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On the Blaschke matrix product and an analogue of the Horwitz-Rubel theorem for the Blaschke matrix product

Jonibek Sh. Abdullayev ⋆ · **Gulmirza Kh. Khudayberganov**

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Abstract. *This article presents the properties of the Blaschke product in the space* $\mathbb{C}[m \times m]$*, which is a matrix model of the multidimensional space* $\mathbb{C}^{m^2}$*. We study the properties of the Blaschke product in the matrix unit circle and in the matrix upper half-plane. A matrix analogue of the Horwitz-Rubel theorem about the Blaschke product in the complex plane is proved.*

Keywords. Blaschke product, matrix unit circle, matrix upper half-plane, Horwitz-Rubel's theorem.

Mathematics Subject Classification (2010): 32A10, 32M15, 32N15

1 Introduction

The Blaschke product plays an important role in many problems of classical complex analysis. Related with the important applications of the Blaschke product in complex analysis, since the second half of the last century, interest in the study of functions of the Blaschke product type has increased and several of its analogues have been obtained (see, for example, [4], [5], [9], [11], [23]).

We recall the Blaschke product in $\mathbb{C}$. Let $\mathbb{U} = \{t \in \mathbb{C} : |t| < 1\}$ be unit circle. The finite Blaschke product is a function of the form

$$B(t) = e^{i\varphi} \prod_{j=1}^{n} \frac{t - t_j}{1 - \bar{t}_j t}, \ |t_j| < 1.$$

The number n of its zeros is called the degree of the Blaschke product. The Blaschke product of degree 0 is a constant, which module is equal to one.

⋆ Corresponding author

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J.Sh. Abdullayev
Urgench State University, Hamid Olimjon street, 14, Urgench, 220100, Uzbekistan
E-mail: j.abdullayev@nuu.uz

G. Khudayberganov
National University of Uzbekistan, University St. 4, 100174, Tashkent, 100174, Uzbekistan
E-mail: gkhudaiberg@mail.ru

The Blaschke product has the following properties (see, for example, [12,21]):
(i) B is continuous up to $\partial\mathbb{U}$;
(ii) $|B| = 1$ to $\partial\mathbb{U}$;
(iii) B has a finite number of zeros in the circle $\mathbb{U}$.
These three properties determine $B(t)$ up to a factor of $e^{i\varphi}$. If a holomorphic function f satisfies (i),(ii) and (iii), and B is a finite Blaschke product with the same zeros, then $|f/B| \leq 1$ and $|B/f| \leq 1$ to $\mathbb{U}$ by the maximum principle, and f/B is constant.
The following theorem holds (see [12]) :

Theorem 1.1 *Let $\{t_n\}$ be a sequence of points on the circle $\mathbb{U}$, such that*

$$\sum_{n=1}^{\infty}(1 - |t_n|) < \infty$$

and m is the number of t_n, which equal to 0. Then the Blaschke product

$$B(t) = t^m \prod_{|t_n|\neq 0} \frac{-t_n}{|t_n|}\frac{t - t_n}{1 - t_n t} \tag{1.1}$$

converges in $\mathbb{U}$. The function $B(t)$ belongs to $H^\infty(\mathbb{U})$[1], and its zeros are exactly $\{t_n\}$, and each zero has multiplicity equal to the number of its occurrence in the sequence $\{t_n\}$. Moreover, $|B(t)| \leq 1$ and

$$\left|B\left(e^{i\theta}\right)\right| = 1$$

almost everywhere.

The following Horwitz-Rubel's theorem expresses one of the most important properties of the Blaschke product (see [4]).

Theorem 1.2 *Let*

$$A_1(t) = \prod_{j=1}^{k} \frac{t - a_j}{1 - \bar{a}_j t},$$

$$A_2(t) = \prod_{j=1}^{k} \frac{t - b_j}{1 - \bar{b}_j t}$$

be the Blaschke products of degree k, where $a_j, b_j \in \mathbb{U} = \{t \in \mathbb{C} : |t| < 1\}$. If $A_1(\lambda_j) = A_2(\lambda_j)$ for k distinct points $\lambda_1, \lambda_2, \ldots, \lambda_k$ from $\mathbb{U}$, then

$$A_1(t) \equiv A_2(t).$$

Note that the specified (canonical) form for $A_1(\lambda_j)$ and $A_2(\lambda_j)$ is essential.
Indeed, let $|c| = 1, c \neq 1$ and

$$A(t) = c\prod_{j=1}^{k} \frac{t - a_j}{1 - \bar{a}_j t},$$

$$B(t) = \frac{t - ca_1}{1 - \overline{ca}_1 t}\prod_{j=1}^{k} \frac{t - a_j}{1 - \bar{a}_j t},$$

[1] The Hardy class $H^\infty(\mathbb{U})$ is the set of all holomorphic and bounded functions in $\mathbb{U}$.

where $a_j \neq 0$ for any j. Then $A\left(a_j\right) = B\left(a_j\right) = 0$ for $j = 2, 3, \dots, k$ and

$$A\left(0\right) = B\left(0\right) = c\prod_{j=1}^{k}(-1)^j a_j,$$

but $A\left(t\right) \not\equiv B\left(t\right)$. Hence, $A \not\equiv kB$ for any constant k. Theorem 1.2 is essentially based on the possibility of representing rational functions $A_1\left(t\right)$ and $A_2\left(t\right)$ in the form:

$$A_1\left(t\right) = \frac{\alpha_0 + \alpha_1 t + \ldots + \alpha_{k-1}t^{k-1} + t^k}{1 + \bar{\alpha}_{k-1}t + \ldots + \bar{\alpha}_1 t^{k-1} + \bar{\alpha}_0 t^k}, \tag{1.2}$$

$$A_2\left(t\right) = \frac{\beta_0 + \beta_1 t + \ldots + \beta_{k-1}t^{k-1} + t^k}{1 + \bar{\beta}_{k-1}t + \ldots + \bar{\beta}_1 t^{k-1} + \bar{\beta}_0 t^k}, \tag{1.3}$$

where the coefficients $\alpha_0, \alpha_1, \dots, \alpha_{k-1}$ and $\beta_0, \beta_1, \dots, \beta_{k-1}$ are expressed by a_j and b_j, $j = 1, 2, \dots, k$, as follows:

$$\alpha_0 = (-1)^k a_1 \cdot \ldots \cdot a_k,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\alpha_{k-2} = a_1 a_2 + a_1 a_3 + \ldots + a_{k-1}a_k,$$
$$\alpha_{k-1} = -\left(a_1 + \ldots + a_k\right),$$

and

$$\beta_0 = (-1)^k b_1 \cdot \ldots \cdot b_k,$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$\beta_{k-2} = b_1 b_2 + b_1 b_3 + \ldots + b_{k-1}b_k,$$
$$\beta_{k-1} = -\left(b_1 + \ldots + b_k\right).$$

Therefore the functions $A_1\left(t\right), A_2\left(t\right)$ are defined uniquely by k different parameters α_0,α_1, $\dots, \alpha_{k-1}$; β_0, β_1,$\dots, \beta_{k-1}$. Further, we introduce the Blaschke product in matrix domains, and then we obtain some of its properties and give analogues of the above Theorem 1.2.

2 Preliminaries and problem statement

Many fundamental theorems of classical complex analysis of one variable still do not have full-fledged generalizations in multidimensional complex analysis. Many formulas in use today are not complete, although each new obtained formula is stronger and more convenient than the previous one. In the 30s and 60s of the twentieth century such scientists as E. Cartan [8], I.I. Pyatetsky-Shapiro [24] and Hua Lo-Keng [16] investigated the problems of multivariate complex analysis using a matrix approach. They conducted research mainly in the classical domains[2] and dealt with questions related to the theory of functions in these domains and the geometry of the domains themselves. Further, B.S. Vladimirov [27,28], A.G. Sergeev [25], [26], [28], Sh. Zhou [30], G. Khenkin [15], S.G.Gindikin [13], Xiao-Ming [29], L. A. Aizenberg [1], [2], [3], G.Khudayberganov [14], [17], [18], [19], and others continued to study the properties of holomorphic functions in matrix domains. In these scientific studies, the biholomorphic equivalence of these domains with bounded domains was widely used in constructing the theory of holomorphic extensions in unbounded domains. In the course of these investigations significant results were obtained in domains

[2] Recall that a bounded domain $D \subset \mathbb{C}^n$ is called classical if the group of its holomorphic automorphisms is a classical Lie group and is transitive on it.

of the space $\mathbb{C}^n$ and in matrix domains. Despite this, many unresolved problems remain in these areas, the solution of which is very important.

Great interest in complex analysis on matrix domains, in recent years, is associated with applications in mathematical physics, the theory of electrical circuits and in solving practical problems using them, including in solving problems of quantum field theory and others, interest in this direction has increased even more complex analysis. About these in the works of V.S. Vladimirov [27,28], A.G.Sergeyeev [25], [26], A.V. Efimova and V.P. Potapova (see [10]), F. Barbaresco [6], E.Bedford and Y.Dadok [7].

The selection of classes of biholomorphically equivalent domains is of great importance in multidimensional complex analysis and its applications. It is fairly well known that any two simply connected domains of the same type on the complex plane map conformally onto each other (Riemann's theorem). But in the multidimensional case, the situation is completely different. For example, two simplest domains, a ball and a polydisk from the space $\mathbb{C}^n-$ do not map biholomorphically onto each other. Therefore, in multivariable complex analysis it is very important to have a stock of domains that are biholomorphically equivalent to each other. Since the set of biholomorphic automorphisms of the matrix unit circle and the matrix upper half-plane are given by linear-fractional functions of matrices, it is natural to consider Blaschke products (for these matrix domains) as a function of matrices. Such products for the matrix circle and for the matrix upper half-plane were introduced in [20].

3 Matrix unit circle and matrix upper half-plane

Consider the space $\mathbb{C}^{m^2}$, the space of m^2 complex variables. In some questions, it is convenient to represent the point Z of this space as $Z = (z_{ij})_{i,j=1}^m$, i.e., as square $[m \times m]$-matrices. With this point representation, the space $\mathbb{C}^{m^2}$ will be denoted by $\mathbb{C}[m \times m]$. Denote by $\mathbb{C}^n[m \times m]$ the direct product of n instances of $[m \times m]$-matrix spaces

$$\underbrace{\mathbb{C}[m \times m] \times \cdots \times \mathbb{C}[m \times m]}_{n}.$$

Let $Z = (Z_1, \dots, Z_n)$ be a vector composed of square matrices Z_j of order m, considered over the field of complex numbers $\mathbb{C}$. Let us write the elements of the vector $Z = (Z_1, \dots, Z_n)$ as points z of the space $\mathbb{C}^{nm^2}$:

$$z = (z_{11}^{(1)}, \dots, z_{1m}^{(1)}, \dots, z_{m1}^{(1)}, \dots, z_{mm}^{(1)}, \dots, z_{11}^{(n)}, \dots, z_{1m}^{(n)}, \dots, z_{m1}^{(n)}, \dots, z_{mm}^{(n)}) \in \mathbb{C}^{nm^2}.$$

Hence, we can assume that Z is an element of the space $\mathbb{C}^n\,[m \times m]$, i.e., we came to the isomorphism $\mathbb{C}^n\,[m \times m] \cong \mathbb{C}^{nm^2}$.

The matrix unit circle (the classical domain of the first type according to E. Cartan's classification [8]) is defined as the set

$$\Re_I\,(m, m) = \{Z \in \mathbb{C}[m \times m] : ZZ^* < I\},$$

where $Z^* = \bar{Z}'$ is conjugate and transposed matrix to Z, notation $ZZ^* < I$ ($I = I_m$ is the identity matrix $[m \times m]$) means that the Hermitian matrix $I - ZZ^*$ is positive definite, thus, all its eigenvalues are positive. It is useful to note that, if $Z \in \mathbb{C}\,[m \times m]$, then

$$\det\,(I - ZZ^*) = \det\,(I - Z^*Z).$$

In addition, the conditions $I - ZZ^* > 0$ and $I - Z^*Z > 0$ is equivalent. This statement is also true for rectangular matrices (see Theorem 2.1.3 in [16]).

The boundary $\mathfrak{R}_I\left(m,m\right)$ consists of the set

$$\partial\mathfrak{R}_I\left(m,m\right)=\left\{Z\in\mathbb{C}[m\times m]:\det(I-ZZ^{*})=0,\;ZZ^{*}\leq I\right\},$$

i.e., from the set of matrices Z, for which the matrix $I-ZZ^{*}$ is non-negative definite but not positive definite Hermitian matrix (its eigenvalues are non-negative and at least one of them is equal to zero).

On the border lies the set

$$\Gamma_{\mathfrak{R}}=\left\{Z\in\mathbb{C}[m\times m]:ZZ^{*}=I\right\},$$

which is called the skeleton of $\mathfrak{R}_I\left(m,m\right)$ (note that, $\Gamma_{\mathfrak{R}}$ is the Shilov boundary for $\mathfrak{R}_I\left(m,m\right)$). It is clear that, the set $\Gamma_{\mathfrak{R}}$ is the set of all unitary $[m\times m]$-matrices (the set of unitary matrices of order n is usually denoted by $U\left(n\right)$). It should be noted that, the set of matrices $\left\{Z:\det\left(I-ZZ^{*}\right)=0\right\}$ contains a bounded component distinguished by the condition $ZZ^{*}\leq I$, and unbounded component, for which $ZZ^{*}\geq I$. These components intersect along the skeleton $\Gamma_{\mathfrak{R}}$ (see [16]).

Note that, the matrix unit circle is a bounded domain centered at O (O is zero matrix). Indeed, if we denote by $z^j=\left(z_1^j,z_2^j,\ldots,z_m^j\right)$ of any j-th row of the matrix Z, then the elements of the main diagonal $I-ZZ^{*}$ will be $1-\left|z^j\right|^2$, and since this matrix is positive definite, hence they are all positive. Therefore

$$|Z|^2=\sum_{j=1}^{m}\left|z^j\right|^2<m$$

and the domain $\mathfrak{R}_I\left(m,m\right)$ lies in the ball $\mathbb{B}=\{|Z|<\sqrt{m}\}$. Further, from [16] it is known that, for any $Z\in\mathbb{C}\left[m\times m\right]$ there exist unitary matrices U of order m and V of order m such that

$$Z=U\begin{pmatrix}\lambda_1&0&\cdots&0\\0&\lambda_2&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&\lambda_m\end{pmatrix}V$$

for some $\lambda_1\geq\lambda_2\geq\ldots\geq\lambda_m\geq0$. Hence it follows that

$$\det\left(I^{(m)}-Z\bar{Z}'\right)=\left(1-\lambda_1^2\right)\ldots\left(1-\lambda_m^2\right)=\det\left(I^{(m)}-\bar{Z}'Z\right).$$

Therefore, for a given $Z\in\mathbb{C}\left[m\times m\right]$ the relation $I^{(m)}-Z\bar{Z}'>0$ holds if and only if $1-\lambda_s^2>0$ or $\lambda_s<1$, $s=1,\ldots,m$. On the other hand, for any $Z\in\mathbb{B}^{m^2}\left(1\right)$, where

$$\mathbb{B}^{m^2}\left(1\right)=\left\{Z=\left(z_{11},\ldots,z_{1m},z_{21}\ldots,z_{2m},\ldots,z_{m1}\ldots,z_{mm}\right)\in\mathbb{C}^{m^2}:\sum_{s=1}^{m}\sum_{j=1}^{m}\left|z_{sj}\right|^2<1\right\}$$

is the ball from space $\mathbb{C}^{m^2}$, we have

$$|Z|^2=\sum_{s=1}^{m}\sum_{j=1}^{m}|z_{sj}|^2=\sum_{s=1}^{m}\sum_{j=1}^{m}z_{sj}\bar{z}_{sj}=Sp\left(Z\bar{Z}'\right)=\sum_{s=1}^{m}\lambda_s^2<1.$$

Whence $\lambda_s<1\left(s=1,\ldots,m\right)$ and $Z\in\mathfrak{R}_I\left(m,m\right)$. Therefore, we have the following relation

$$\mathbb{B}^{m^2}\left(1\right)\subset\mathfrak{R}_I\left(m,m\right).$$

For $m = 1$ the domain $\mathfrak{R}_I\,(m, m)$ coincides with the unit circle, and the $\Gamma_{\mathfrak{R}}$ is the unit circle in the complex plane $\mathbb{C}$.

For $m = 2$ the domian τ admits the representation

$$\tau = \{Z \in \mathbb{C}[2 \times 2] : \ \psi(Z) < 0\},$$

where

$$\psi(Z) = \max\left[|z_{11}|^2 + |z_{12}|^2 - 1, \ |z_{21}|^2 + |z_{22}|^2 - 1, \ \psi_0\,(Z)\right],$$

$$\psi_0(Z) = \det ZZ^* + SpZZ^* - 1,$$

and $Sp\,(Z)$ is the trace (hole) of the matrix Z (this representation can be obtained from Sylvestr's criterion for a positive definite matrix).

The matrix upper half-plane is defined as the set of matrices

$$\mathfrak{I} = \{Z \in \mathbb{C}\,[m \times m] : \operatorname{Im} Z > 0\},$$

where $\operatorname{Im} Z = \frac{1}{2i}\,(Z - Z^*)$. It is the usual upper half-plane when $m = 1$.

The boundary $\partial\mathfrak{I}$ of this domain consists of matrices Z, for which $\operatorname{Im} Z$ is a non-negative definite but non-positive definite Hermitian matrix (its eigenvalues are non-negative and at least one of them is equal to zero). Since the vanishing of the eigenvalues of a Hermitian matrix is expressed by a real analytic equality, then $\partial\mathfrak{I}$ consists of pieces of real analytic surfaces of dimension $2m^2 - 1$.

The set

$$S(\mathfrak{I}) = \{Z : \operatorname{Im} Z = 0\}\ \ ,$$

which lies on $\partial\mathfrak{I}$, is called the skeleton of the upper half-plane $\mathfrak{I}$. It consists of all Hermitian matrices. The Hermitianity condition is expressed by m^2 independent equations, so the real dimension of $S(\mathfrak{I})$ is equal to m^2.

For $m = 2$ this special case has the form

$$\mathfrak{I} = \left\{Z \in \mathbb{C}\,[2 \times 2] : \begin{pmatrix} \operatorname{Im} z_{11} & \frac{z_{12} - \bar{z}_{21}}{2i} \\ \frac{z_{21} - \bar{z}_{12}}{2i} & \operatorname{Im} z_{22} \end{pmatrix} > 0\right\}.$$

This domain is defined by the inequalities

$$\mathfrak{I} = \left\{\operatorname{Im} z_{11} > 0, \operatorname{Im} z_{11} \operatorname{Im} z_{22} - \frac{1}{4}|z_{12} - \bar{z}_{21}|^2 > 0\right\},$$

and its boundary by the equation

$$\partial\mathfrak{I} = \left\{\operatorname{Im} z_{11} \operatorname{Im} z_{22} = \frac{1}{4}|z_{12} - \bar{z}_{21}|^2\right\},$$

where the skeleton is the real four-dimensional plane

$$S(\mathfrak{I}) = \{\operatorname{Im} z_{11} = \operatorname{Im} z_{22} = 0, z_{12} = \bar{z}_{21}\}.$$

Note that, the non-degenerate affine transformation

$$\Phi\,(Z) = \begin{pmatrix} z_{11} + z_{22} & z_{12} + z_{21} \\ i\,(z_{21} - z_{12}) & z_{11} - z_{22} \end{pmatrix}$$

maps $\mathfrak{I}$ to the domain defined by the inequalities

$$\tau(2) = \left\{\operatorname{Im} z_{11} > 0, (\operatorname{Im} z_{11})^2 > (\operatorname{Im} z_{12})^2 + (\operatorname{Im} z_{21})^2 + (\operatorname{Im} z_{22})^2\right\},$$

which called the pipe of the future (see [28]), i.e. into the tubular domain $T = \mathbb{R}^4(x) + iC$ over the cone

$$C = \left\{y_{11}^2 - y_{22}^2 - y_{12}^2 - y_{21}^2 > 0\right\},$$

more precisely, over one cavity C_+ of this cone for which $y_{11} > 0$ (we put $z_{jk} = x_{jk} + iy_{jk}$). Then the boundary $\partial\mathfrak{I}$ maps $\partial C_+ \times \mathbb{R}^4(x)$, and the skeleton becomes into the real subspace $\mathbb{R}^4(x)$, more precisely, into the product of the vertex of the cone C_+ by it. These implementations are given in more detailed in [27], [28].

4 Blaschke matrix product

In this section, using the Blaschke product for the matrix unit circle and the matrix upper half-plane, some results are obtained for one variable.

A finite Blaschke matrix product of degree k in $\mathfrak{R}_I(m,m)$ we will call a function of matrices of the form (see [20])

$$B(Z) = \prod_{j=1}^{k} (Z - P_j)\left(I - P_j^* Z\right)^{-1},$$

where $P_j \in \mathfrak{R}_I(m,m)$, $j = 1, 2, ..., k$.

Theorem 4.1 *The Blaschke product for the matrix unit circle has the following properties:*
(i) The function $B(Z)$ is continuous up to the boundary $\mathfrak{R}_I(m,m)$;
(ii) $B(Z)B^*(Z) = I$ on $\Gamma_{\mathfrak{R}}$;
(iii) $B(P_i) = \mathrm{O}, \mathrm{O} \in \mathfrak{R}_I(m,m), j = 1, 2, \ldots, k$.

Proof. i) follows from the fact that $\det\left(I - P_j^* Z\right) \neq 0$. $P_j \in \mathfrak{R}_I(m,m)$ by condition. Since the conditions $I - ZZ^* > 0$ and $I - Z^*Z > 0$ are equivalent we have $P_j^* \in \mathfrak{R}_I(m,m)$. From the relation

$$I - \left(P_j^* Z\right)\left(P_j^* Z\right)^* = I - P_j^* Z Z^* P_j = \left(I - P_j^* P_j\right) + P_j^*\left(I - ZZ^*\right)P_j > 0$$

follows that $\left(P_j^* Z\right) \in \mathfrak{R}_I(m,m)$. Therefore, all eigenvalues of the matrix $\left(P_j^* Z\right) \in \mathfrak{R}_I(m,m)$ lie inside the unit circle. And so $\det\left(I - P_j^* Z\right) \neq 0$.

ii) According to [16], the automorphism $\mathfrak{R}_I(m,m)$, which maps an arbitrary point $P \in \mathfrak{R}_I(m,m)$ to the origin has the form

$$W = A(Z - P)(I - P^*Z)^{-1}B^{-1}, \tag{4.1}$$

where A, B are $[m \times m]$ -square matrices such that

$$\bar{A}\left(I - \bar{P}P'\right)A' = I, \bar{B}\left(I - P'\bar{P}\right)B' = I.$$

The inverse mapping to (4.1) has the form

$$Z = \left(I - A^{-1}WBP^*\right)^{-1}\left(A^{-1}WB + P\right). \tag{4.2}$$

Since for $P = \mathrm{O}$, it follows from (4.2) that $Z = A^{-1}WB$, then the automorphism (4.1) can be represented as

$$\zeta = (Z - P)(I - P^*Z)^{-1}. \tag{4.3}$$

It is cear that, the unitary matrix $Z \in \Gamma_R$ we can take as $Z = VU^*$ and the matrix $P \in \mathfrak{R}_I(m,m)$ as $P = VSU^*$ Here V, U are unitary matrices.

Indeed, let $P \in \mathfrak{R}_I(m,m)$ be, then for all unitary matrices U, V matrix S we have

$$I - (VSU^*)(VSU^*)^* = I - VSU^*US^*V^* = V(I - SS^*)V^* > 0\,.$$

Therefore, this means that $P = VSU^* \in \mathfrak{R}_I(m,m)$. This is one of those looks we have a relationship:

$$P_j^* P_j = Z^* P_j P_j^* Z. \tag{4.4}$$

To prove the equality

$$B(Z)\,B^*(Z) = I$$

we will show that, the equality $B^*(Z)\,B(Z) = I$ holds, which is equivalent to mentioned equality above. Let's look at the difference between the right and left sides of this last expression, by virtue of (4.3):

$$B^*(Z)\,B(Z) - I = \left[(Z - P_j)\left(I - P_j^* Z\right)^{-1}\right]^* \left[(Z - P_j)\left(I - P_j^* Z\right)^{-1}\right] - I =$$

$$= \left[\left(I - P_j^* Z\right)^{-1}\right]^* [(Z - P_j)]^* \left[(Z - P_j)\left(I - P_j^* Z\right)^{-1}\right] - I =$$

$$= \left[\left(I - P_j^* Z\right)^*\right]^{-1} \left(Z^* - P_j^*\right)(Z - P_j)\left(I - P_j^* Z\right)^{-1} - I =$$

$$= (I - Z^* P_j)^{-1}\left(Z^* - P_j^*\right)(Z - P_j)\left(I - P_j^* Z\right)^{-1} - I =$$

$$= (I - Z^* P_j)^{-1}\left(Z^* - P_j^*\right)(Z - P_j)\left(I - P_j^* Z\right)^{-1} - I =$$

$$= (I - Z^* P_j)^{-1}\left[\left(Z^* - P_j^*\right)(Z - P_j) - (I - Z^* P_j)\left(I - P_j^* Z\right)\right]\left(I - P_j^* Z\right)^{-1} =$$

$$= (I - Z^* P_j)^{-1}\left[P_j^* P_j - Z^* P_j P_j^* Z\right]\left(I - P_j^* Z\right)^{-1}$$

The second multiplier in the last multiplication is zero according to (4.4) multiplication. Hence, $B(Z)\,B^*(Z) = I$ is true on $\Gamma_{\mathfrak{R}}$.

iii) Since the function $B(Z)$ vanishes at the points $Z = P_j$ we get reality of the statement iii.

5 The Blaschke product for the matrix upper half-plane

Let $\mathfrak{I}_\sigma = \{Z \in \mathbb{C}[m \times m] : \operatorname{Im} Z > -\sigma I\}$ be the matrix half-plane of square $[m \times m]$ matrix in the space $\mathbb{C}[m \times m]$, where $\sigma > 0$.

The next theorem is true.

Theorem 5.1 *Transformation*

$$W = (Z - A^* + 2i\sigma I)^{-1}(Z - A)\,, \tag{5.1}$$

where $A \in \mathfrak{I}_\sigma, A - A^* = cI, c \in \mathbb{C}$, *biholomorphically maps* $\mathfrak{I}_\sigma$ *to the matrix unit circle*

$$\mathfrak{R}_I(m,m) = \{W \in \mathbb{C}[m \times m] : WW^* < I\}\,.$$

Proof. First of all, let us show that the next matrix is invertible

$$\left(Z - A^* + 2i\sigma I\right).$$

Let ρ be column vector of length m and

$$\left(Z - A^* + 2i\sigma I\right)\rho = 0.$$

Hence

$$\left(Z + i\sigma I\right)\rho = \left(A^* - i\sigma I\right)\rho,$$
$$\rho^*\left(Z^* - i\sigma I\right) = \rho^*\left(A + i\sigma I\right).$$

Multiplying both sides of the first equality on the left by ρ^*, the second equality on the right by ρ and subtracting the second equation from the first, we get

$$\rho^*\left(Z - Z^* + 2i\sigma I\right)\rho = -\rho^*\left(A - A^* + 2i\sigma I\right)\rho,$$

Consequently,

$$\rho^*\left(\operatorname{Im} Z + \sigma I\right)\rho = -\rho^*\left(\operatorname{Im} A + \sigma I\right)\rho.$$

Since

$$\operatorname{Im} Z + \sigma I > 0,$$
$$\operatorname{Im} A + \sigma I > 0,$$

then $\rho = 0$. This means that the matrix

$$\left(Z - A^* + 2i\sigma I\right)$$

is nondegenerate, i.e. it is invertible and therefore the map (5.1) is holomorphic in $\mathfrak{I}_\sigma$. Now we need to show that (5.1) maps $\mathfrak{I}_\sigma$ to $\mathfrak{R}_I\left(m, m\right)$. This follows from the following relationship:

$$I - WW^* = I - \left(Z - A^* + 2i\sigma I\right)^{-1}\left(Z - A\right)\left(Z^* - A^*\right)\left(Z^* - A - 2i\sigma I\right)^{-1}$$
$$= \left(Z - A^* + 2i\sigma I\right)^{-1}\left[\left(Z - A^* + 2i\sigma I\right)\left(Z^* - A - 2i\sigma I\right) - \left(Z - A\right)\left(Z^* - A^*\right)\right]$$
$$\times\left(\left(Z - A^* + 2i\sigma I\right)^{-1}\right)^*$$
$$= G[ZZ^* - A^*Z^* + 2i\sigma Z^* - ZA + A^*A - 2i\sigma AI - 2i\sigma Z + 2iA^*\sigma I + 4\sigma^2 I$$
$$-ZZ^* + AZ^* + ZA^* - AA^*]G^* = G[2i\left(\frac{A - A^*}{2i} + \sigma\right)Z^* - Z\left(\frac{A - A^*}{2i} + \sigma\right)2i$$
$$-2i\left(\frac{A - A^*}{2i} + \sigma\right)2i\sigma]G^* = G[2i\left(\frac{A - A^*}{2i} + \sigma\right)\times\left(\frac{Z - Z^*}{2i} + \sigma\right)(-2i)]G^*$$
$$= 4G\left(\frac{A - A^*}{2i} + \sigma I\right)\left(\frac{Z - Z^*}{2i} + \sigma I\right)G^* = 4G\left(\operatorname{Im} A + \sigma I\right)\left(\operatorname{Im} Z + \sigma I\right)G^*,$$

where $G = \left(Z - A^* + 2i\sigma I\right)^{-1}$.

It is known that, the Hermitian matrices B and ABA^* are both positive definite if A is nondegenerate. Since $A - A^* = cI, c \in \mathbb{C}$, then

$$B = \left(\operatorname{Im} A + \sigma I\right)\left(\operatorname{Im} Z + \sigma I\right) = \left[\left(\operatorname{Im} A + \sigma I\right)\left(\operatorname{Im} Z + \sigma I\right)\right]^* = B^*,$$

i.e. B is Hermitian matrix. Therefore, the matrices $I - WW^*$ and

$$B = \left(\operatorname{Im} A + \sigma I\right)\left(\operatorname{Im} Z + \sigma I\right) = \theta\left(\operatorname{Im} Z + \sigma I\right),$$

are simultaneously positive definite where θ is a positive number. This says that (5.1) maps $\mathfrak{I}_\sigma$ to $\mathfrak{R}_I(m,m)$.

Now from (5.1) we will find the inverse mapping

$$Z = [A - (A^* - 2i\sigma I)W](I - W)^{-1}. \tag{5.2}$$

Since for $W \in \mathfrak{R}_I(m,m)$ all its eigenvalues are less than one, then $\det(I - W) \neq 0$, i.e. the matrix $(I - W)$ is nondegenerate. This implies that the map (5.2) is holomorphic in $\mathfrak{R}_I(m,m)$. Due to the relation

$$I - WW^* = 4(Z - A^* + 2i\sigma I)^{-1}(\operatorname{Im} A + \sigma I) \times$$
$$\times (\operatorname{Im} Z + \sigma I)((Z - A^* + 2i\sigma I)^{-1})^*,$$

it can be seen that for $W \in \mathfrak{R}_I(m,m)$, i.e. $I - WW^* > 0$, the matrix $G = (Z - A^* + 2i\sigma I)^{-1}$, is non degenerate also and hence, $\operatorname{Im} Z + \sigma I > 0$, i.e. (5.1) maps $\mathfrak{R}_I(m,m)$ to $\mathfrak{I}_\sigma$. The statement is proved.

In a special case, when $A = iI$ and $\sigma = 0$, the transformation (5.1) is the well-known Cayley transformation. In addition, (5.1) transforms the skeleton

$$\Gamma_\sigma = \{Z \in \mathbb{C}[m \times m] : \operatorname{Im} Z = -\sigma I\}$$

of the matrix half-plane $\mathfrak{I}_\sigma$ the set

$$\Gamma_{\mathfrak{R}} = \{W : WW^* = I\}$$

the skeleton (Shilov boundary) $\mathfrak{R}_I(m,m)$, which consists of unitary $[m \times m]$ matrices.

By a finite Blaschke matrix product of degree k in $\mathfrak{I}_\sigma$ we say the function of matrices (mapping $\mathbb{C}[m \times m] \to \mathbb{C}[m \times m]$ of the form (5.1))

$$B_k(Z) = \prod_{j=1}^{k} \left[\left(Z - A_j^* + 2i\sigma I\right)^{-1}(Z - A_j) \right]^{q_j}, \tag{5.3}$$

where $A_j \in H_\sigma, A_j - A_j^* = c_j I, c_j \in \mathbb{C}, q_j$ are positive integers, $j = 1, 2, ..., k$.

Based on Theorem 5.1 for the Blaschke product of the form (5.3)*, we have the following*

Corollary 5.1 *The Blaschke product of the form* (5.3) *has the following properties:*
i. $B_k(Z)$ *is continuous up to the boundary* $\mathfrak{I}_\sigma$*;*
ii. $B_k(Z)B_j^*(Z) = I$ *on* Γ_σ;
iii. $B_k(A_j) = \mathrm{O}, \mathrm{O} \in \mathfrak{R}_I(m,m), j = 1, 2, \ldots, k;$
iv. Transformation $B(Z)$ *is a map of domain* $\mathfrak{I}_\sigma$ *to* $\overline{\mathfrak{R}_I}(m,m)$.

Proof. The first assertion i) follows from the invertibility of the matrices

$$\left(Z - A_j^* + 2i\sigma I\right)$$

for all $Z, A_j^* \in \mathfrak{I}_\sigma$.

Properties ii) is a consequence of the fact that the mapping

$$W = \left(Z - A_j^* + 2i\sigma I\right)^{-1}(Z - A_j), A_j \in \mathfrak{I}_\sigma,$$

maps $\mathfrak{I}_\sigma$ to $\mathfrak{R}_I(m,m)$. In this case, the skeleton $\mathfrak{I}_\sigma$ maps to the skeleton $\Gamma_{\mathfrak{R}}$.

iii) It is clear that the function $B_k(Z)$ vanishes at the points $Z = A_j$, which makes the third statement true.

iv) Note that, $Z, W \in \mathfrak{R}_I(m,m)$ implies that $(ZW) \in \mathfrak{R}_I(m,m)$. The validity of this assertion follows from the statement above and from the Theorem 5.1.

6 The Horwitz-Rubel theorem for the Blaschke matrix product

In this subsection, we prove an analogue of the Horwitz-Rubel theorem for the Blaschke matrix product. It should be noted that in this case, Blaschke matrix products cannot be represented in the form (1.2), (1.3), due to the noncommutativity of matrices.

Let

$$B_1(Z) = \prod_{j=1}^{k} \left[(Z - P_j) \left(I - P_j^* Z \right)^{-1} \right],$$

$$B_2(Z) = \prod_{j=1}^{k} \left[(Z - Q_j) \left(I - Q_j^* Z \right)^{-1} \right],$$

where $P_j, Q_j \in \mathfrak{R}_I(m,m), j = 1, 2, \ldots, k$, Blaschke matrix products of degree k for the matrix unit disc.

The following theorem is true.

Theorem 6.1 *Let be P_j, Q_j be diagonal matrices. If $B_1(\nu_j) = B_2(\nu_j)$ for diagonal matrices ν_j from $\mathfrak{R}_I(m,m)$, such that* $\det(\nu_j) \neq 0,\ j = 1, \ldots, 2$, *then*

$$B_1(Z) \equiv B_2(Z).$$

To prove Theorem 6.1, we first prove the following.

Lemma 6.1 *If P_j, Q_j are diagonal matrices, then for B_1 and B_2 there exists at least one matrix $U \in \Gamma_{\mathfrak{R}}$, such that*

$$B_1(U) = B_2(U).$$

Proof. The equality $B_1(Z) = B_2(Z)$ is equivalent to the equality

$$\prod_{j=1}^{k} \left[(Z - P_j) \left(I - P_j^* Z \right)^{-1} \right] = \prod_{j=1}^{k} \left[(Z - Q_j) \left(I - Q_j^* Z \right)^{-1} \right]. \tag{6.1}$$

For $ZZ^* = I$, i.e. $Z \in \Gamma_{\mathfrak{R}}$, the matrices $Z - P_j$ and $Z - Q_j$ are nondegenerate, i.e. $\det(Z - P_j) \neq 0, \det(Z - Q_j) \neq 0, j = 1, 2, \ldots, k$, so from (6.1) we get

$$\prod_{j=1}^{k} \left[(Z - P_j) \left(I - P_j^* Z \right)^{-1} \right] \left[\prod_{j=1}^{k} \left[(Z - Q_j) \left(I - Q_j^* Z \right)^{-1} \right] \right]^{-1} = I. \tag{6.2}$$

Consider a matrix function of a scalar variable $t \in \mathbb{C}$ of the form

$$\varphi(t) = \prod_{j=1}^{k} \left[(tI - P_j) \left(I - P_j^* t \right)^{-1} \right] \left[\prod_{j=1}^{k} \left[(tI - Q_j) \left(I - Q_j^* t \right)^{-1} \right] \right]^{-1}.$$

Note that, the elements of $\varphi(t)$ are holomorphic in the disc $\{|t| < 1\} = \mathbb{U}$ and continuous functions on $\overline{\mathbb{U}}$.

If $t\bar{t} = 1$ then, due to the diagonality of P_j, Q_j, the function $\varphi(t)$ can be presented in the form

$$\varphi(t) = \prod_{j=1}^{k} \left[(tI - P_j)(tI - Q_j)^{-1} \right] \left[\overline{(tI - P_j)(tI - Q_j)^{-1}} \right]^{-1}.$$

Then the elements of $\varphi(t)$ are functions of the form

$$\varphi_{ss}(t) = \prod_{j=1}^{k} \left[(tI - P_j)(tI - Q_j)^{-1} \right]_{ss} \left[\overline{(tI - P_j)(tI - Q_j)^{-1}} \right]_{ss}^{-1},$$

$s = 1, 2, ..., m$. Assuming that

$$P_j = \left(p_j^{ss}\right), Q_j = \left(q_j^{ss}\right), i = 1, 2, \ldots, m, j = 1, 2, \ldots, k,$$

we can rewrite the function $\varphi_{ss}(t)$ in the form

$$\varphi_{ss}(t) = \prod_{j=1}^{k} \left[\frac{t - p_j^{ss}}{t - q_j^{ss}} / \overline{\left(\frac{t - p_j^{ss}}{t - q_j^{ss}} \right)}^{-1} \right] = \frac{g_{ss}(t)}{\overline{g_{ss}(t)}},$$

where

$$g_{ss}(t) = \prod_{j=1}^{k} \frac{t - p_j^{ss}}{t - q_j^{ss}}.$$

If we represent $g_{ss}(t)$ in exponential form, i.e.

$$g_{ss}(t) = |g_{ss}(t)| e^{i \arg g_{ss}(t)},$$

then $\varphi_{ss}(t) = e^{2i \arg g_{ss}(t)}$ (here we assume that $\arg g_{ss}(t)$ exists). We are interested in the values $t_s, |t_s| = 1$, such that $\varphi_{ss}(t_s) = 1$ for each fixed s, i.e.

$$\cos(2 \arg g_{ss}(t_s)) + i \sin(2 \arg g_{ss}(t_s)) = 1.$$

It follows

$$\cos(2 \arg g_{ss}(t_s)) = 1,$$

$$\sin(2 \arg g_{ss}(t_s)) = 0.$$

Hence, $\arg g_{ss}(t_i) = \pi n$ for some n. Thus,

$$\varphi_{ss}\left(e^{i\psi_s}\right) = 1,$$

where

$$e^{i\psi_s} = t_s, 0 \leq \psi_s \leq 2\pi.$$

Let us now show that for every fixed s, $s = 1, 2, \ldots, m$, there exists $\arg g_{ss}(t)$. Denote

$$\delta = \min_{\substack{t \in \left\{p_j^{ss}, q_j^{ss}\right\} \\ \zeta \in \partial \mathbb{U}}} |t - \zeta| > 0.$$

Then the functions $g_{ss}(t)$ are holomorphic and do not vanish in

$$\mathbb{U}_\delta = \{t \in \mathbb{C} : |t| > 1 - \delta\},$$

since $P_j, Q_j \in \mathfrak{R}_I(m, m)$, thats why $p_j^{ss}, q_j^{ss} \in \mathbb{U} = \{|t| < 1\}$. Consider the analytic function

$$\ln g_{ss}(t) = h_{ss}(t),$$

which admits a holomorphic branch in $\mathbb{U}_\delta$, corresponding to the function $g_{ss}(t)$ on ∞. Since $g_{ss}(\infty)=1$, then we choose a branch with the condition $h_{ss}(\infty)=0$. Let $T(t)=\frac{1}{t}$. Then

$$T(\mathbb{U}_\delta)=\left\{t\in\mathbb{C}:|t|<\frac{1}{1-\delta}\right\}\supset\mathbb{U}, T(\partial\mathbb{U})=\partial\mathbb{U}$$

and

$$T(\infty)=0.$$

It's clear that

$$h_{ss}\left(\frac{1}{t}\right)\in\mathcal{O}\left(\left\{t\in\mathbb{C}:|t|<\frac{1}{1-\delta}\right\}\right).$$

Therefore, by virtue of the mean value theorem for harmonic functions (we apply this theorem in $\mathbb{U}$ for harmonic function $\operatorname{Im}h_{ss}\left(\frac{1}{t}\right)$) we get

$$\operatorname{Im}h_{ss}(\infty)=\frac{1}{2\pi}\int\limits_0^{2\pi}\operatorname{Im}h_{ss}\left(e^{i\psi}\right)d\psi.$$

From here

$$0=\frac{1}{2\pi}\int\limits_0^{2\pi}\operatorname{Im}h_{ss}\left(e^{i\psi}\right)d\psi.$$

Therefore, there exists a value ψ_s, $0\le\psi_s\le 2\pi$, such that $\operatorname{Im}h_{ss}\left(e^{i\psi_s}\right)=0$. Since

$$h_{ss}(t)=\ln g_{ss}(t)=\ln|g_{ss}(t)|+i\arg g_{ss}(t)=\operatorname{Re}h_{ss}(t)+i\cdot\operatorname{Im}h_{ss}(t),$$

then

$$h_{ss}\left(e^{i\psi_s}\right)=\operatorname{Re}h_{ss}\left(e^{i\psi_s}\right)+i\cdot\operatorname{Im}h_{ss}\left(e^{i\psi_s}\right)=\ln\left|g_{ss}\left(e^{i\psi_s}\right)\right|=\ln g_{ss}\left(e^{i\psi_s}\right)=0.$$

Thus $\varphi_{ss}\left(e^{i\psi_s}\right)=1, s=1,2,\ldots,m$. Now, using the Cauchy formula for functions of matrices (see e.g. [22]) for all $Z\in\mathfrak{R}_I(m,m)$, we have

$$\varphi(Z)=\frac{1}{2\pi i}\int\limits_{\partial\mathbb{U}}\varphi(t)(tI-Z)^{-1}dt. \tag{6.3}$$

The left side of (6.3) is the same as the left side of (6.2) and it is continuous on $\Gamma_{\mathfrak{R}}$. At the points $t_s=e^{i\psi_s}, 0\le\psi_s\le 2\pi$, the function $\varphi_{ss}(t)$ takes the value 1, hence at the points

$$U=\begin{pmatrix}e^{i\psi_1}&0&\ldots&0\\0&e^{i\psi_2}&\ldots&0\\\ldots&\ldots&\ldots&\ldots\\0&0&\ldots&e^{i\psi_m}\end{pmatrix}\in\Gamma_{\mathfrak{R}}$$

the value of the matrix function $\varphi(z)$ equals to I, i.e.,

$$\varphi(Z)=\prod_{j=1}^{k}\left[(Z-P_j)\left(I-P_j^*Z\right)^{-1}\right]\left[\prod_{j=1}^{k}\left[(Z-Q_j)\left(I-Q_j^*Z\right)^{-1}\right]\right]^{-1}=I.$$

This means that

$$B_1(U)=B_2(U).$$

The lemma is proved.

7 Proof of Theorem 6.1

Let

$$R(Z) = B_1(Z) \cdot B_2^{-1}(Z).$$

Then

$$R(Z) = \prod_{j=1}^{k} \left[(Z - P_j)\left(I - P_j^* Z\right)^{-1} \right] \left[\prod_{j=1}^{k} \left[(Z - Q_j)\left(I - Q_j^* Z\right)^{-1} \right] \right]^{-1}.$$

Since $B_1(Z)$ and $B_2(Z)$ are automorphisms of $\mathfrak{R}_I(m, m)$, then

$$R(Z) R^*(Z) = I$$

on $\varGamma_{\mathfrak{R}}$. Hence

$$R(Z) R^*\left(Z^{*^{-1}}\right) = I$$

on $\varGamma_{\mathfrak{R}}$. By the condition of the theorem we have $R(\nu_j) = I$. Hence, $R\left(\nu_j^{*^{-1}}\right) = I$ for all $j = 1, 2, \dots, k$. Consider the matrix function

$$R(t) = (R_{ss}(t))_1^m, \; t \in \mathbb{C}.$$

Put

$$\nu_j = \begin{pmatrix} (\nu_j)_{11} & 0 & \dots & 0 \\ 0 & (\nu_j)_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & (\nu_j)_{mm} \end{pmatrix}.$$

Then $R_{ss}\left((\nu_j)_{ss}\right) = 1$ and $R_{ss}\left(\frac{1}{(\overline{\nu_j})_{ss}}\right) = 1$ for all $s = 1, \dots, m, j = 1, \dots, k$.

By virtue of Lemma 6.1, we have $R(U) = I$ for some

$$U = \begin{pmatrix} e^{i\psi_1} & 0 & \dots & 0 \\ 0 & e^{i\psi_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e^{i\psi_m} \end{pmatrix} \in \varGamma_{\mathfrak{R}}$$

i.e. $R_{ss}\left(e^{i\psi_s}\right) = 1, s = 1, \dots, m$. Therefore, for every fixed s, the rational function $R_{ss}(t)$ of degree $2k$ takes the value 1 at $2k+1$ points $e^{i\psi_s}; (\nu_1)_{ss}, \dots, (\nu_k)_{ss}; \frac{1}{(\overline{\nu}_1)_{ss}}, \dots, \frac{1}{(\overline{\nu}_k)_{ss}}$. Since $R_{ss}(t) \equiv 1$, we have

$$B_1(Z) \equiv B_2(Z).$$

The Theorem 6.1 is proved.

8 Conclusion

This paper has advanced the theory of Blaschke products in the context of matrix-valued functions, establishing key properties and proving a matrix analogue of the Horwitz-Rubel theorem. By extending classical results from complex analysis to the of matrix domains, we have developed a framework that preserves the essential characteristics of Blaschke products while accommodating the complexities of higher-dimensional spaces.

We rigorously defined finite Blaschke products in the matrix unit circle $\mathfrak{R}_I(m,m)$ and the matrix upper half-plane $\mathfrak{I}_\sigma$, proving their continuity, boundary behavior, and zero structure. These results generalize the well-known properties of scalar Blaschke products while accounting for the noncommutative nature of matrix multiplication.

We established explicit transformations between the matrix upper half-plane and the matrix unit circle, extending the classical Cayley transform to the matrix setting. These mappings preserve the structural properties of Blaschke products and provide a foundation for further investigations in matrix complex analysis.

A central result of this work is the matrix analogue of the Horwitz-Rubel theorem, which guarantees the uniqueness of Blaschke products when they coincide on a sufficiently large set of diagonal matrices. This theorem highlights the delicate interplay between algebraic constraints and analytic behavior in matrix domains. In particular, for $m = 1$ Theorem 6.1 completely coincides with the Horwitz–Rubel theorem [4].

The implications of our findings extend beyond pure mathematics, with potential applications in mathematical physics, operator theory, and multidimensional signal processing. Future research directions include:

Investigating whether the uniqueness theorem holds for non-diagonal matrices or more general classes of operators;

Developing convergence criteria and factorization theorems for infinite Blaschke products in matrix unit circle $\mathfrak{R}_I(m,m)$ and the matrix upper half-plane $\mathfrak{I}_\sigma$;

Exploring the role of matrix Blaschke products in the spectral theory of linear operators and their applications in system theory.

In conclusion, this work bridges classical complex analysis with modern matrix theory, offering new insights into the behavior of holomorphic functions in noncommutative spaces. The results presented here not only enrich the theoretical landscape but also pave the way for novel applications in both mathematics and related disciplines. Further exploration of these ideas promises to uncover deeper connections between complex analysis, matrix algebra, and functional analysis.

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