathcal{X}_2(\varpi, 0) = e^{-\varpi} \end{cases}, \tag{5.2}$$

where $\mathcal{X}_1 = \mathcal{X}_1(\varpi, \varphi), \mathcal{X}_2 = \mathcal{X}_2(\varpi, \varphi)$ and $\mathbb{D}_{\varphi}^{\alpha,\rho}$ is the Caputo-Katugampola fractional derivative of order α, ρ with $0 < \alpha \leq 1$ and $\rho > 0$.

To resolve the system (5.1) with the initial conditions (5.2), we follow the same steps presented in Section 3.

First, we assume that the solution of (5.1)-(5.2) is of the form

$$\begin{cases} \mathcal{X}_1(\varpi, \varphi) = \sum\limits_{n=0}^{\infty} \mathcal{X}_{1n}(\varpi, \varphi) \\ \mathcal{X}_2(\varpi, \varphi) = \sum\limits_{n=0}^{\infty} \mathcal{X}_{2n}(\varpi, \varphi) \end{cases}. \tag{5.3}$$

Next, we consider the following system

$$\begin{cases} \mathbb{D}_{\varphi}^{\alpha,\rho}\mathcal{X}_{1\lambda_1} = \lambda_1 \left[1 - \mathcal{X}_{1\lambda_1} - \mathcal{X}_{2\lambda_1}\mathcal{X}_{1\lambda_1\varpi}\right] \\ \mathbb{D}_{\varphi}^{\alpha,\rho}\mathcal{X}_{2\lambda_2} = \lambda_2 \left[-1 + \mathcal{X}_{2\lambda_2} - \mathcal{X}_{1\lambda_2}\mathcal{X}_{2\lambda_2\varpi}\right] \end{cases}, \lambda_1, \lambda_2 \in [0, 1], \tag{5.4}$$

and the solution of the system (5.4) can be expressed as

$$\begin{cases} \mathcal{X}_{1\lambda_1}(\varpi,\varphi)=\sum\limits_{n=0}^{\infty}\lambda_1^n\mathcal{X}_{1n}(\varpi,\varphi) \\ \mathcal{X}_{2\lambda_2}(\varpi,\varphi)=\sum\limits_{n=0}^{\infty}\lambda_2^n\mathcal{X}_{2n}(\varpi,\varphi) \end{cases}. \tag{5.5}$$

Applying the ρ-Khalouta transform on (5.4) with respect to the variable "φ" and Theorem 2.3, we get

$$\begin{cases} \mathbb{KH}_\rho\left[\mathcal{X}_{1\lambda_1}\right]=e^{\varpi}+\left(\frac{\gamma\eta}{s}\right)^\alpha\lambda_1\mathbb{KH}_\rho\left[1-\mathcal{X}_{1\lambda_1}-\mathcal{X}_{2\lambda_1}\mathcal{X}_{1\lambda_1\varpi}\right] \\ \mathbb{KH}_\rho\left[\mathcal{X}_{2\lambda_2}\right]=e^{-\varpi}+\left(\frac{\gamma\eta}{s}\right)^\alpha\lambda_2\mathbb{KH}_\rho\left[-1+\mathcal{X}_{2\lambda_2}-\mathcal{X}_{1\lambda_2}\mathcal{X}_{2\lambda_2\varpi}\right] \end{cases}, \tag{5.6}$$

Taking the inverse ρ-Khalouta transform on (5.6), we get

$$\begin{cases} \mathcal{X}_{1\lambda_1}(\varpi,\varphi)=e^{\varpi}+\lambda_1\mathbb{KH}_\rho^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^\alpha\mathbb{KH}_\rho\left[1-\mathcal{X}_{1\lambda_1}-\mathcal{X}_{2\lambda_1}\mathcal{X}_{1\lambda_1\varpi}\right]\right] \\ \mathcal{X}_{2\lambda_2}(\varpi,\varphi)=e^{-\varpi}+\lambda_2\mathbb{KH}_\rho^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^\alpha\mathbb{KH}_\rho\left[-1+\mathcal{X}_{2\lambda_2}-\mathcal{X}_{1\lambda_2}\mathcal{X}_{2\lambda_2\varpi}\right]\right] \end{cases}, \tag{5.7}$$

Application of the new decomposition method [16] implies

$$\begin{cases} \sum\limits_{n=0}^{\infty}\lambda_1^n\mathcal{X}_{1n}(\varpi,\varphi)=e^{\varpi}+\lambda_1\mathbb{KH}_\rho^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^\alpha\mathbb{KH}_\rho\left[\begin{array}{c}1-\sum\limits_{n=0}^{\infty}\lambda_1^n\mathcal{X}_{1n}\\ -\sum\limits_{n=0}^{\infty}\lambda_1^nK_{1n}\end{array}\right]\right] \\ \sum\limits_{n=0}^{\infty}\lambda_2^n\mathcal{X}_{2n}(\varpi,\varphi)=e^{-\varpi}+\lambda_2\mathbb{KH}_\rho^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^\alpha\mathbb{KH}_\rho\left[\begin{array}{c}-1+\sum\limits_{n=0}^{\infty}\lambda_2^n\mathcal{X}_{2n}\\ -\sum\limits_{n=0}^{\infty}\lambda_2^nK_{2n}\end{array}\right]\right] \end{cases}, \tag{5.8}$$

where K_{1n} and K_{2n} are polynomials which respectively represent the nonlinear terms $\mathcal{X}_2\mathcal{X}_{1\varpi}$ and $\mathcal{X}_1\mathcal{X}_{2\varpi}$.

According to the relation (3.13), the first few components of the polynomials K_{1n} and K_{2n} are given by

$$\begin{aligned} K_{10}&=\mathcal{X}_{20}\mathcal{X}_{10\varpi},\\ K_{11}&=\mathcal{X}_{20}\mathcal{X}_{11\varpi}+\mathcal{X}_{21}\mathcal{X}_{10\varpi},\\ K_{12}&=\mathcal{X}_{20}\mathcal{X}_{12\varpi}+\mathcal{X}_{21}\mathcal{X}_{11\varpi}+\mathcal{X}_{22}\mathcal{X}_{10\varpi},\\ &\vdots \end{aligned} \tag{5.9}$$

and

$$\begin{aligned} K_{20}&=\mathcal{X}_{10}\mathcal{X}_{20\varpi},\\ K_{21}&=\mathcal{X}_{10}\mathcal{X}_{21\varpi}+\mathcal{X}_{11}\mathcal{X}_{20\varpi},\\ K_{22}&=\mathcal{X}_{10}\mathcal{X}_{22\varpi}+\mathcal{X}_{11}\mathcal{X}_{21\varpi}+\mathcal{X}_{12}\mathcal{X}_{20\varpi},\\ &\vdots \end{aligned} \tag{5.10}$$

By equating the terms in (5.8) with identical powers of λ_1 and λ_1, the following relation is obtained.

$$\begin{cases} \mathcal{X}_{10}(\varpi,\varphi) = e^{\varpi} \\ \mathcal{X}_{20}(\varpi,\varphi) = e^{-\varpi} \end{cases}, \quad \begin{cases} \mathcal{X}_{11}(\varpi,\varphi) = \mathbb{KH}_{\rho}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}_{\rho}\left[1-\mathcal{X}_{10}-K_{10}\right]\right] \\ \mathcal{X}_{21}(\varpi,\varphi) = \mathbb{KH}_{\rho}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}_{\rho}\left[-1+\mathcal{X}_{20}-K_{20}\right]\right] \end{cases}, \quad \begin{cases} \mathcal{X}_{1n}(\varpi,\varphi) = \mathbb{KH}_{\rho}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}_{\rho}\left[-\mathcal{X}_{1(n-1)}-K_{1(n-1)}\right]\right] \\ \mathcal{X}_{2n}(\varpi,\varphi) = \mathbb{KH}_{\rho}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}_{\rho}\left[\mathcal{X}_{2(n-1)}-K_{2(n-1)}\right]\right] \end{cases} \tag{5.11}$$

Now, according to (5.3) and (5.5), we have

$$\begin{cases} \mathcal{X}_{1}(\varpi,\varphi) = \lim\limits_{\lambda_1\to 1}\mathcal{X}_{1\lambda_1}(\varpi,\varphi) = \mathcal{X}_{10}(\varpi,\varphi)+\mathcal{X}_{11}(\varpi,\varphi)+\sum\limits_{n=2}^{\infty}\mathcal{X}_{1n}(\varpi,\varphi) \\ \mathcal{X}_{2}(\varpi,\varphi) = \lim\limits_{\lambda_2\to 1}\mathcal{X}_{2\lambda_2}(\varpi,\varphi) = \mathcal{X}_{20}(\varpi,\varphi)+\mathcal{X}_{21}(\varpi,\varphi)+\sum\limits_{n=2}^{\infty}\mathcal{X}_{2n}(\varpi,\varphi) \end{cases}. \tag{5.12}$$

Thus, the following approximations are obtained successively

$$\begin{cases} \mathcal{X}_{10}(\varpi,\varphi) = e^{\varpi} \\ \mathcal{X}_{20}(\varpi,\varphi) = e^{-\varpi} \end{cases}, \quad \begin{cases} \mathcal{X}_{11}(\varpi,\varphi) = -\frac{1}{\Gamma(\alpha+1)}\frac{\varphi^{\alpha\rho}}{\rho^{\alpha}}e^{\varpi} \\ \mathcal{X}_{21}(\varpi,\varphi) = \frac{1}{\Gamma(\alpha+1)}\frac{\varphi^{\alpha\rho}}{\rho^{\alpha}}e^{-\varpi} \end{cases}, \quad \begin{cases} \mathcal{X}_{12}(\varpi,\varphi) = \frac{1}{\Gamma(2\alpha+1)}\frac{\varphi^{2\alpha\rho}}{\rho^{2\alpha}}e^{\varpi} \\ \mathcal{X}_{22}(\varpi,\varphi) = \frac{1}{\Gamma(2\alpha+1)}\frac{\varphi^{2\alpha\rho}}{\rho^{2\alpha}}e^{-\varpi} \end{cases}, \quad \begin{cases} \mathcal{X}_{13}(\varpi,\varphi) = -\frac{1}{\Gamma(3\alpha+1)}\frac{\varphi^{3\alpha\rho}}{\rho^{3\alpha}}e^{\varpi} \\ \mathcal{X}_{23}(\varpi,\varphi) = \frac{1}{\Gamma(3\alpha+1)}\frac{\varphi^{3\alpha\rho}}{\rho^{3\alpha}}e^{-\varpi} \end{cases}, \quad \vdots \tag{5.13}$$

The solution is finally expressed by

$$\begin{cases} \mathcal{X}_{1}(\varpi,\varphi) = \left(1-\frac{1}{\Gamma(\alpha+1)}\frac{\varphi^{\alpha\rho}}{\rho^{\alpha}}+\frac{1}{\Gamma(2\alpha+1)}\frac{\varphi^{2\alpha\rho}}{\rho^{2\alpha}}-\frac{1}{\Gamma(3\alpha+1)}\frac{\varphi^{3\alpha\rho}}{\rho^{3\alpha}}+...\right)e^{\varpi} \\ \mathcal{X}_{2}(\varpi,\varphi) = \left(1+\frac{1}{\Gamma(\alpha+1)}\frac{\varphi^{\alpha\rho}}{\rho^{\alpha}}+\frac{1}{\Gamma(2\alpha+1)}\frac{\varphi^{2\alpha\rho}}{\rho^{2\alpha}}+\frac{1}{\Gamma(3\alpha+1)}\frac{\varphi^{3\alpha\rho}}{\rho^{3\alpha}}+...\right)e^{-\varpi} \end{cases}. \tag{5.14}$$

Taking $\rho = 1$ in (5.14), the following solution is obtained.

$$\begin{cases} \mathcal{X}_{1}(\varpi,\varphi) = \left(1-\frac{\varphi^{\alpha}}{\Gamma(\alpha+1)}+\frac{\varphi^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{\varphi^{3\alpha}}{\Gamma(3\alpha+1)}+...\right)e^{\varpi} \\ \mathcal{X}_{2}(\varpi,\varphi) = \left(1+\frac{\varphi^{\alpha}}{\Gamma(\alpha+1)}+\frac{\varphi^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{\varphi^{3\alpha}}{\Gamma(3\alpha+1)}+...\right)e^{-\varpi} \end{cases}, \tag{5.15}$$

which is the solution of the system (5.1) based on the Caputo fractional derivative.

Taking $\alpha = 1$ in (5.14), the following solution is obtained.

$$\begin{cases} \mathcal{X}_{1}(\varpi,\varphi) = \left(1-\frac{\varphi^{\rho}}{\rho}+\frac{\varphi^{2\rho}}{2!\rho^{2}}-\frac{\varphi^{3\alpha}}{3!\rho^{3}}+...\right)e^{\varpi} \\ \mathcal{X}_{2}(\varpi,\varphi) = \left(1+\frac{\varphi^{\alpha}}{\rho}+\frac{\varphi^{2\alpha}}{2!\rho^{2}}+\frac{\varphi^{3\alpha}}{\rho^{3}}+...\right)e^{-\varpi} \end{cases}, \tag{5.16}$$

which is the solution of the system (5.1) based on the conformable fractional derivative.

Taking $\alpha = \rho = 1$ in (5.14), the following solution is obtained.

$$\begin{cases} \mathcal{X}_1(\varpi,\varphi) = \left(1-\varphi+\frac{\varphi^2}{2!}-\frac{\varphi^3}{3!}+...\right)e^{\varpi} = e^{-\varphi+\varpi} \\ \mathcal{X}_2(\varpi,\varphi) = \left(1+\varphi+\frac{\varphi^2}{2!}+\frac{\varphi^3}{3!}+...\right)e^{-\varpi} = e^{\varphi-\varpi} \end{cases}, \tag{5.17}$$

which is the exact solution available in the literature [8].

Example 2 Consider the system of nonlinear homogeneous time-fractional partial differential equations of the form

$$\begin{cases} \mathbb{D}_{\varphi}^{\alpha,\rho}\mathcal{X}_1 - \mathcal{X}_{1\varpi\varpi} - 2\mathcal{X}_1\mathcal{X}_{1\varpi} + (\mathcal{X}_1\mathcal{X}_2)_{\varpi} = 0 \\ \mathbb{D}_{\varphi}^{\alpha,\rho}\mathcal{X}_2 - \mathcal{X}_{2\varpi\varpi} - 2\mathcal{X}_2\mathcal{X}_{2\varpi} + (\mathcal{X}_1\mathcal{X}_2)_{\varpi} = 0 \end{cases}, \tag{5.18}$$

with the initial conditions

$$\begin{cases} \mathcal{X}_1(\varpi,0) = \sin(\varpi) \\ \mathcal{X}_2(\varpi,0) = \sin(\varpi) \end{cases}, \tag{5.19}$$

where $\mathcal{X}_1 = \mathcal{X}_1(\varpi,\varphi), \mathcal{X}_2 = \mathcal{X}_2(\varpi,\varphi)$ and $\mathbb{D}_{\varphi}^{\alpha,\rho}$ is the Caputo-Katugampola fractional derivative of order α, ρ with $0 < \alpha \leq 1$ and $\rho > 0$.

To resolve the system (5.18) with the initial conditions (5.19), we follow the same steps presented in Section 3.

First, we assume that the solution of (5.18)-(5.19) is of the form

$$\begin{cases} \mathcal{X}_1(\varpi,\varphi) = \sum\limits_{n=0}^{\infty} \mathcal{X}_{1n}(\varpi,\varphi) \\ \mathcal{X}_2(\varpi,\varphi) = \sum\limits_{n=0}^{\infty} \mathcal{X}_{2n}(\varpi,\varphi) \end{cases}. \tag{5.20}$$

Next, we consider the following system

$$\begin{cases} \mathbb{D}_{\varphi}^{\alpha,\rho}\mathcal{X}_{1\lambda_1} = \lambda_1\left[\mathcal{X}_{1\lambda_1\varpi\varpi} + 2\mathcal{X}_{1\lambda_1}\mathcal{X}_{1\lambda_1\varpi} - (\mathcal{X}_{1\lambda_1}\mathcal{X}_{2\lambda_1})_{\varpi}\right] \\ \mathbb{D}_{\varphi}^{\alpha,\rho}\mathcal{X}_{2\lambda_2} = \lambda_2\left[\mathcal{X}_{2\lambda_2\varpi\varpi} + 2\mathcal{X}_{2\lambda_2}\mathcal{X}_{2\lambda_2\varpi} - (\mathcal{X}_{1\lambda_2}\mathcal{X}_{2\lambda_2})_{\varpi}\right] \end{cases}, \lambda_1, \lambda_2 \in [0,1], \tag{5.21}$$

and the solution of the system (5.4) can be expressed as

$$\begin{cases} \mathcal{X}_{1\lambda_1}(\varpi,\varphi) = \sum\limits_{n=0}^{\infty} \lambda_1^n\mathcal{X}_{1n}(\varpi,\varphi) \\ \mathcal{X}_{2\lambda_2}(\varpi,\varphi) = \sum\limits_{n=0}^{\infty} \lambda_2^n\mathcal{X}_{2n}(\varpi,\varphi) \end{cases}. \tag{5.22}$$

Applying the ρ-Khalouta transform on (5.21) with respect to the variable "φ" and Theorem 2.3, we get

$$\begin{cases} \mathbb{KH}_{\rho}\left[\mathcal{X}_{1\lambda_1}\right] = \sin(\varpi) + \left(\frac{\gamma\eta}{s}\right)^{\alpha}\lambda_1\mathbb{KH}_{\rho}\left[\mathcal{X}_{1\lambda_1\varpi\varpi} + 2\mathcal{X}_{1\lambda_1}\mathcal{X}_{1\lambda_1\varpi} - (\mathcal{X}_{1\lambda_1}\mathcal{X}_{2\lambda_1})_{\varpi}\right] \\ \mathbb{KH}_{\rho}\left[\mathcal{X}_{2\lambda_2}\right] = \sin(\varpi) + \left(\frac{\gamma\eta}{s}\right)^{\alpha}\lambda_2\mathbb{KH}_{\rho}\left[\mathcal{X}_{2\lambda_2\varpi\varpi} + 2\mathcal{X}_{2\lambda_2}\mathcal{X}_{2\lambda_2\varpi} - (\mathcal{X}_{1\lambda_2}\mathcal{X}_{2\lambda_2})_{\varpi}\right] \end{cases}, \tag{5.23}$$

Taking the inverse ρ-Khalouta transform on (5.23), we get

$$\begin{cases} \mathcal{X}_{1\lambda_1}(\varpi,\varphi) = \sin(\varpi) + \lambda_1\mathbb{KH}_{\rho}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}_{\rho}\left[\mathcal{X}_{1\lambda_1\varpi\varpi} + 2\mathcal{X}_{1\lambda_1}\mathcal{X}_{1\lambda_1\varpi} - (\mathcal{X}_{1\lambda_1}\mathcal{X}_{2\lambda_1})_{\varpi}\right]\right] \\ \mathcal{X}_{2\lambda_2}(\varpi,\varphi) = \sin(\varpi) + \lambda_2\mathbb{KH}_{\rho}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}_{\rho}\left[\mathcal{X}_{2\lambda_2\varpi\varpi} + 2\mathcal{X}_{2\lambda_2}\mathcal{X}_{2\lambda_2\varpi} - (\mathcal{X}_{1\lambda_2}\mathcal{X}_{2\lambda_2})_{\varpi}\right]\right] \end{cases}, \tag{5.24}$$

Application of the new decomposition method [16] implies

$$\begin{cases} \sum\limits_{n=0}^{\infty} \lambda_1^n \mathcal{X}_{1n}(\varpi,\varphi) = \sin(\varpi) + \lambda_1 \mathbb{KH}_\rho^{-1} \left[\left(\frac{\gamma\eta}{s}\right)^\alpha \mathbb{KH}_\rho \left[\begin{array}{c} \sum\limits_{n=0}^{\infty} \lambda_1^n \mathcal{X}_{1n\varpi\varpi} + 2\sum\limits_{n=0}^{\infty} \lambda_1^n K_{1n} \\ -\sum\limits_{n=0}^{\infty} \lambda_1^n K_{3n} \end{array} \right] \right] \\ \sum\limits_{n=0}^{\infty} \lambda_2^n \mathcal{X}_{2n}(\varpi,\varphi) = \sin(\varpi) + \lambda_2 \mathbb{KH}_\rho^{-1} \left[\left(\frac{\gamma\eta}{s}\right)^\alpha \mathbb{KH}_\rho \left[\begin{array}{c} \sum\limits_{n=0}^{\infty} \lambda_2^n \mathcal{X}_{2n\varpi\varpi} + 2\sum\limits_{n=0}^{\infty} \lambda_2^n K_{2n} \\ -\sum\limits_{n=0}^{\infty} \lambda_2^n K_{3n} \end{array} \right] \right] \end{cases}, \tag{5.25}$$

where K_{1n}, K_{2n} and K_{3n} are polynomials which respectively represent the nonlinear terms $\mathcal{X}_1\mathcal{X}_{1\varpi}, \mathcal{X}_2\mathcal{X}_{2\varpi}$ and $(\mathcal{X}_1\mathcal{X}_2)_\varpi$.

According to the relation (3.13), the first few components of the polynomials K_{1n}, K_{2n} and K_{3n} are given by

$$\begin{aligned} K_{10} &= \mathcal{X}_{10}\mathcal{X}_{10\varpi}, \\ K_{11} &= \mathcal{X}_{10}\mathcal{X}_{11\varpi} + \mathcal{X}_{11}\mathcal{X}_{10\varpi}, \\ K_{12} &= \mathcal{X}_{10}\mathcal{X}_{12\varpi} + \mathcal{X}_{11}\mathcal{X}_{11\varpi} + \mathcal{X}_{12}\mathcal{X}_{10\varpi}, \\ &\vdots \end{aligned} \tag{5.26}$$

$$\begin{aligned} K_{20} &= \mathcal{X}_{20}\mathcal{X}_{20\varpi}, \\ K_{21} &= \mathcal{X}_{20}\mathcal{X}_{21\varpi} + \mathcal{X}_{21}\mathcal{X}_{20\varpi}, \\ K_{22} &= \mathcal{X}_{20}\mathcal{X}_{22\varpi} + \mathcal{X}_{21}\mathcal{X}_{21\varpi} + \mathcal{X}_{22}\mathcal{X}_{20\varpi}, \\ &\vdots \end{aligned} \tag{5.27}$$

and

$$\begin{aligned} K_{30} &= (\mathcal{X}_{10}\mathcal{X}_{20})_\varpi, \\ K_{31} &= (\mathcal{X}_{11}\mathcal{X}_{20} + \mathcal{X}_{10}\mathcal{X}_{21})_\varpi, \\ K_{32} &= (\mathcal{X}_{20}\mathcal{X}_{22} + \mathcal{X}_{21}\mathcal{X}_{21} + \mathcal{X}_{22}\mathcal{X}_{20})_\varpi, \\ &\vdots \end{aligned} \tag{5.28}$$

By equating the terms in (5.25) with identical powers of λ_1 and λ_1, the following relation is obtained.

$$\begin{aligned} &\begin{cases} \mathcal{X}_{10}(\varpi,\varphi) = \sin(\varpi) \\ \mathcal{X}_{20}(\varpi,\varphi) = \sin(\varpi) \end{cases}, \\ &\begin{cases} \mathcal{X}_{11}(\varpi,\varphi) = \mathbb{KH}_\rho^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^\alpha \mathbb{KH}_\rho\left[\mathcal{X}_{10\varpi\varpi} + 2K_{10} - K_{30}\right]\right] \\ \mathcal{X}_{21}(\varpi,\varphi) = \mathbb{KH}_\rho^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^\alpha \mathbb{KH}_\rho\left[\mathcal{X}_{20\varpi\varpi} + 2K_{20} - K_{30}\right]\right] \end{cases}, \\ &\begin{cases} \mathcal{X}_{1n}(\varpi,\varphi) = \mathbb{KH}_\rho^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^\alpha \mathbb{KH}_\rho\left[\left[\mathcal{X}_{1(n-1)\varpi\varpi} + 2K_{1(n-1)} - K_{3(n-1)}\right]\right]\right] \\ \mathcal{X}_{2n}(\varpi,\varphi) = \mathbb{KH}_\rho^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^\alpha \mathbb{KH}_\rho\left[\mathcal{X}_{2(n-1)\varpi\varpi} + 2K_{2(n-1)} - K_{3(n-1)}\right]\right] \end{cases} \end{aligned} \tag{5.29}$$

Now, according to (5.20) and (5.22), we have

$$\begin{cases} \mathcal{X}_1(\varpi,\varphi) = \lim\limits_{\lambda_1\to 1} \mathcal{X}_{1\lambda_1}(\varpi,\varphi) = \mathcal{X}_{10}(\varpi,\varphi) + \mathcal{X}_{11}(\varpi,\varphi) + \sum\limits_{n=2}^{\infty} \mathcal{X}_{1n}(\varpi,\varphi) \\ \mathcal{X}_2(\varpi,\varphi) = \lim\limits_{\lambda_2\to 1} \mathcal{X}_{2\lambda_2}(\varpi,\varphi) = \mathcal{X}_{20}(\varpi,\varphi) + \mathcal{X}_{21}(\varpi,\varphi) + \sum\limits_{n=2}^{\infty} \mathcal{X}_{2n}(\varpi,\varphi) \end{cases}. \tag{5.30}$$

Thus, the following approximations are obtained successively

$$
\begin{aligned}
&\begin{cases} \mathcal{X}_{10}(\varpi,\varphi) = \sin(\varpi) \\ \mathcal{X}_{20}(\varpi,\varphi) = \sin(\varpi) \end{cases}, \\
&\begin{cases} \mathcal{X}_{11}(\varpi,\varphi) = -\frac{1}{\Gamma(\alpha+1)}\frac{\varphi^{\alpha\rho}}{\rho^{\alpha}}\sin(\varpi) \\ \mathcal{X}_{21}(\varpi,\varphi) = -\frac{1}{\Gamma(\alpha+1)}\frac{\varphi^{\alpha\rho}}{\rho^{\alpha}}\sin(\varpi) \end{cases}, \\
&\begin{cases} \mathcal{X}_{12}(\varpi,\varphi) = \frac{1}{\Gamma(2\alpha+1)}\frac{\varphi^{2\alpha\rho}}{\rho^{2\alpha}}\sin(\varpi) \\ \mathcal{X}_{22}(\varpi,\varphi) = \frac{1}{\Gamma(2\alpha+1)}\frac{\varphi^{2\alpha\rho}}{\rho^{2\alpha}}\sin(\varpi) \end{cases}, \\
&\begin{cases} \mathcal{X}_{13}(\varpi,\varphi) = -\frac{1}{\Gamma(3\alpha+1)}\frac{\varphi^{3\alpha\rho}}{\rho^{3\alpha}}\sin(\varpi) \\ \mathcal{X}_{23}(\varpi,\varphi) = -\frac{1}{\Gamma(3\alpha+1)}\frac{\varphi^{3\alpha\rho}}{\rho^{3\alpha}}\sin(\varpi) \end{cases}, \\
&\vdots
\end{aligned}
\tag{5.31}
$$

The solution is finally expressed by

$$
\begin{cases} \mathcal{X}_{1}(\varpi,\varphi) = \left(1 - \frac{1}{\Gamma(\alpha+1)}\frac{\varphi^{\alpha\rho}}{\rho^{\alpha}} + \frac{1}{\Gamma(2\alpha+1)}\frac{\varphi^{2\alpha\rho}}{\rho^{2\alpha}} - \frac{1}{\Gamma(3\alpha+1)}\frac{\varphi^{3\alpha\rho}}{\rho^{3\alpha}} + ...\right)\sin(\varpi) \\ \mathcal{X}_{2}(\varpi,\varphi) = \left(1 - \frac{1}{\Gamma(\alpha+1)}\frac{\varphi^{\alpha\rho}}{\rho^{\alpha}} + \frac{1}{\Gamma(2\alpha+1)}\frac{\varphi^{2\alpha\rho}}{\rho^{2\alpha}} - \frac{1}{\Gamma(3\alpha+1)}\frac{\varphi^{3\alpha\rho}}{\rho^{3\alpha}} + ...\right)\sin(\varpi) \end{cases}. \tag{5.32}
$$

Taking $\rho = 1$ in (5.32), the following solution is obtained.

$$
\begin{cases} \mathcal{X}_{1}(\varpi,\varphi) = \left(1 - \frac{\varphi^{\alpha}}{\Gamma(\alpha+1)} + \frac{\varphi^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{\varphi^{3\alpha}}{\Gamma(3\alpha+1)} + ...\right)\sin(\varpi) \\ \mathcal{X}_{2}(\varpi,\varphi) = \left(1 - \frac{\varphi^{\alpha}}{\Gamma(\alpha+1)} + \frac{\varphi^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{\varphi^{3\alpha}}{\Gamma(3\alpha+1)} + ...\right)\sin(\varpi) \end{cases}, \tag{5.33}
$$

which is the solution of the system (5.18) based on the Caputo fractional derivative.
Taking $\alpha = 1$ in (5.32), the following solution is obtained.

$$
\begin{cases} \mathcal{X}_{1}(\varpi,\varphi) = \left(1 - \frac{\varphi^{\rho}}{\rho} + \frac{\varphi^{2\rho}}{2!\rho^{2}} - \frac{\varphi^{3\alpha}}{3!\rho^{3}} + ...\right)\sin(\varpi) \\ \mathcal{X}_{2}(\varpi,\varphi) = \left(1 - \frac{\varphi^{\alpha}}{\rho} + \frac{\varphi^{2\alpha}}{2!\rho^{2}} - \frac{\varphi^{3\alpha}}{\rho^{3}} + ...\right)\sin(\varpi) \end{cases}, \tag{5.34}
$$

which is the solution of the system (5.18) based on the conformable fractional derivative.
Taking $\alpha = \rho = 1$ in (5.32), the following solution is obtained.

$$
\begin{cases} \mathcal{X}_{1}(\varpi,\varphi) = \left(1 - \varphi + \frac{\varphi^{2}}{2!} - \frac{\varphi^{3}}{3!} + ...\right)\sin(\varpi) = e^{-\varphi}\sin(\varpi) \\ \mathcal{X}_{2}(\varpi,\varphi) = \left(1 - \varphi + \frac{\varphi^{2}}{2!} - \frac{\varphi^{3}}{3!} + ...\right)\sin(\varpi) = e^{-\varphi}\sin(\varpi) \end{cases}, \tag{5.35}
$$

which is the exact solution available in the literature [3].

Example 3 Consider the system of nonlinear nonhomogeneous time-fractional partial differential equations of the form

$$
\begin{cases} \mathbb{D}_{\varphi}^{\alpha,\rho}\mathcal{X}_{1} + 2\mathcal{X}_{2}\mathcal{X}_{1\varpi} - \mathcal{X}_{1} = 2 \\ \mathbb{D}_{\varphi}^{\alpha,\rho}\mathcal{X}_{2} - 3\mathcal{X}_{1}\mathcal{X}_{2\varpi} + \mathcal{X}_{2} = 3 \end{cases}, \tag{5.36}
$$

with the initial conditions

$$
\begin{cases} \mathcal{X}_{1}(\varpi,0) = e^{\varpi} \\ \mathcal{X}_{2}(\varpi,0) = e^{-\varpi} \end{cases}, \tag{5.37}
$$

where $\mathcal{X}_1 = \mathcal{X}_1(\varpi,\varphi), \mathcal{X}_2 = \mathcal{X}_2(\varpi,\varphi)$ and $\mathbb{D}_{\varphi}^{\alpha,\rho}$ is the Caputo-Katugampola fractional derivative of order α,ρ with $0<\alpha\leq 1$ and $\rho>0$.

To resolve the system (5.36) with the initial conditions (5.37), we follow the same steps presented in Section 3.

First, we assume that the solution of (5.36)-(5.37) is of the form

$$\begin{cases} \mathcal{X}_1(\varpi,\varphi) = \sum\limits_{n=0}^{\infty} \mathcal{X}_{1n}(\varpi,\varphi) \\ \mathcal{X}_2(\varpi,\varphi) = \sum\limits_{n=0}^{\infty} \mathcal{X}_{2n}(\varpi,\varphi) \end{cases}. \tag{5.38}$$

Next, we consider the following system

$$\begin{cases} \mathbb{D}_{\varphi}^{\alpha,\rho}\mathcal{X}_{1\lambda_1} = \lambda_1\left[2+\mathcal{X}_{1\lambda_1}-2\mathcal{X}_{2\lambda_1}\mathcal{X}_{1\lambda_1\varpi}\right] \\ \mathbb{D}_{\varphi}^{\alpha,\rho}\mathcal{X}_{2\lambda_2} = \lambda_2\left[3-\mathcal{X}_{2\lambda_2}+3\mathcal{X}_{1\lambda_2}\mathcal{X}_{2\lambda_2\varpi}\right] \end{cases}, \lambda_1,\lambda_2\in[0,1], \tag{5.39}$$

and the solution of the system (5.39) can be expressed as

$$\begin{cases} \mathcal{X}_{1\lambda_1}(\varpi,\varphi) = \sum\limits_{n=0}^{\infty} \lambda_1^n\mathcal{X}_{1n}(\varpi,\varphi) \\ \mathcal{X}_{2\lambda_2}(\varpi,\varphi) = \sum\limits_{n=0}^{\infty} \lambda_2^n\mathcal{X}_{2n}(\varpi,\varphi) \end{cases}. \tag{5.40}$$

Applying the ρ-Khalouta transform on (5.39) with respect to the variable "φ" and Theorem 2.3, we get

$$\begin{cases} \mathbb{KH}_{\rho}\left[\mathcal{X}_{1\lambda_1}\right] = e^{\varpi} + \left(\frac{\gamma\eta}{s}\right)^{\alpha}\lambda_1\mathbb{KH}_{\rho}\left[2+\mathcal{X}_{1\lambda_1}-2\mathcal{X}_{2\lambda_1}\mathcal{X}_{1\lambda_1\varpi}\right] \\ \mathbb{KH}_{\rho}\left[\mathcal{X}_{2\lambda_2}\right] = e^{-\varpi} + \left(\frac{\gamma\eta}{s}\right)^{\alpha}\lambda_2\mathbb{KH}_{\rho}\left[3-\mathcal{X}_{2\lambda_2}+3\mathcal{X}_{1\lambda_2}\mathcal{X}_{2\lambda_2\varpi}\right] \end{cases}. \tag{5.41}$$

Taking the inverse ρ-Khalouta transform on (5.42), we get

$$\begin{cases} \mathcal{X}_{1\lambda_1}(\varpi,\varphi) = e^{\varpi} + \lambda_1\mathbb{KH}_{\rho}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}_{\rho}\left[2+\mathcal{X}_{1\lambda_1}-2\mathcal{X}_{2\lambda_1}\mathcal{X}_{1\lambda_1\varpi}\right]\right] \\ \mathcal{X}_{2\lambda_2}(\varpi,\varphi) = e^{-\varpi} + \lambda_2\mathbb{KH}_{\rho}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}_{\rho}\left[3-\mathcal{X}_{2\lambda_2}+3\mathcal{X}_{1\lambda_2}\mathcal{X}_{2\lambda_2\varpi}\right]\right] \end{cases}. \tag{5.42}$$

Application of the new decomposition method [16] implies

$$\begin{cases} \sum\limits_{n=0}^{\infty} \lambda_1^n\mathcal{X}_{1n}(\varpi,\varphi) = e^{\varpi} + \lambda_1\mathbb{KH}_{\rho}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}_{\rho}\left[\begin{array}{c} 2+\sum\limits_{n=0}^{\infty}\lambda_1^n\mathcal{X}_{1n} \\ -2\sum\limits_{n=0}^{\infty}\lambda_1^n K_{1n} \end{array}\right]\right] \\ \sum\limits_{n=0}^{\infty} \lambda_2^n\mathcal{X}_{2n}(\varpi,\varphi) = e^{-\varpi} + \lambda_2\mathbb{KH}_{\rho}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}_{\rho}\left[\begin{array}{c} 3-\sum\limits_{n=0}^{\infty}\lambda_2^n\mathcal{X}_{2n} \\ +3\sum\limits_{n=0}^{\infty}\lambda_2^n K_{2n} \end{array}\right]\right] \end{cases}, \tag{5.43}$$

where K_{1n} and K_{2n} are polynomials which respectively represent the nonlinear terms $\mathcal{X}_2\mathcal{X}_{1\varpi}$ and $\mathcal{X}_1\mathcal{X}_{2\varpi}$.

According to the relation (3.13), the first few components of the polynomials K_{1n} and K_{2n} are given by

$$\begin{aligned} K_{10} &= \mathcal{X}_{20}\mathcal{X}_{10\varpi}, \\ K_{11} &= \mathcal{X}_{20}\mathcal{X}_{11\varpi} + \mathcal{X}_{21}\mathcal{X}_{10\varpi}, \\ K_{12} &= \mathcal{X}_{20}\mathcal{X}_{12\varpi} + \mathcal{X}_{21}\mathcal{X}_{11\varpi} + \mathcal{X}_{22}\mathcal{X}_{10\varpi}, \\ &\vdots \end{aligned} \tag{5.44}$$

and

$$\begin{aligned} K_{20} &= \mathcal{X}_{10}\mathcal{X}_{20\varpi}, \\ K_{21} &= \mathcal{X}_{10}\mathcal{X}_{21\varpi} + \mathcal{X}_{11}\mathcal{X}_{20\varpi}, \\ K_{22} &= \mathcal{X}_{10}\mathcal{X}_{22\varpi} + \mathcal{X}_{11}\mathcal{X}_{21\varpi} + \mathcal{X}_{12}\mathcal{X}_{20\varpi}, \\ &\vdots \end{aligned} \tag{5.45}$$

By equating the terms in (5.43) with identical powers of λ_1 and λ_1, the following relation is obtained.

$$\begin{aligned} &\begin{cases} \mathcal{X}_{10}(\varpi,\varphi) = e^{\varpi} \\ \mathcal{X}_{20}(\varpi,\varphi) = e^{-\varpi} \end{cases}, \\ &\begin{cases} \mathcal{X}_{11}(\varpi,\varphi) = \mathbb{KH}_{\rho}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}_{\rho}\left[2+\mathcal{X}_{10}-2K_{10}\right]\right] \\ \mathcal{X}_{21}(\varpi,\varphi) = \mathbb{KH}_{\rho}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}_{\rho}\left[3-\mathcal{X}_{20}+3K_{20}\right]\right] \end{cases}, \\ &\begin{cases} \mathcal{X}_{1n}(\varpi,\varphi) = \mathbb{KH}_{\rho}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}_{\rho}\left[\mathcal{X}_{1(n-1)}-2K_{1(n-1)}\right]\right] \\ \mathcal{X}_{2n}(\varpi,\varphi) = \mathbb{KH}_{\rho}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}_{\rho}\left[-\mathcal{X}_{2(n-1)}+3K_{2(n-1)}\right]\right] \end{cases} \end{aligned} \tag{5.46}$$

Now, according to (5.38) and (5.40), we have

$$\begin{cases} \mathcal{X}_{1}(\varpi,\varphi) = \lim\limits_{\lambda_1\to 1}\mathcal{X}_{1\lambda_1}(\varpi,\varphi) = \mathcal{X}_{10}(\varpi,\varphi) + \mathcal{X}_{11}(\varpi,\varphi) + \sum\limits_{n=2}^{\infty}\mathcal{X}_{1n}(\varpi,\varphi) \\ \mathcal{X}_{2}(\varpi,\varphi) = \lim\limits_{\lambda_2\to 1}\mathcal{X}_{2\lambda_2}(\varpi,\varphi) = \mathcal{X}_{20}(\varpi,\varphi) + \mathcal{X}_{21}(\varpi,\varphi) + \sum\limits_{n=2}^{\infty}\mathcal{X}_{2n}(\varpi,\varphi) \end{cases}. \tag{5.47}$$

Thus, the following approximations are obtained successively

$$\begin{aligned} &\begin{cases} \mathcal{X}_{10}(\varpi,\varphi) = e^{\varpi} \\ \mathcal{X}_{20}(\varpi,\varphi) = e^{-\varpi} \end{cases}, \\ &\begin{cases} \mathcal{X}_{11}(\varpi,\varphi) = \frac{1}{\Gamma(\alpha+1)}\frac{\varphi^{\alpha\rho}}{\rho^{\alpha}}e^{\varpi} \\ \mathcal{X}_{21}(\varpi,\varphi) = -\frac{1}{\Gamma(\alpha+1)}\frac{\varphi^{\alpha\rho}}{\rho^{\alpha}}e^{-\varpi} \end{cases}, \\ &\begin{cases} \mathcal{X}_{12}(\varpi,\varphi) = \frac{1}{\Gamma(2\alpha+1)}\frac{\varphi^{2\alpha\rho}}{\rho^{2\alpha}}e^{\varpi} \\ \mathcal{X}_{22}(\varpi,\varphi) = \frac{1}{\Gamma(2\alpha+1)}\frac{\varphi^{2\alpha\rho}}{\rho^{2\alpha}}e^{-\varpi} \end{cases}, \\ &\begin{cases} \mathcal{X}_{13}(\varpi,\varphi) = \frac{1}{\Gamma(3\alpha+1)}\frac{\varphi^{3\alpha\rho}}{\rho^{3\alpha}}e^{\varpi} \\ \mathcal{X}_{23}(\varpi,\varphi) = -\frac{1}{\Gamma(3\alpha+1)}\frac{\varphi^{3\alpha\rho}}{\rho^{3\alpha}}e^{-\varpi} \end{cases}, \\ &\vdots \end{aligned} \tag{5.48}$$

The solution is finally expressed by

$$\begin{cases} \mathcal{X}_{1}(\varpi,\varphi) = \left(1+\frac{1}{\Gamma(\alpha+1)}\frac{\varphi^{\alpha\rho}}{\rho^{\alpha}}+\frac{1}{\Gamma(2\alpha+1)}\frac{\varphi^{2\alpha\rho}}{\rho^{2\alpha}}+\frac{1}{\Gamma(3\alpha+1)}\frac{\varphi^{3\alpha\rho}}{\rho^{3\alpha}}+...\right)e^{\varpi} \\ \mathcal{X}_{2}(\varpi,\varphi) = \left(1-\frac{1}{\Gamma(\alpha+1)}\frac{\varphi^{\alpha\rho}}{\rho^{\alpha}}+\frac{1}{\Gamma(2\alpha+1)}\frac{\varphi^{2\alpha\rho}}{\rho^{2\alpha}}-\frac{1}{\Gamma(3\alpha+1)}\frac{\varphi^{3\alpha\rho}}{\rho^{3\alpha}}+...\right)e^{-\varpi} \end{cases}. \tag{5.49}$$

Taking $\rho = 1$ in (5.49), the following solution is obtained.

$$\begin{cases} \mathcal{X}_{1}(\varpi,\varphi) = \left(1+\frac{\varphi^{\alpha}}{\Gamma(\alpha+1)}+\frac{\varphi^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{\varphi^{3\alpha}}{\Gamma(3\alpha+1)}+...\right)e^{\varpi} \\ \mathcal{X}_{2}(\varpi,\varphi) = \left(1-\frac{\varphi^{\alpha}}{\Gamma(\alpha+1)}+\frac{\varphi^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{\varphi^{3\alpha}}{\Gamma(3\alpha+1)}+...\right)e^{-\varpi} \end{cases}, \tag{5.50}$$

which is the solution of the system (5.36) based on the Caputo fractional derivative.

Taking $\alpha = 1$ in (5.49), the following solution is obtained.

$$\begin{cases} \mathcal{X}_1(\varpi,\varphi) = \left(1 + \frac{\varphi^{\rho}}{\rho} + \frac{\varphi^{2\rho}}{2!\rho^2} + \frac{\varphi^{3\alpha}}{3!\rho^3} + ...\right) e^{\varpi} \\ \mathcal{X}_2(\varpi,\varphi) = \left(1 - \frac{\varphi^{\alpha}}{\rho} + \frac{\varphi^{2\alpha}}{2!\rho^2} - \frac{\varphi^{3\alpha}}{\rho^3} + ...\right) e^{-\varpi} \end{cases}, \tag{5.51}$$

which is the solution of the system (5.36) based on the conformable fractional derivative.

Taking $\alpha = \rho = 1$ in (5.49), the following solution is obtained.

$$\begin{cases} \mathcal{X}_1(\varpi,\varphi) = \left(1 + \varphi + \frac{\varphi^2}{2!} + \frac{\varphi^3}{3!} + ...\right) e^{\varpi} = e^{\varphi+\varpi} \\ \mathcal{X}_2(\varpi,\varphi) = \left(1 - \varphi + \frac{\varphi^2}{2!} - \frac{\varphi^3}{3!} + ...\right) e^{-\varpi} = e^{-\varphi-\varpi} \end{cases}, \tag{5.52}$$

which is the exact solution available in the literature [20].

6 Conclusions

In this paper, we investigated the approximate analytical solutions of a system of nonlinear time-fractional partial differential eqyations based on the Khalouta-Caputo-Katugampola fractional derivative using ρ-KHDM. Theoretical and numerical results clearly reveal the full reliability and effectiveness of the proposed method. As a future direction, this method can be particularly considered to solve a variety of classes of nonlinear fractional partial differential equation system involving different types of fractional derivative operators analytically and numerically.

References

1. Agarwal, G., Yadav, L.K., Albalawi, W., Abdel-Aty, A-H, Nisar, K.S., Shefeeq, T.: *Two Analytical Approaches for Space- and Time-Fractional Coupled Burger's Equations via Elzaki Transform,* Progr. Fract. Differ. Appl., **8**(1), 177-190 (2022).
2. Alfaqeih, S., Misirli, E.: *On convergence analysis and analytical solutions of the conformable fractional FitzHugh–Nagumo model using the conformable Sumudu decomposition method*, Symmetry, **13**(2), 243 p. (2021).
3. Ali, H.M., Ahmad, H., Askar, S., Ameen, I.G.: *Efficient Approaches for Solving Systems of Nonlinear Time-Fractional Partial Differential Equations*, Fractal and Fractional, **6**(1), 32 p. (2022).
4. Al-Sawalha, M.M., Shah, R., Khan, A., Ababneh, O.Y., Botmart, T.: *Fractional view analysis of Kersten-Krasil'shchik coupled KdV-mKdV systems with non-singular kernel derivatives,* AIMS Mathematics, 7(10), 18334-18359 (2022).
5. Bektas, U., Anaç, H.: *hybrid method to solve a fractional-order Newell–Whitehead–Segel equation*, Bound. Value Probl., **2024**(38), 1-13 (2024).
6. Diethelm, K.: *The mean value theorems and a Nagumo-type uniqueness theorem for Caputo's fractional calculus*, Fract. Calc. Appl. Anal., **15**, 304-313 (2012).
7. Din, A., Khan, F.M., Khan, Z.U., Yusuf, A., Munir, T.: *The mathematical study of climate change model under nonlocal fractional derivative*, Partial Differ. Equ. Appl. Math., **5**, 100204 (2022).
8. Elzaki, M.T., Alamri, B.A.S.: *Projected differential transform method and Elzaki transform for solving system of nonlinear partial differential equations*, World Appl. Sci. J., **32**(9), 1974-1979 (2014).

9. El-Shahed, M., Nieto, J.J., Ahmed, A.: *Fractional-order model for biocontrol of the lesser date moth in palm trees and its discretization*, Adv. Difference Equ., **2017**(295), (2017).
10. Erturk, V.S., Godwe, E., Baleanu, D., Kumar, P., Asad, J., Jajarmi, A.: *Novel fractional-order Lagrangian to describe motion of beam on nanowire*, Acta Phys. Pol. A, **140**(3), 265-272 (2021).
11. Jarad, F., Abdeljawad, T., Baleanu, D.: *On the generalized fractional derivatives and their Caputo modification*, J. Nonlinear Sci. Appl., **10**(5), 2607-2619 (2017).
12. Granas, A., Dugundji, J.: Fixed Point Theory, *Springer-Verlag, New York*, 2003.
13. Katugampola, U.N.: *New approach to generalized fractional integral*, Appl. Math. Comput., **218**, 860–865 (2011).
14. Keskin, Y., Oturanç, G.: *Reduced Differential Transform Method for Partial Differential Equations*, Int. J. Nonlinear Sci. Numer. Simul., **10**(6), 741–749 (2009).
15. Khalouta, A.: *Conformable fractional Khalouta transform and its applications to fractional differential equations*, Nonlinear Dyn. Syst. Theory, **24**(5), 495–504 (2024).
16. Khalouta, A.: *A new decomposition transform method for solving nonlinear fractional logistic differential equation*, J. Supercomput, **80**, 8179–8201 (2024),
17. Khalouta, A.: *A new identification of Lagrange multipliers to study solutions of nonlinear Caputo–Fabrizio fractional problems,* Partial Differ. Equ. Appl. Math., **10**, 100711 (2024).
18. Khalouta, A.: *New approaches for solving Caputo time-fractional nonlinear system of equations describing the unsteady flow of a polytropic gas,* Int. J. Nonlinear Anal. Appl., **14**(3), 33-46 (2023).
19. Martynyuk, V., Ortigueira, M.: *Fractional model of an electrochemical capacitor*, Signal Processing, **107**, 355-360 (2015).
20. Olayiwola, M.O.: *Variational iteration method: a computational tool for solving coupled system of nonlinear partial differential equations*, Journal of Science and Arts, **3**(36), 243–248 (2016).
21. Oldham, K.B.: *Fractional differential equations in electrochemistry*, Adv. Eng. Softw., **41**(1), 9-12 (2010).
22. Sadaf, M., Perveen, Z., Akram, Gh., Habiba, U., Abbas, M., Emadifar, H.: *Solution of time-fractional gas dynamics equation using Elzaki decomposition method with Caputo-Fabrizio fractional derivative*, PLoS ONE, **19**(5), e0300436 (2024).
23. Salah, E., Qazza, A., Saadeh, R., El-Ajou, A.: *A hybrid analytical technique for solving multi-dimensional time-fractional Navier-Stokes system*, AIMS Mathematics, **8**(1), 1713-1736 (2022).
24. Yang, Q., Chen, D., Zhao, T., Chen, Y.: *Fractional calculus in image processing: A review*, Fract. Calc. Appl. Anal., **19**(5), 1222-1249 (2016).
25. Niyozov, I.E., Juraev, D.A., Efendiev, R.F., Jalalov, M.J.: *The Cauchy problem for the system of elasticity* Journal of Contemporary Applied Mathematics **14** (2), 92-107 (2024).
26. Niyozov, I.E., Juraev, D.A., Efendiev, R.F., Abdalla, M.. *Cauchy problem for the biharmonic equation*, J.Umm Al-Qura Univ. Appll. Sci., 1-12 (2025).
27. Efendiev, R.F., Annaghili, Sh. *Inverse Spectral Problem of Discontinuous Non-self-adjoint Operator Pencil with Almost Periodic Potentials*, Azerb. J. Math. **13** (1), (2023).