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On some properties of grand sequence spaces

Migdad I. Ismailov* · **Chinara M. Hasanova** · **Hagigat Z. Ashirova**

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Abstract. *In this paper we introduce the grand space $l_{p)},\ p > 1$ of sequences of numbers. Their completeness is proved, and their connections with ordinary spaces l_p of number sequences are studied. The closure $g_{p)}$ of the set c_{00} of finite number sequences in the space $l_{p)}$ is considered. The basis property of the system $e_n = \{\delta_{nm}\}_{m\in N}$, $n \in N$ in the subspace $g_{p)}$ is proved.*

Keywords. spaces of sequences of numbers, grand Lebesgue spaces, closure, completeness, basicity, orthogonality, Fourier coefficients.

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1 Introduction

Recently, in connection with important applications of the theory of partial differential equations, in the theory of optimal control, etc., interest in research in non-standard Banach function spaces has increased greatly. Such spaces include Morrey spaces, Lebesgue spaces with variable summability exponent, grand Lebesgue spaces, Orlicz spaces, etc. The issues of harmonic analysis and approximation theory in these spaces are the subject of works, for example, [1-13].

Note that grand Lebesgue spaces $L_{p)}(\Omega)$ for a bounded set $\Omega \subset R^n$ were introduced in [14] as a space of functions $f \in L_1(\Omega)$ satisfying the condition

$$\|f\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{|\Omega|} \int_{\Omega} |f(t)|^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}} < +\infty.$$

* Corresponding author

M.I. Ismailov
Azerbaijan State University of Oil and Industry, Baku, Azerbaijan
E-mail: migdad-ismailov@rambler.ru

Ch.M. Hasanova
Ganja State University, Ganja, Azerbaijan
E-mail: Chinareh@gmail.com

H.Z. Ashirova
Odlar Yurdu University, Baku, Azerbaijan
E-mail: hagigatashirova@gmail.com

From the results of [14, 15], it follows that the integrability of the Jacobian of the mapping is reduced to the belonging of its components to grand Lebesgue spaces. The basis property of systems of exponentials and their perturbations in grand Lebesgue spaces and in their weighted versions was considered in [16-19]. When studying discrete operators in grand Lebesgue spaces, one has to consider the corresponding grand spaces of sequences of numbers. Concerning grand spaces of sequences of numbers, the work [20] is known. In [20], grand spaces $l_{p),\theta}$, $p > 1$, $\theta > 0$ of sequences of numbers $x = \{x_n\}_{n\in Z}$ with a finite norm

$$\| \{x_n\}_{n\in Z} \|_{l_{p),\theta}} = \sup_{0<\varepsilon<p-1} \varepsilon^{\theta} \left(\sum_{n=-\infty}^{+\infty} |x_n|^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} < +\infty$$

are introduced, it is proved that the set of finite sequences is dense in the subspace of sequences $x = \{x_n\}_{n\in Z} \in l_{p),\theta}$ satisfying the condition

$$\lim_{\varepsilon\to+0} \varepsilon^{\theta} \sum_{n=-\infty}^{+\infty} |x_n|^{p(1+\varepsilon)} = 0.$$

In this paper, another grand space $l_{p)}$, $p > 1$ of sequences of numbers is introduced, its completeness is proved, and the structure of a subspace in which the set of finite sequences of numbers is dense is given. Also, in this paper an analogue of the Hausdorff-Young theorem in grand spaces of sequences $l_{p)}$, $p > 1$ is established.

2 Grand space $l_{p)}$ sequences of number

Let l_p, $p \geq 1$ be the space of sequences $a = \{a_n\}_{n\in N}$ of numbers satisfying the condition

$$\sum_{n=1}^{\infty} |a_n|^p < +\infty, \;\; p < +\infty, \;\; \sup_{n\in N} |a_n| < +\infty, \;\; p = +\infty \; .$$

The space l_p, $p \geq 1$ is a complete with the norm

$$\|a\|_{l_p} = \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{\frac{1}{p}}, \;\; p< +\infty;$$

$$\|a\|_{l_\infty} = \sup_{n\in N} |a_n|, \;\; p = +\infty.$$

For $1 < p < q < +\infty$ the following continuous strict embedding holds:

$$l_1 \subset l_p \subset l_q \subset l_\infty,$$

such that

$$\|a\|_{l_q} \leq \|a\|_{l_p}, \; \|a\|_{l_\infty} = \lim_{p\to\infty} \|a\|_{l_p}.$$

Consider the following grand space $l_{p)}$ of sequences of scalars $\{a_n\}_{n\in Z}$, such that

$$\| \{a_n\}_{n\in N} \|_{l_{p)}} = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p(\varepsilon)}} \left(\sum_{n=1}^{\infty} |a_n|^{p'(\varepsilon)} \right)^{\frac{1}{p'(\varepsilon)}} < +\infty, \tag{2.1}$$

where $p(\varepsilon) = p - \varepsilon$, $p^{'}(\varepsilon) = \frac{p(\varepsilon)}{p(\varepsilon)-1}$. The function $\|\cdot\|_{l_{p)}} : l_{p)} \to R$ defined by formula (2.1) is a norm, and thus $l_{p)}$ becomes a normed space.

The space $l_{p)}$ is complete. Indeed, let $a(k) = \{a_n(k)\}_{n\in N}$, $k \in N$ be a fundamental sequence in $l_{p)}$ i.e. for any $\eta > 0$ there exists a number $k_\eta \in N$ such that for any $k \geq k_\eta$ and $m \in N$ the relation

$$\| \{a_n(k) - a_n(k+m)\}_{n\in N} \|_{l_{p)}} =$$

$$= \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p(\varepsilon)}} \left(\sum_{n=1}^{+\infty} |a_n(k) - a_n(k+m)|^{p^{'}(\varepsilon)} \right)^{\frac{1}{p^{'}(\varepsilon)}} < \eta \tag{2.2}$$

is satisfied.

It follows that for an arbitrary $\varepsilon \in (0, p-1)$ we have

$$\varepsilon^{\frac{1}{p(\varepsilon)}} |a_n(k) - a_n(k+m)| < \eta, \; n \in N,$$

i.e. for any $n \in N$ the sequence $a_n(k)$, $k \in N$ is fundamental. Let $a_n = \lim\limits_{k\to\infty} a_n(k)$. Let us take an arbitrary number $M \in N$. Then according to (2.2) we have

$$\varepsilon^{\frac{1}{p(\varepsilon)}} \left(\sum_{n=1}^{M} |a_n(k) - a_n(k+m)|^{p^{'}(\varepsilon)} \right)^{\frac{1}{p^{'}(\varepsilon)}} < \eta. \tag{2.3}$$

Passing to the limit in (2.3) as $m \to \infty$ we get

$$\varepsilon^{\frac{1}{p(\varepsilon)}} \left(\sum_{n=1}^{M} |a_n(k) - a_n|^{p^{'}(\varepsilon)} \right)^{\frac{1}{p^{'}(\varepsilon)}} \leq \eta \; .$$

Due to the arbitrariness of $M \in N$, we have

$$\varepsilon^{\frac{1}{p(\varepsilon)}} \left(\sum_{n=1}^{+\infty} |a_n(k) - a_n|^{p^{'}(\varepsilon)} \right)^{\frac{1}{p^{'}(\varepsilon)}} \leq \eta, \;\; \varepsilon \in (0, p-1).$$

Consequently,

$$\| \{a_n(k) - a_n\}_{n\in N} \|_{l_{p)}} = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p(\varepsilon)}} \left(\sum_{n=1}^{+\infty} |a_n(k) - a_n|^{p^{'}(\varepsilon)} \right)^{\frac{1}{p^{'}(\varepsilon)}} \leq \eta \; .$$

Therefore, $a = \{a_n\}_{n\in Z} \in l_{p)}$ and the sequence $a(k) = \{a_n(k)\}_{n\in Z}$ converges to a in the space $l_{p)}$. Thus, the space $l_{p)}$ is complete.

The following embeddings are valid:

$$l_{p'} \subset l_{p)}, \quad p^{'} = \frac{p}{p-1}, \quad p \leq 2;$$

$$l_{p'} \subset l_{p)} \subset l_2, \;\; p > 2.$$

We will demonstrate the strictness of these embeddings using the following example. Let $a_n = n^{-\frac{1}{p'}}$, $n \in N$. For $p \leq 2$ we have

$$\|a\|_{l_{p)}} = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p(\varepsilon)}} \left(\sum_{n=1}^{+\infty} n^{-\frac{p^{'}(\varepsilon)}{p^{'}}} \right)^{\frac{1}{p^{'}(\varepsilon)}}$$

$$= \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p(\varepsilon)}} \left(1+\sum_{n=2}^{+\infty} n^{-\frac{p'(\varepsilon)}{p'}}\right)^{\frac{1}{p'(\varepsilon)}}$$

$$\leq \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p(\varepsilon)}} \left(1+\sum_{n=2}^{+\infty} \int_{n-1}^{n} x^{-\frac{p'(\varepsilon)}{p'}} dx\right)^{\frac{1}{p'(\varepsilon)}}$$

$$= \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p(\varepsilon)}} \left(1+\frac{p'}{p'(\varepsilon)-p'}\sum_{n=2}^{+\infty}((n-1)^{-\frac{p'(\varepsilon)-p'}{p'}} - n^{-\frac{p'(\varepsilon)-p'}{p'}})\right)^{\frac{1}{p'(\varepsilon)}}$$

$$= \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p(\varepsilon)}} \left(1+\frac{p'}{p'(\varepsilon)-p'}\right)^{\frac{1}{p'(\varepsilon)}} = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p(\varepsilon)}} \left(1+\frac{p(p-\varepsilon-1)}{\varepsilon}\right)^{\frac{1}{p'(\varepsilon)}}$$

$$= \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}-\frac{p-\varepsilon-1}{p-\varepsilon}} ((p-1)(p-\varepsilon))^{\frac{p-\varepsilon-1}{p-\varepsilon}} \leq c \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{2-p+\varepsilon}{p-\varepsilon}} = c,$$

where $c = \max\{p(p-1), 1\}$.

Now let $p > 2$. Then

$$\sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p(\varepsilon)}} \left(\sum_{n=1}^{+\infty} n^{-\frac{p'(\varepsilon)}{p'}}\right)^{\frac{1}{p'(\varepsilon)}} \geq \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p(\varepsilon)}} \left(\sum_{n=1}^{+\infty} \int_{n}^{n+1} x^{-\frac{p'(\varepsilon)}{p'}} dx\right)^{\frac{1}{p'(\varepsilon)}}$$

$$= \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p(\varepsilon)}} \left(\frac{p'}{p'(\varepsilon)-p'}\sum_{n=1}^{+\infty}(n^{-\frac{p'(\varepsilon)-p'}{p'}} - (n+1)^{-\frac{p'(\varepsilon)-p'}{p'}})\right)^{\frac{1}{p'(\varepsilon)}}$$

$$= \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p(\varepsilon)}} \left(\frac{p'}{p'(\varepsilon)-p'}\right)^{\frac{1}{p'(\varepsilon)}} = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p(\varepsilon)}} \left(\frac{p(p-\varepsilon-1)}{\varepsilon}\right)^{\frac{1}{p'(\varepsilon)}}$$

$$= \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}-\frac{p-\varepsilon-1}{p-\varepsilon}} (p(p-\varepsilon-1))^{\frac{p-\varepsilon-1}{p-\varepsilon}} \geq \lim_{\varepsilon\to 0} \varepsilon^{\frac{2-p+\varepsilon}{p-\varepsilon}} = +\infty.$$

By $g_{p)}$ we denote the closure of the set c_{00} in the space $l_{p)}$. It is clear that $g_{p)}$ coincides with the closure of $l_{p'}$ in the space $l_{p)}$.

The following statement studies the structure of the subspace $g_{p)}$.

Theorem 2.1 *Let $a = \{a_n\}_{n\in N} \in l_{p)}$. Then the following conditions are equivalent:*

1) $a = \{a_n\}_{n\in N}$ belongs to the space $g_{p)}$;

2) the equality

$$\lim_{\varepsilon\to+0} \varepsilon^{\frac{1}{p(\varepsilon)}} \left(\sum_{n=1}^{+\infty} |a_n|^{p'(\varepsilon)}\right)^{\frac{1}{p'(\varepsilon)}} = 0 \tag{2.4}$$

holds;

3) the equality

$$\lim_{m\to+\infty} \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p(\varepsilon)}} \left(\sum_{n=m+1}^{+\infty} |a_n|^{p'(\varepsilon)}\right)^{\frac{1}{p'(\varepsilon)}} = 0 \tag{2.5}$$

holds.

Proof. First, we show that 1) implies 2). Let $a \in g_{p)}$ be an arbitrary sequence and $\delta > 0$ an arbitrary number. Then there exists a sequence $b = \{b_n\}_{n\in N} \in l_q$ such that

$$\|a-b\|_{l_{p)}} = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p(\varepsilon)}} \left(\sum_{n=1}^{+\infty} |a_n - b_n|^{p'(\varepsilon)} \right)^{\frac{1}{p'(\varepsilon)}} < \delta.$$

We have

$$\varepsilon^{\frac{1}{p(\varepsilon)}} \left(\sum_{n=1}^{+\infty} |a_n|^{p'(\varepsilon)} \right)^{\frac{1}{p'(\varepsilon)}} \le \varepsilon^{\frac{1}{p(\varepsilon)}} \left(\sum_{n=1}^{+\infty} |a_n - b_n|^{p'(\varepsilon)} \right)^{\frac{1}{p'(\varepsilon)}} + \varepsilon^{\frac{1}{p(\varepsilon)}} \left(\sum_{n=1}^{+\infty} |b_n|^{p'(\varepsilon)} \right)^{\frac{1}{p'(\varepsilon)}}$$

$$\le \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p(\varepsilon)}} \left(\sum_{n=1}^{+\infty} |a_n - b_n|^{p'(\varepsilon)} \right)^{\frac{1}{p'(\varepsilon)}} + \varepsilon^{\frac{1}{p(\varepsilon)}} \left(\sum_{n=1}^{+\infty} |b_n|^{p'} \right)^{\frac{1}{p'}} < \delta + \varepsilon^{\frac{1}{p(\varepsilon)}} \left(\sum_{n=1}^{+\infty} |b_n|^{p'} \right)^{\frac{1}{p'}}.$$

From here we get that

$$\lim_{\varepsilon\to+0} \varepsilon^{\frac{1}{p(\varepsilon)}} \left(\sum_{n=1}^{+\infty} |a_n|^{p'(\varepsilon)} \right)^{\frac{1}{p'(\varepsilon)}} < \delta.$$

Therefore, due to the arbitrariness of the number δ we obtain equality (2.4).

Now let us establish 2) implies 3). Let us take an arbitrary number $\delta > 0$. Then there exists a number ε_0 such that for $\forall \varepsilon : \varepsilon < \varepsilon_0$ we have

$$\varepsilon^{\frac{1}{p(\varepsilon)}} \left(\sum_{n=1}^{+\infty} |a_n|^{p'(\varepsilon)} \right)^{\frac{1}{p'(\varepsilon)}} < \frac{\delta}{2}.$$

Since $a \in l_{p'(\varepsilon_0)}$, there exists a number $m_0 \in N$ such that for $\forall m > m_0$ we have

$$\left(\sum_{n=m+1}^{+\infty} |a_n|^{p'(\varepsilon_0)} \right)^{\frac{1}{p'(\varepsilon_0)}} < \frac{\delta}{2(p-1)}.$$

Thus, for $\forall m > m_0$ we have

$$\sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p(\varepsilon)}} \left(\sum_{n=m+1}^{+\infty} |a_n|^{p'(\varepsilon)} \right)^{\frac{1}{p'(\varepsilon)}}$$

$$\le \sup_{0<\varepsilon<\varepsilon_0} \varepsilon^{\frac{1}{p(\varepsilon)}} \left(\sum_{n=m+1}^{+\infty} |a_n|^{p'(\varepsilon)} \right)^{\frac{1}{p'(\varepsilon)}} + \sup_{\varepsilon_0\le\varepsilon<p-1} \varepsilon^{\frac{1}{p(\varepsilon)}} \left(\sum_{n=m+1}^{+\infty} |a_n|^{p'(\varepsilon)} \right)^{\frac{1}{p'(\varepsilon)}}$$

$$< \frac{\delta}{2} + \sup_{\varepsilon_0\le\varepsilon<p-1} \varepsilon^{\frac{1}{p(\varepsilon)}} \left(\sum_{n=m+1}^{+\infty} |a_n|^{p'(\varepsilon_0)} \right)^{\frac{1}{p'(\varepsilon_0)}} < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

From the last relation it follows that (2.5) is true.

Finally, let us establish 3) implies 1). Let $\delta > 0$ be an arbitrary number. Then there exists a number $m_0 \in N$ such that for $\forall m > m_0$ we have

$$\sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p(\varepsilon)}} \left(\sum_{n=m+1}^{+\infty} |a_n|^{p'(\varepsilon)} \right)^{\frac{1}{p'(\varepsilon)}} < \delta.$$

Let us set $b_m = (a_1, a_2, ..., a_m, 0, 0, ...)$. For $\forall m > m_0$ we have

$$\|a - b_m\|_{l_{p)}} = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p(\varepsilon)}} \left(\sum_{n=m+1}^{+\infty} |a_n|^{p'(\varepsilon)} \right)^{\frac{1}{p'(\varepsilon)}} < \delta,$$

i.e. the inclusion $a \in g_{p)}$ holds. The theorem is proved.

Theorem 2.2 *The system* $e_n = \{\delta_{nm}\}_{m\in N}$, $n \in N$ *forms a basis in the space* $g_{p)}$.

Proof. Take an arbitrary sequence $a \in g_{p)}$. For any $m \in N$ we have

$$\|a - \sum_{k=1}^{m} a_k e_k\|_{l_{p)}} = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p(\varepsilon)}} \left(\sum_{n=m+1}^{+\infty} |a_n|^{p'(\varepsilon)} \right)^{\frac{1}{p'(\varepsilon)}} .$$

Hence, by Theorem 2.1, we obtain the decomposition $a = \sum_{k=1}^{+\infty} a_k e_k$. The uniqueness of the decomposition is obvious. The theorem is proved.

Remark 2.1 It follows from the proven theorem that $g_{p)}$ is a separable subspace of the space $l_{p)}$.

3 Analogue of the Hausdorff-Young theorem in grand spaces of sequences

Let $L_p(a, b)$, $p \geq 1$, be the Lebesgue space of measurable on $[a, b]$ functions f with finite norm

$$\|f\|_{L_p} = \left(\int_a^b |f(t)|^p dt \right)^{\frac{1}{p}} .$$

Suppose that $\{\varphi_n\}_{n\in N}$ is an orthonormal sequence of measurable functions such that almost everywhere $|\varphi_n(t)| \leq M < +\infty$, $n \in N$. It is known ([21, Theorem 2.8, p. 154]) that a generalization of the Hausdorff-Young theorem in spaces $L_p(a, b)$ is the Riesz theorem, which states that if $f \in L_p(a, b)$, $1 < p \leq 2$ and $a_n = \int_a^b f(t)\varphi_n(t)dt$, $n \in N$ then $\{a_n\}_{n\in N} \in l_{p'}$ and the inequality

$$\| \{a_n\}_{n\in N} \|_{l_{p'}} \leq M^{\frac{2-p}{p}} \|f\|_{L_p},$$

holds, and if $\{a_n\}_{n\in N} \in l_p$, $1 < p \leq 2$, then $\exists f \in L_{p'}(a, b)$, such that $a_n = \int_a^b f(t)\varphi_n(t)dt$, $n \in N$ and the inequality

$$\|f\|_{L_{p'}} \leq M^{\frac{2-p}{p}} \| \{a_n\}_{n\in N} \|_{l_p}$$

holds.

Let $p > 1$, $L_{p)}(a, b)$ be a grand Lebesgue space, i.e. the space of measurable on $[a, b]$ functions f with finite norm

$$\|f\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{b-a} \int_a^b |f(t)|^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}} .$$

The space $L_{p)}(a,b)$ is a non-reflexive and non-separable Banach function space. The connection of these spaces with Lebesgue spaces is expressed by the following continuous embedding

$$L_p(a,b) \subset L_{p)}(a,b) \subset L_{p-\varepsilon}(a,b),\ \ \varepsilon \in (0,p-1).$$

The following theorem establishes an analogue of the Hausdorff-Young theorem in grand sequence spaces.

Theorem 3.1 *The following statements are true:*

1) if $f \in L_{p)}(a,b)$, $1 < p \leq 2$ and $a_n = \int\limits_a^b f(t)\varphi_n(t)dt$, $n \in N$, then $\{a_n\}_{n\in N} \in l_{p)}$ and we have

$$\| \{a_n\}_{n\in N} \|_{l_{p)}} \leq M_1 \|f\|_{p)}, \tag{3.1}$$

where M_1 does not depend on f;

2) if $\{\varphi_n\}_{n\in N}$ is total and $\{a_n\}_{n\in N} \in l_{p)}$, $p > 2$, then $\exists f \in L_{p)}(a,b)$ such that $a_n = \int\limits_a^b f(t)\varphi_n(t)dt$, $n \in N$ and we have

$$\|f\|_{p)} \leq M_2 \| \{a_n\}_{n\in N} \|_{l_{p)}}, \tag{3.2}$$

where M_2 does not depend on $\{a_n\}_{n\in N}$.

Proof. Let $f \in L_{p)}(a,b)$, $1 < p \leq 2$. Take an arbitrary number $\varepsilon \in (0,p-1)$. Then $f \in L_{p-\varepsilon}(a,b)$ and $1 < p-\varepsilon < 2$. Therefore, by the Riesz theorem, the relation

$$\left(\sum_{n=1}^{+\infty} |a_n|^{p'(\varepsilon)}\right)^{\frac{1}{p'(\varepsilon)}} \leq M^{\frac{2-p+\varepsilon}{p-\varepsilon}} \left(\int\limits_a^b |f(t)|^{p-\varepsilon} dt\right)^{\frac{1}{p-\varepsilon}}$$

holds. From this we obtain that

$$\varepsilon^{\frac{1}{p-\varepsilon}} \left(\sum_{n=1}^{+\infty} |a_n|^{p'(\varepsilon)}\right)^{\frac{1}{p'(\varepsilon)}} \leq \left((b-a)M^{2-p+\varepsilon}\right)^{\frac{1}{p-\varepsilon}} \left(\frac{\varepsilon}{b-a}\int\limits_a^b |f(t)|^{p-\varepsilon} dt\right)^{\frac{1}{p-\varepsilon}}.$$

Let $M_1 = \sup\limits_{\varepsilon\in(0,p-1)} \left((b-a)M^{2-p+\varepsilon}\right)^{\frac{1}{p-\varepsilon}}$. From the last relation we obtain

$$\varepsilon^{\frac{1}{p-\varepsilon}} \left(\sum_{n=1}^{+\infty} |a_n|^{p'(\varepsilon)}\right)^{\frac{1}{p'(\varepsilon)}} \leq M_1 \left(\frac{\varepsilon}{b-a}\int\limits_a^b |f(t)|^{p-\varepsilon} dt\right)^{\frac{1}{p-\varepsilon}}.$$

Passing here to the upper bound on the possible values of $\varepsilon \in (0,p-1)$ we obtain the required inequality (3.1).

Let $\{\varphi_n\}_{n\in N}$ now be total and $\{a_n\}_{n\in N} \in l_{p)}$, $p > 2$. Then there exists a number $\varepsilon_0 \in (0,p-1)$ such that for any $\varepsilon \in (0,\varepsilon_0]$ we have $p'(\varepsilon) \leq 2$. Then, according to the convergence of the series $\sum_{n=1}^{+\infty}|a_n|^{p'(\varepsilon)}$ by the Riesz theorem $\exists f \in L_{p-\varepsilon}(a,b)$, $a_n = \int_a^b f(t)\varphi_n(t)dt$, $n \in N$ and the inequality

$$\left(\int_a^b |f(t)|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}} \leq M^{\frac{2-p'(\varepsilon)}{p'(\varepsilon)}} \left(\sum_{n=1}^{+\infty}|a_n|^{p'(\varepsilon)}\right)^{\frac{1}{p'(\varepsilon)}}, \ \ \varepsilon \in (0,\varepsilon_0].$$

Note that here the non-dependence of the function f on $\varepsilon \in (0,\varepsilon_0]$ follows from the condition of totality of the system $\{\varphi_n\}_{n\in N}$. Let us multiply both parts of the last inequality by $\left(\frac{\varepsilon}{b-a}\right)^{\frac{1}{p-\varepsilon}}$. We have

$$\left(\frac{\varepsilon}{b-a}\int_a^b |f(t)|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}} \leq (b-a)^{-\frac{1}{p(\varepsilon)}} M^{\frac{2-p'(\varepsilon)}{p'(\varepsilon)}} \varepsilon^{\frac{1}{p-\varepsilon}} \left(\sum_{n=1}^{+\infty}|a_n|^{p'(\varepsilon)}\right)^{\frac{1}{p'(\varepsilon)}}$$

$$\leq C_1 \| \{a_n\}_{n\in N} \|_{l_{p)}}, \ \ \varepsilon \in (0,\varepsilon_0]. \tag{3.3}$$

Further, for $\varepsilon \in (\varepsilon_0, p-1)$ using Holder's inequality with the exponent $\frac{p(\varepsilon_0)}{p(\varepsilon)}$ we get

$$\left(\int_a^b |f(t)|^{p(\varepsilon)}dt\right)^{\frac{1}{p(\varepsilon)}} \leq \left(\int_a^b |f(t)|^{p(\varepsilon_0)}dt\right)^{\frac{1}{p(\varepsilon_0)}} (b-a)^{\frac{\varepsilon-\varepsilon_0}{p(\varepsilon_0)p(\varepsilon)}}.$$

Consequently, for $\varepsilon \in (\varepsilon_0, p-1)$ we get

$$\left(\frac{\varepsilon}{b-a}\int_a^b |f(t)|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}} \leq \varepsilon^{\frac{1}{p(\varepsilon)}} \varepsilon_0^{-\frac{1}{p(\varepsilon_0)}} \left(\frac{\varepsilon_0}{b-a}\int_a^b |f(t)|^{p(\varepsilon_0)}dt\right)^{\frac{1}{p(\varepsilon_0)}}$$

$$\leq (b-a)^{-\frac{1}{p(\varepsilon)}} \varepsilon^{\frac{1}{p(\varepsilon)}} \varepsilon_0^{-\frac{1}{p(\varepsilon_0)}} C_1 \| \{a_n\}_{n\in N} \|_{p)} \leq C_2 \| \{a_n\}_{n\in N} \|_{p)}. \tag{3.4}$$

So, using (3.3) and (3.4), we obtain

$$\|f\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{b-a}\right)^{\frac{1}{p-\varepsilon}} \|f\|_{p-\varepsilon} \leq M_2 \| \{a_n\}_{n\in N} \|_{p)},$$

where $M_2 = 2\max\{C_1, C_2\}$, i.e. inequality (3.2) is true.

The theorem is proved.

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