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Riesz property criterion for the system of eigen and associated functions of second order ordinary differential operator

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Abstract. *On the interval $G = (0, 1)$ an ordinary differential operator of second order with summable coefficients is considered. Riesz property for the system of eigen and associated functions of the given operator is studied. The criterion of Riesz property in $L_p(G), 1 < p \leq 2$, of eigen and associated functions system is established.*

Keywords. differential operator · eigen and associated functions · Riesz property criterion.

Mathematics Subject Classification (2010): 34L10, 42A20

1 Introduction and statement of main result

On the interval $G = (0, 1)$ consider the ordinary differential operator

$$Lu = u^{''} + q_1(x)\, u^{'} + q_2(x)\, u,$$

with summable coefficients $q_i(x) \in L_1(G)$, $i = 1, 2$.

The root functions (i.e. eigenfunctions and associated function) of the operator L are understood in the generalized sense (irrespective of the boundary conditions) [3].

Consider an arbitrary system $\{u_k(x)\}_{k=1}^{\infty}$ consisting of eigen and associated functions (root functions) of the operator L. Let $\{\lambda_k\}_{k=1}^{\infty}$ be the corresponding system of eigenvalues of this operator. We require that, together with each associated function of order s, $s \geq 1$, the system $\{u_k(x)\}_{k=1}^{\infty}$ also contains the corresponding eigenfunction and all associated functions of order less than s. This means that each element of the system $\{u_k(x)\}_{k=1}^{\infty}$ is identically non-zero, is absolutely continuous together with its first order derivative on $\overline{G}$, and almost everywhere in G satisfies the equation $Lu_k + \lambda_k u_k = \theta_k u_{k-1}$, where θ_k equals 0 (in this case $u_k(x)$ is an eigenfunction), or 1 (in this case $u_k(x)$ is an associated functions of order $r \geq 1$, and $u_{k-1}(x)$ is an associated functions of order $r - 1$, $\lambda_k = \lambda_{k-1} = \cdots = = \lambda_{k-r}$, $\theta_k = \theta_{k-1} = \theta_{k-r+1} = 1$ and $\theta_{k-r} = 0$).

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The highest order of root functions corresponding to the given eigenfunction will be called the rank of this eigenfunction.

In such a generalized understanding of eigen and associated functions (root functions) Ilin V.A.(see [3]) first established the Bessel property unconditional basic property criterion in $L_2(G)$ of the system of root functions of the operator L for $q_1(x) \equiv 0,\ q_2(x) \in L_1(G)$.

In the papers (see [2],[4],[6-9], [16]) these problems were studied for a higher order ordinary differential operator. This time, for Bessel property it was always assumed that $q_1(x) \in L_2(G),\ \ q_2(x) \in L_1(G),\ \ f(x) \in L_2(G),$ for Riesz property it was always assumed that $q_1(x) \in L_p(G),\ 1 < p < 2,\ \ f(x) \in L_p(G)$.

It should be note that the Riess inequality (in particular, the Bessel inequality) is widely used to study the convergence of spectral expansions of functions (see [1-4], [9-16]).

In the present paper we study the Riesz property (in case $p = 2$ Bessel property) for the system of root functions of the given operator with the coefficients $q_i(x) \in L_1(G), i = 1, 2$, and the criterion of Riesz property of root functions system is established.

Definition 1.1 *A system $\{\varphi_k(x)\}_{k=1}^{\infty} \subset L_q(G)$ is called Riesz system if there exists a constant $M = M(p)$ such that for each function $f(x) \in L_p(G),\ 1 < p \leq 2,\ q = p/(p-1)$, one has the inequality*

$$\left(\sum_{k=1}^{\infty} |(f, \varphi_k)|^q\right)^{1/q} \leq M\|f\|_p.$$

Denote $\mu_k = \sqrt{\lambda_k}$, $Re\mu_k \geq 0$.

The main result of this work is the following theorem.

Theorem (Riesz property criterion). Let $q_i(x) \in L_1(G), i = 1, 2$, the ranks of eigenfunctions be uniformly bounded, and assume that there exists a constant C_0 such that

$$|Im\mu_k| \leq C_0,\ \ k = 1, 2, \ldots. \tag{1.1}$$

Then a necessary and sufficient condition for the Riesz property of the system $\left\{u_k(x)\|u_k\|_q^{-1}\right\}_{k=1}^{\infty}$ in $L_p(G),\ \ 1 < p \leq 2,\ \ \ q = p/(p-1)$ is that there a constant M_1 such that

$$\sum_{\tau \leq Re\mu_k \leq \tau+1} 1 \leq M_1,\ \ \forall \tau \geq 0. \tag{1.2}$$

Remark 1.1 In the sufficient part of the Theorem, the uniform boundedness of the ranks of eigenfunctions holds automatically as a consequence of inequality (1.2).

2 Proof of the Theorem

Necessity: We fix an arbitrary number $t \geq 0$. Introduce following set

$$\Omega_\tau = \{k \in N:\ \tau \leq Re\mu_k\ \leq \tau + 1, |Im\mu_k| \leq C_0\ ,\ \ |\mu_k| \geq 1\},$$

and number

$$R_0 = (n_0(1 + C_0)^{1+1/q})^{-1},$$

where $n_0 \geq 1$ is chosen so that $R_0 \leq 1/4$ and for any set $E \subset \overline{G}$, $mesE \leq 2R_0$, the following is fulfilled:

$$\omega(R_0) = \sup_{E \subset \overline{G}} \left\{\|q_i\|_{1,E},\ \ i = 1, 2\right\} \leq N_0^{-1}, \tag{2.1}$$

where $\|q_i\|_{1,E} = \int_E |q_i(x)|\, dx$, $\left(\|\cdot\|_1 = \|\cdot\|_{1,G}\right)$; N_0- is a positive number, the choice of the value of which will be determined later.

Let $x \in \left[0, \frac{1}{2}\right]$, $k \in \Omega_\tau$. We write the mean value formula (see [3], [9], [10]) for the points $x, x+t, x+2t$ at $t \le R_0$:

$$u_k(x) = 2u_k(x+t)\cos\mu_k t - u_k(x+2t) = {\mu_k}^{-1} \int\limits_x^{x+2t} \left\{ q_1(\xi)\, u_k'(\xi) + \right.$$

$$\left. + q_2(\xi)\, u_k(\xi) - \theta_k u_{k-1}(\xi) \right\} \sin\mu_k(|x+t-\xi| - t)\, d\xi. \tag{2.2}$$

Since, according to the condition of the theorem, the rank of eigenfunctions is uniformly bounded, then it suffices to consider only the eigen-functions $u_k(x)$, i.e. the case $\theta_k = 0$. Then adding and subtracting on the right hand of formula (2.2) the expression $2u_k(x+t)\cos\tau t$ and by using the identity

$$\cos a - \cos\beta = 2\sin\frac{a+\beta}{2}\,\sin\frac{\beta - a}{2},$$

then applying the operation $R_0^{-1}\int_0^{R_0} dt$, in the result we can obtain

$$u_k(x) = R_0^{-1}(u_k, v) + 4{R_0}^{-1} \int_0^{R_0} u_k(x+t)\sin\frac{\mu_k + \tau}{2}\, t\sin\frac{\tau - \mu_k}{2}\, tdt +$$

$$+ (R_0\mu_k)^{-1} \int\limits_0^{R_0} \int\limits_x^{x+2t} \left\{ q_1(\xi)\, u_k'(\xi) + q_2(\xi)\, u_k(\xi) \right\} \sin\mu_k\, (|x+t-\xi| - t)\, d\xi dt, \tag{2.3}$$

where $v(t) = 2\cos t\,(x-t) - \frac{1}{2}$ for $t \in [x, x+R_0]$, $v(t) = -1/2$ for $t \in [x+R_0, x+2R_0]$ and $v(t) = 0$ for $t \notin [x, x+2R]$.

In the second addend on the right hand side of the formula (2.3) applying to the expression $u_k(x+t)$ following shift formula (see [3], [9], [10])

$$u_k(x \pm t) = u_k(x)\cos\mu_k t \pm \mu_k^{-1}\sin\mu_k\, t u_k'(x) \pm$$

$$\pm \mu_k^{-1} \int_x^{x \pm t} \left\{ q_1(\xi)\, u_k'(\xi) + q_2(\xi)\, u_k(\xi) - \theta_k u_{k-1}(\xi) \right\} \sin\mu_k\, (|x - \xi| - t)\, d\xi \tag{2.4}$$

for $\theta_k = 0$, we get

$$u_k(x) = R_0^{-1}(u_k, v) + 4R_0^{-1} u_k(x) \int_0^{R_0} \cos\mu_k\, t\sin\frac{\mu_k + \tau}{2}\, t\sin\frac{\tau - \mu_k}{2}\, tdt +$$

$$+ 4(\mu_k R_0)^{-1} u_k'(x) \int_0^{R_0} \sin\mu_k\, t\sin\frac{\mu_k + \tau}{2}\, t\sin\frac{\tau - \mu_k}{2}\, tdt +$$

$$+ 4(\mu_k R_0)^{-1} \int_0^{R_0} \sin\frac{\mu_k + \tau}{2}\, t\sin\frac{\tau - \mu_k}{2}\, dt \int_x^{x+t} \left\{ q_1(\xi)\, u_k'(\xi) + \right.$$

$$\left. + q_2(\xi)\, u_k(\xi) \right\} \sin\mu_k\, (|x - \xi| - t)\, d\xi dt +$$

$$+\left(\mu_k R_0\right)^{-1}\int_0^{R_0}\int_x^{x+2t}\left\{q_1\left(\xi\right)u_k^{'}\left(\xi\right)+q_2\left(\xi\right)u_k\left(\xi\right)\right\}\sin\mu_k\ \left(\left|x+t-\xi\right|-t\right)d\xi dt=$$

$$=R_0^{-1}\left(u_k,v\right)+\sum_{j=1}^{4}T_j. \tag{2.5}$$

Let estimate the integrals $T_j,\quad j=\overline{1,4}$. Taking into account $k\in\Omega_\tau$ and using the inequality

$$\left|\sin z\ \right|\leq 2,\ \left|\cos z\ \right|\leq 2,\ \left|\sin z\ \right|\leq 2\left|z\right|\ \text{ for } Imz\leq 1$$

we get

$$\left|T_1\right|\leq 8R_0\left|\tau-\mu_k\right|\left|u_k\left(x\right)\right|\leq 8R_0\left(1+C_0\right)\left|u_k\left(x\right)\right|\leq 8n_0^{-1}\left|u_k\left(x\right)\right|,$$

$$\left|T_2\right|\leq 8R_0\left|\tau-\mu_k\right|\left|\mu_k^{-1}\right|\left|u_k^{'}\left(x\right)\right|\leq 8R_0\left(1+C_0\right)\left|\mu_k^{-1}\right|\left|u_k^{'}\left(x\right)\right|\leq 8n_0^{-1}|\mu_k|^{-1}\left|u_k^{'}\left(x\right)\right|$$

We estimate T_3 for $k\in\Omega_\tau$. Using the estimations (see [8],[9])

$$\left\|u_k^{(s)}\right\|_\infty\leq C_1(1+|\mu_k|)^s(1+|Im\mu_k\ |)^{1/p}\times$$

$$\times\left\{\|u_k\|_p+\theta_k|\mu_k|^{-1}(1+|Im\mu_k\ |)^{-1}\|\mu_{k-1}\|_p\right\},1\leq p<\infty,s=0,1; \tag{2.6}$$

$$\left\|u_k^{'}\right\|_p\leq C_2\left(1+|\mu_k|\right)\left\{\|u_k\|_p+\theta_k|\mu_k|^{-1}(1+|Im\mu_k\ |)^{-1}\|u_{k-1}\|_p\right\},\ \ p\geq 1, \tag{2.7}$$

for $\theta_k=0$ and applying the above elementary inequalities, we have

$$\left|T_3\right|\leq 32R_0\left|\mu_k^{-1}\right|\int_0^{R_0}\int_x^{x+t}\left|q_1\left(\xi\right)\right|\left|u_k^{'}\left(\xi\right)\right|d\xi dt+$$

$$+32R_0\int_0^{R_0}\int_x^{x+t}|q_2\left(\xi\right)|d\xi dt\|u_k\|_\infty\leq 32\omega\left(R_0\right)|\mu_k|^{-1}\left\|u_k^{'}\right\|_\infty+16R_0\omega\left(R_0\right)\|u_k\|_\infty\leq$$

$$\leq 32\omega\left(R_0\right)C_1\left(1+|\mu_k|^{-1}\right)(1+|Im\mu_k\ |)^{1/q}\|u_k\|_q+$$

$$+16R_0\omega\left(R_0\right)C_1(1+|Im\mu_k\ |)^{1/q}\|u_k\|_q\leq$$

$$\leq 64\omega\left(R_0\right)(1+C_0)^{1/2}C_1\|u_k\|_q+16C_1(1+C_0)^{1/2}R_0\omega\left(R_0\right)\|u_k\|_q\leq$$

$$\leq C_3\omega\left(R_0\right)\|u_k\|_q\leq C_4N_0^{-1}\|u_k\|_q.$$

The same estimation is fulfilled also for the integral T_4 for $k\in\Omega_t$.
Consequently, from (2.5) we get

$$\left|u_k\left(x\right)\right|\|u_k\|_q^{-1}\leq R_0^{-1}\left|\left(u_k\|u_k\|_q^{-1},v\right)\right|+$$

$$+8n_0^{-1}\left|u_k\left(x\right)\right|\|u_k\|_q^{-1}+8n_0^{-1}|\mu_k|^{-1}\left|u_k^{'}\left(x\right)\right|\|u_k\|_q^{-1}+C_4N_0^{-1} \tag{2.8}$$

By virtue of symmetry (see formulas (2.2), (2.4)) this inequality is valid in the case $x\in\left[0,\frac{1}{2}\right]$, as well. This time the function $v\left(t\right)$ is determined by the formula: $v\left(t\right)=\frac{1}{2}$ for $t\in\left[x-2R_0,x-R_0\right]$, $v\left(t\right)=2\cos t\ \left(x-t\right)-\frac{1}{2}$ for $t\in\left[x-R_0,x\right]$, $v\left(t\right)=0$ for $t\notin\left[x-2R_0,\ x\right]$.

Hence, for $n_0 \geq 16$ it follows that

$$|u_k(x)|\,\|u_k\|_q^{-1} \leq 2R_0^{-1}\left|\left(u_k\|u_k\|_q^{-1}, v\right)\right| +$$

$$+16n_0^{-1}|\mu_k|^{-1}\left|u_k'(x)\right|\|u_k\|_q^{-1} + 2C_4N_0^{-1}. \qquad (2.9)$$

We raised to the power q every part of inequality (2.9), applying inequality

$$\left(\sum_{i=1}^{3} a_i\right)^q \leq 3^{q-1}\sum_{i=1}^{3} a_i^q, \quad a_i \geq 0,$$

integrating given result with respect to x from 0 to 1, we get

$$1 \leq 2^q 3^{q-1} R_0^{-q}\left|\left(u_k\|u_k\|_q^{-1}, v\right)\right|^q +$$

$$+3^{q-1}16^q n_0^{-q}|\mu_k|^{-q}\left\|u_k'\right\|_q^q\|u_k\|_q^{-q} + 3^{q-1}\left(2C_4N_0^{-1}\right)^q$$

applying inequality (2.7) for $p = q,\ \theta_k = 0$, we obtain

$$1 \leq 2^q 3^{q-1} R_0^{-q}\left|\left(u_k\|u_k\|_q^{-1}, v\right)\right|^q + 3^{q-1}16^q n_0^{-q}C_2^q\left(1 + |\mu_k|^{-1}\right)^q +$$

$$+3^{q-1}\left(2C_4N_0^{-1}\right)^q$$

Summing above inequality over $k \in \Omega_\tau$, we get

$$\sum_{k\in\Omega_\tau} 1 \leq 2^q 3^{q-1} R_0^{-q}\sum_{k\in\Omega_\tau}\left|\left(u_k\|u_k\|_q^{-1}, v\right)\right|^q +$$

$$+3^{q-1}16^q n_0^{-q}C_2^q\sum_{k\in\Omega_\tau}\left(1 + |\mu_k|^{-1}\right)^q + 3^{q-1}\left(2C_4N_0^{-1}\right)^q\sum_{k\in\Omega_\tau} 1.$$

since, $\left\{u_k(x)\,\|u_k\|_q^{-1}\right\}$ is Riesz system, and $1 + |\mu_k|^{-1} \leq 2$ for $\mathrm{k} \in \Omega_\tau$, then

$$\sum_{k\in\Omega_\tau} 1 \leq 2^q 3^{q-1} R_0^{-q}\|v\|_p^q + 3^{q-1}\left(\left(\frac{32C_2}{n_0}\right)^q + \left(2C_4N_0^{-1}\right)^q\right)\sum_{k\in\Omega_\tau} 1.$$

Taking into account estimate $\|v\|_p^q \leq (6R_0)^{q/p}$, we have

$$\sum_{k\in\Omega_\tau} 1 \leq 6^{q/p}R_0^{-1} + \left\{C_5 n_0^{-q} + C_6 N_0^{-q}\right\}\sum_{k\in\Omega_\tau} 1,$$

where $C_5 = 3^{q-1}(32C_2)^q$, $C_6 = 3^{q-1}(2C_4)^q$.

Choosing the numbers $n_0\,(n_0 \geq 16)$ and N_0 so that $C_5 n_0^{-q} + C_6 N_0^{-q} \leq \frac{1}{2}$, we have at the inequality

$$\sum_{k\in\Omega_\tau} 1 \leq const.$$

Consequently, for $|\mu_k| \geq 1$ necessity of condition (1.2) is established.

For $|\mu_k| < 1$ the validity of condition (1.2) is proved in the following way. We consider the equation $Lu_k - 2u_k + \lambda_k' u_k = 0$, where $\lambda_k' = \lambda_k + 2$, $|\lambda_k| < 1$. Then $\left|Re\lambda_k'\right| \geq 1$

and the system $\{u_k(x)\}$ does not change. Therefore inequality (1.2) is fulfilled in the case $|\lambda_k| < 1$ as well. The necessity of condition (1.2) is established.

Sufficiency. Let conditions (1.1) and (1.2) be fulfilled. Prove that $\left\{u_k(x)\,\|u_k\|_q^{-1}\right\}_{k=1}^{\infty}$ is Riesz system. Firstly we prove that $\left\{u_k(x)\,\|u_k\|_2^{-1}\right\}_{k=1}^{\infty}$ is Bessel system in $L_2(G)$. By formula (2.4), conditions (1.1) and (1.2) for the convergence of the series $\sum\limits_{k=1}^{\infty}\left|\left(f, u_k\|u_k\|_2^{-1}\right)\right|^2$ for any $f \in L_2(G)$, it suffices to prove the validity of the following inequalities:

$$\sum_{|\mu_k|\geq 1}\left|\int_0^1 \overline{f(t)}\cos\mu_k\, tdt\right|^2 \|u_k\|^{-2}|u_k(0)|^2 \leq C\|f\|_2^2; \tag{2.10}$$

$$\sum_{|\mu_k|\geq 1}|\mu_k|^{-2}\left|\int_0^1 \overline{f(t)}\sin\mu_k\, tdt\right|^2 |u_k(0)|^2 \leq C\|f\|_2^2; \tag{2.11}$$

$$\sum_{|\mu_k|\geq 1}|\mu_k|^{-2}\left|\int_0^1 \overline{f(t)}\int_0^t q_1(\xi)\,u_k'(\xi)\sin\mu_k\,(\xi-t)\,d\xi dt\right|^2 \|u_k\|_2^{-2} \leq C\,\|f\|_2^2; \tag{2.12}$$

$$\sum_{|\mu_k|\geq 1}|\mu_k|^{-2}\left|\int_0^1 \overline{f(t)}\int_0^t q_2(\xi)\,u_k(\xi)\sin\mu_k(\xi-t)\,d\xi dt\right|^2 \|u_k\|_2^{-2} \leq C\,\|f\|_2^2; \tag{2.13}$$

$$\sum_{|\mu_k|\geq 1}\xi_k\,|\mu_k|^{-2}\left|\int_0^1 \overline{f(t)}\int_0^t u_{k-1}(\xi)\sin\mu_k\,(\xi-t)\,d\xi dt\right|^2 \|u_k\|_2^{-2} \leq C\|f\|_2^2, \tag{2.14}$$

where $f \in L_2(G)$.

Under conditions (1.1), (1.2), the validity of inequalities (2.10), (2.11), (2.13) follows from [3].

Prove the validity of the inequality (2.12). Denote

$$g(t,\xi) = \begin{cases} f(t+\xi), & 0 \leq t \leq 1-\xi, \\ 0, & 1-\xi < t \leq 1, \end{cases} \quad \text{where } \xi \in [0,1].$$

Then

$$S_k(f) = |\mu_k|^{-2}\left|\int_0^1 \overline{f(t)}\int_0^t q_1(\xi)\,u_k'(\xi)\sin\mu_k\,(\xi-t)d\xi dt\right|^2 \|u_k\|_2^{-2} =$$

$$= \int_0^t q_1(\xi)\,u_k'(\xi)\,\mu_k^{-1}\|u_k\|_2^{-1}\int_0^1 \overline{g(t,\xi)}\sin\mu_k\, tdtd\xi\times$$

$$\times\int_0^1 \overline{q_1(z)}\overline{u_k'(z)}\overline{\mu_k^{-1}}\|u_k\|_2^{-1}\int_0^1 g(r,z)\,\overline{\sin\mu_k\, r}drdz =$$

$$= \int_0^1\int_0^1 q_1(\xi)\,\overline{q_1(z)}u_k'(\xi)\,\mu_k^{-1}\|u_k\|_2^{-1}\overline{u_k'(z)}\overline{\mu_k^{-1}}\|u_k\|_2^{-1}\times$$

$$\times\int_0^1 \overline{g(t,\xi)}\sin\mu_k\, tdt\int_0^1 g(r,z)\,\overline{\sin\mu_k\, r}drd\xi dz$$

Applying here the estimation (see [5])

$$\left\|u_k'\right\|_\infty \le C_7\left(1+|\mu_k|\right)\|u_k\|_2$$

we get that at $|\mu_k| \ge 1$ for $S_k(f)$ the following inequality is fulfilled:

$$S_k(f) \le C_8 \int_0^1\int_0^1 |q_1(\xi)|\,|q_1(z)| \left|\int_0^1 \overline{g(t,\xi)}\sin\mu_k\, tdt\right| \left|\int_0^1 \overline{g(r,z)}\sin\mu_k\, rdr\right| d\xi dz.$$

For an arbitrary finite subset J' of the set of indices $J=\{k: |\mu_k|\ge 1\}$, we obtain

$$\sum_{k\in J'} S_k(f) \le C_8 \int_0^1\int_0^1 |q_1(\xi)|\,|q_1(z)| \left(\sum_{k\in J'}\left|\int_0^1 \overline{g(t,\xi)}\sin\mu_k\, tdt\right|^2\right)^{1/2} \times$$

$$\times\left(\sum_{k\in J'}\left|\int_0^1 \overline{g(r,z)}\sin\mu_k\, rdr\right|^2\right)^{1/2} d\xi dz \le$$

$$\le C_9 \int_0^1\int_0^1 |q_1(\xi)\,q_1(z)|\|g(\cdot,\xi)\|_2\,\|g(\cdot,z)\|\,d\xi dz.$$

Taking into account that for any $\xi\in[0,1]$ we have the inequality $\|g(\cdot,\xi)\|_2 \le \|f\|_2$, we obtain

$$\sum_{k\in J'} S_k(f) \le C_{10}\|q_1\|_1^2\|f\|_2^2.$$

Hence, from arbitrariness of $J'\subset J$, it follows the inequality
Therefore, for an arbitrary function $f(x)\in L_2(G)$ the following Bessel inequality is satisfied:

$$\left(\sum_{k=1}^{\infty}\left|\left(f, u_k\|u_k\|_2^{-1}\right)\right|^2\right)^{1/2} \le M\|f\|_2. \qquad (2.15)$$

On the other hand, for any $f\in L_1(G)$

$$\sup_k\left(f, u_k\|u_k\|_2^{-1}\right) \le \mathop{sup}_k\left(\|u_k\|_\infty\ \|u_k\|_2^{-1}\right)\|f\|_1$$

Hence, by virtue of the inequality (see [5])

$$\|u_k\|_r \le c_{11}\,\|u_k\|_\gamma,\quad \gamma\ge 1,\quad r\ge 1 \qquad (2.16)$$

we get that for any function $f\in L_1(G)$ the following inequality is valid:

$$\mathop{sup}_k\left(f, u_k\|u_k\|_2^{-1}\right)^{1/2} \le M_2\|f\|_1. \qquad (2.17)$$

By virtue of the Riesz-Torin interpolation theorem (see [17], p.144) from inequalities (2.15) and (2.17) it follows that the system $\left\{u_k(x)\,\|u_k\|_2^{-1}\right\}_{k=1}^{\infty}$ satisfies the Riesz inequality, i.e.

$$\left(\sum_{k=1}^{\infty}\left|\left(f, u_k(x)\,\|u_k\|_2^{-1}\right)\right|^q\right)^{1/q} \le M(p)\,\|f\|_q$$

for $1 < p \leq 2, \ \ q = p/(p-1)$.

Since

$$u_k(x)\|u_k\|_2^{-1} = u_k(x)\|u_k\|_q^{-1}\|u_k\|_q\|u_k\|_2^{-1}$$

and by virtue of inequality (2.16), the estimation

$$1 \leq \|u_k\|_q\|u_k\|_2^{-1} \leq C_{12}$$

is fulfilled, then the system $\left\{u_k(x)\|u_k\|_q^{-1}\right\}_{k=1}^{\infty}$ satisfies the Riesz inequality as well, i.e. the inequality

$$\left(\sum_{k=1}^{\infty}\left|\left(f, u_k(x)\|u_k\|_q^{-1}\right)\right|^q\right)^{1/q} \leq M_2(p)\|f\|_q, \qquad q = p/(p-1)$$

is fulfilled for any $f \in L_p(G), \ 1 < p \leq 2$.

Theorem is completely proved.

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