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Commutator of Marcinkiewicz integral on total mixed Morrey spaces

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Abstract. *In this paper, we study the boundedness of the Marcinkiewicz operator* μ_Ω *and its commutator* $\mu_{b,\Omega}$ *on total mixed Morrey spaces* $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$.

Keywords. Total mixed Morrey spaces, Marcinkiewicz operator, commutators, BMO.

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1 Introduction

In 1961, Benedek and Panzone [7] introduced Lebesgue spaces $L^{\vec{p}}$ with mixed norm over Euclifean spaces, which extend Lebesgue spaces and their related properties. In 1975, Bagby [6] investigated the boundedness of the Hardy-Littlewood maximal operator for functions taking values in spaces $l^{\vec{p}}(\mathbb{R}^n)$. Since then, manu papers focus various mixed norm spaces and the bounded properties of integral operators on spaces with mixed norm. In 2019, Nogayama [22,23] considered a new Morrey space, with the L^p norm replaced by the mixed Lebesgue norm $L^{\vec{p}}(\mathbb{R}^n)$, which is call mixed Morrey spaces.

Classical Morrey spaces $L^{p,\lambda}$ were originally introduced by Morrey in [21] to study the local behavior of solutions of second-order elliptic partial differential equations. In 2022, Guliyev [12] introduced a variant of Morrey spaces called total Morrey spaces $L^{p,\lambda,\mu}(\mathbb{R}^n)$,

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$0 < p < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. Total Morrey spaces generalize the classical Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$ so that $L^{p,\lambda,\lambda}(\mathbb{R}^n) \equiv L^{p,\lambda}(\mathbb{R}^n)$ and the modified Morrey spaces $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ so that $L^{p,\lambda,0}(\mathbb{R}^n) = \widetilde{L}^{p,\lambda}(\mathbb{R}^n)$. Necessary and sufficient conditions for the boundedness of the maximal commutator operator M_b and the commutator of the maximal operator $[b, M]$ on $L^{p,\lambda,\mu}(\mathbb{R}^n)$ when b belongs to the spaces $BMO(\mathbb{R}^n)$, are given in [12, Theorems 3 and 4], see also [9, 14–16, 24, 25].

In [16], the authors consider the total mixed Morrey spaces $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ introduced by Guliyev in [12] in the case $\vec{p} = (p, \ldots, p)$. These spaces generalize mixed Morrey spaces so that $L^{\vec{p},\lambda,\lambda}(\mathbb{R}^n) \equiv L^{\vec{p},\lambda}(\mathbb{R}^n)$ and the modified mixed Morrey spaces so that $L^{\vec{p},\lambda,0}(\mathbb{R}^n) = \widetilde{L}^{\vec{p},\lambda}(\mathbb{R}^n)$. The main properties of the spaces $L^{\vec{p},\lambda,\lambda}(\mathbb{R}^n)$ were presented and some embeddings into the Morrey space $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ were studied. Necessary and sufficient conditions for the boundedness of the maximal commutator operator M_b and the commutator of the maximal operator $[b, M]$ on $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ were also presented. New characteristics for some subclasses of $BMO(\mathbb{R}^n)$ were obtained.

For $x \in \mathbb{R}^n$, and $r > 0$, let $B(x, r)$ be the open ball centered at x with the radius r, and $B^c(x, r)$ be its complement. Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ is the unit sphere of $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure. Suppose that Ω satisfies the following conditions.

(i) Ω is a homogeneous function of degree zero on $\mathbb{R}^n$. That is,

$$\Omega(tx) = \Omega(x) \tag{1.1}$$

for all $t > 0$ and $x \in \mathbb{R}^n$.

(ii) Ω has mean zero on S^{n-1}. That is,

$$\int_{S^{n-1}} \Omega(x')dx' = 0, \tag{1.2}$$

where $x' = x/|x|$ for any $x \neq 0$.

The Marcinkiewicz integral operator of higher dimension μ_Ω is defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_{\Omega,t}(f)(x) = \int_{B(x,t)} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

It is well known that the Littlewood-Paley g-function is a very important tool in harmonic analysis and the Marcinkiewicz integral is essentially a Littlewood-Paley g-function. In this paper, we will also consider the commutator $\mu_{\Omega,b}$ which is given by the following expression

$$\mu_{\Omega,b}f(x) = \left(\int_0^\infty |F^b_{\Omega,t}(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F^b_{\Omega,t}(x) = \int_{B(x,t)} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x) - b(y)] f(y)dy.$$

On the other hand, the study of Schrödinger operator $L = -\Delta + V$ recently attracted much attention. In particular, Shen [26] considered L^p estimates for Schrödinger operators L with certain potentials which include Schrödinger Riesz transforms $R^L_j = \frac{\partial}{\partial x_j} L^{-\frac{1}{2}}$, $j = 1, \ldots, n$. Then, Dziubański and Zienkiewicz [10] introduced the Hardy type space

$H_L^1(\mathbb{R}^n)$ associated with the Schrödinger operator L, which is larger than the classical Hardy space $H^1(\mathbb{R}^n)$, see also [2–5, 8, 13, 17, 18].

Similar to the classical Marcinkiewicz function, we define the Marcinkiewicz functions $\mu_{j,\Omega}$ associated with the Schrödinger operator L by

$$\mu_{j,\Omega}^L f(x) = \left(\int_0^\infty \left| \int_{B(x,t)} |\Omega(x-y)| K_j^L(x,y) f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2},$$

where $K_j^L(x,y) = \widetilde{K_j^L}(x,y)|x-y|$ and $\widetilde{K_j^L}(x,y)$ is the kernel of $R_j = \frac{\partial}{\partial x_j} L^{-\frac{1}{2}}$, $j = 1, \ldots, n$. In particular, when $V = 0$, $K_j^{\Delta}(x,y) = \widetilde{K_j^{\Delta}}(x,y)|x-y| = \frac{(x-y)_j/|x-y|}{|x-y|^{n-1}}$ and $\widetilde{K_j^{\Delta}}(x,y)$ is the kernel of $R_j = \frac{\partial}{\partial x_j} \Delta^{-\frac{1}{2}}$, $j = 1, \ldots, n$. In this paper, we write $K_j(x,y) = K_j^{\Delta}(x,y)$ and

$$\mu_{j,\Omega} f(x) = \left(\int_0^\infty \left| \int_{B(x,t)} |\Omega(x-y)| K_j(x,y) f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2}.$$

Obviously, $\mu_{j,\Omega} f$ are classical Marcinkiewicz functions with rough kernel. Therefore, it will be an interesting to study the properties of the operator $\mu_{j,\Omega}^L$. The main goal of this paper is to show that Marcinkiewicz operators with rough kernel associated with the Schrödinger operators $\mu_{j,\Omega}^L$, $j = 1, \ldots, n$, are bounded on the total mixed Morrey space $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$, $1 < \vec{p} < \infty$, $0 \le \lambda \le n$, $0 \le \mu \le n$.

The commutator of the classical Marcinkiewicz function with rough kernel is defined by

$$\mu_{j,\Omega,b} f(x) = \left(\int_0^\infty \left| \int_{B(x,t)} |\Omega(x-y)| K_j(x,y) [b(x) - b(y)] f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2}.$$

The commutator $\mu_{j,\Omega,b}^L$ formed by $b \in BMO(\mathbb{R}^n)$ and the Marcinkiewicz function with rough kernel $\mu_{j,\Omega}^L$ is defined by

$$\mu_{j,\Omega,b}^L f(x) = \left(\int_0^\infty \left| \int_{B(x,t)} |\Omega(x-y)| K_j^L(x,y) [b(x) - b(y)] f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2}.$$

The well-known classical Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy,$$

where $f \in L_{loc}^1(\mathbb{R}^n)$ and $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$. As we know, the Hardy-Littlewood maximal operator M is bounded on $L^{\vec{p}}(\mathbb{R}^n)$, $1 < \vec{p} < \infty$ (see [22, 23]), but there is no complete boundedness results for some other operators on the mixed Lebesgue spaces.

We find the conditions with $b \in BMO(\mathbb{R}^n)$ which ensures the boundedness of the operators $\mu_{j,\Omega,b}^L$, $j = 1, \ldots, n$ on total mixed Morrey space $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$, $1 < \vec{p} < \infty$, $0 \le \lambda \le n$, $0 \le \mu \le n$.

By $A \lesssim B$, we mean that $A \le CB$ for some constant $C > 0$, and $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2 Definitions and preliminaries

For any $r > 0$ and $x \in \mathbb{R}^n$, let $B(x,r) = \{y : |y-x| < r\}$ be the ball centered at x with radius r. Let $\mathrm{B} = \{B(x,r) : x \in \mathbb{R}^n, r > 0\}$ be the set of all such balls. We also use χ_E and $|E|$ to denote the characteristic function and the Lebesgue measure of a measurable set E.

We first recall the definition of mixed Lebesgue space defined in [7].

Let $\vec{p} = (p_1, \cdots, p_n) \in (0, \infty]^n$. Then the mixed Lebesgue norm $\|\cdot\|_{L^{\vec{p}}}$ or $\|\cdot\|_{L^{(p_1,\ldots,p_n)}}$ is defined by

$$\begin{aligned}\|f\|_{L^{\vec{p}}} &= \|f\|_{L^{(p_1,\ldots,p_n)}}\\ &= \Big(\int_{\mathbb{R}} \cdots \Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |f(x_1, x_2, \ldots, x_n)|^{p_1} dx_1\Big)^{\frac{p_2}{p_1}} dx_2\Big)^{\frac{p_3}{p_2}} \ldots dx_n\Big)^{\frac{1}{p_n}}\end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{C}$ is a measurable function. If $p_j = \infty$ for some $j = 1, n$, then we have to make appropriate modifications. We define the mixed Lebesgue space $L^{\vec{p}}(\mathbb{R}^n) = L^{(p_1,\cdots,p_n)}(\mathbb{R}^n)$ to be the set of all locally integrable functions f with $\|f\|_{L^{\vec{p}}} < \infty$.

Definition 2.1 *Let $0 < \vec{p} < \infty$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $[t]_1 = \min\{1, t\}$, $t > 0$. We denote by $L^{\vec{p},\lambda}(\mathbb{R}^n)$ the mixed Morrey space [23], by $\widetilde{L}^{\vec{p},\lambda}(\mathbb{R}^n)$ the modified mixed Morrey space [11], and by $L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ the total mixed Morrey space the set of all classes of locally integrable functions f with the finite norms*

$$\begin{aligned}\|f\|_{L^{\vec{p},\lambda}} &= \sup_{x\in\mathbb{R}^n,\, t>0} t^{-\frac{\lambda}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} \|f\|_{L^{\vec{p}}(B(x,t))},\\ \|f\|_{\widetilde{L}^{\vec{p},\lambda}} &= \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} \|f\|_{L^{\vec{p}}(B(x,t))},\\ \|f\|_{L^{\vec{p},\lambda,\mu}} &= \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} [1/t]_1^{\frac{\mu}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} \|f\|_{L^{\vec{p}}(B(x,t))},\end{aligned}$$

respectively.

Definition 2.2 *Let $0 < \vec{p} < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. We define the weak mixed Morrey space $WL^{\vec{p},\lambda}(\mathbb{R}^n)$ [23], the weak modified mixed Morrey space $W\widetilde{L}^{\vec{p},\lambda}(\mathbb{R}^n)$ [11] and the weak total mixed Morrey space $WL^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$ as the set of all locally integrable functions f with finite norms*

$$\begin{aligned}\|f\|_{WL^{\vec{p},\lambda}} &= \sup_{x\in\mathbb{R}^n,\, t>0} t^{-\frac{\lambda}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} \|f\|_{WL^{\vec{p}}(B(x,t))},\\ \|f\|_{W\widetilde{L}^{\vec{p},\lambda}} &= \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} \|f\|_{WL^{\vec{p}}(B(x,t))},\\ \|f\|_{WL^{\vec{p},\lambda,\mu}} &= \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} [1/t]_1^{\frac{\mu}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} \|f\|_{WL^{\vec{p}}(B(x,t))},\end{aligned}$$

respectively.

Note that

$$L^{\vec{p},0,0}(\mathbb{R}^n) = \widetilde{L}^{\vec{p},0}(\mathbb{R}^n) = L^{\vec{p},0}(\mathbb{R}^n) = L^{\vec{p}}(\mathbb{R}^n),$$
$$WL^{\vec{p},0,0}(\mathbb{R}^n) = W\widetilde{L}^{\vec{p},0}(\mathbb{R}^n) = WL^{\vec{p},0}(\mathbb{R}^n) = WL^{\vec{p}}(\mathbb{R}^n),$$
$$L^{\vec{p},\lambda,\lambda}(\mathbb{R}^n) = L^{\vec{p},\lambda}(\mathbb{R}^n), \quad L^{\vec{p},\lambda,0}(\mathbb{R}^n) = \widetilde{L}^{\vec{p},\lambda}(\mathbb{R}^n),$$
$$\|f\|_{WL^{\vec{p},\lambda,\mu}} \le \|f\|_{L^{\vec{p},\lambda,\mu}} \text{ and therefore } L^{\vec{p},\lambda,\mu}(\mathbb{R}^n) \subset WL^{\vec{p},\lambda,\mu}(\mathbb{R}^n)$$

and

$$L^{\vec{p},\lambda,\mu}(\mathbb{R}^n) \subset_{\succ} L^{\vec{p},\lambda}(\mathbb{R}^n),\ \mu \le \lambda \text{ and } \|f\|_{L^{\vec{p},\lambda}} \le \|f\|_{L^{\vec{p},\lambda,\mu}},$$
$$L^{\vec{p},\lambda,\mu}(\mathbb{R}^n) \subset_{\succ} L^{\vec{p},\mu}(\mathbb{R}^n),\ \mu \le \lambda \text{ and } \|f\|_{L^{\vec{p},\mu}} \le \|f\|_{L^{\vec{p},\lambda,\mu}}$$
$$\widetilde{L}^{\vec{p},\lambda}(\mathbb{R}^n) \subset_{\succ} L^{\vec{p}}(\mathbb{R}^n) \text{ and } \|f\|_{L^{\vec{p}}} \le \|f\|_{\widetilde{L}^{\vec{p},\lambda}}$$

and if $\lambda < 0$ or $\lambda > n$, then

$$L^{\vec{p},\lambda}(\mathbb{R}^n) = \widetilde{L}^{\vec{p},\lambda}(\mathbb{R}^n) = WL^{\vec{p},\lambda}(\mathbb{R}^n) = W\widetilde{L}^{\vec{p},\lambda}(\mathbb{R}^n) = \Theta,$$

where $\Theta \equiv \Theta(\mathbb{R}^n)$ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

Lemma 2.1 *[16] If $0 < \vec{p} < \infty$, $0 \le \mu \le \lambda \le n$, then*

$$L^{\vec{p},\lambda,\mu}(\mathbb{R}^n) = L^{\vec{p},\lambda}(\mathbb{R}^n) \cap L^{\vec{p},\mu}(\mathbb{R}^n)$$

and

$$\|f\|_{L^{\vec{p},\lambda,\mu}(\mathbb{R}^n)} = \max\left\{\|f\|_{L^{\vec{p},\lambda}(\mathbb{R}^n)}, \|f\|_{L^{\vec{p},\mu}(\mathbb{R}^n)}\right\}.$$

Lemma 2.2 *[16] If $0 < \vec{p} < \infty$, $0 \le \mu \le \lambda \le n$, then*

$$WL^{\vec{p},\lambda,\mu}(\mathbb{R}^n) = WL^{\vec{p},\lambda}(\mathbb{R}^n) \cap WL^{\vec{p},\mu}(\mathbb{R}^n)$$

and

$$\|f\|_{WL^{\vec{p},\lambda,\mu}(\mathbb{R}^n)} = \max\left\{\|f\|_{WL^{\vec{p},\lambda}}, \|f\|_{WL^{\vec{p},\mu}}\right\}.$$

Remark 2.1 If $0 < \vec{p} < \infty$, and $\mu < 0$ or $\lambda > n$, then

$$L^{\vec{p},\lambda,\mu}(\mathbb{R}^n) = WL^{\vec{p},\lambda,\mu}(\mathbb{R}^n) = \Theta(\mathbb{R}^n).$$

3 Marcinkiewicz operator μ_Ω in total mixed Morrey spaces

In this section, we investigate the boundedness of Marcinkiewicz operator μ_Ω satisfies the conditions (1.1), (1.2) and $\Omega \in L^\infty(S^{n-1})$ on the total mixed Morrey space $L^{\vec{p},\lambda,\mu}$.

We first use one lemma, which give us the explicit estimates for the $L^{\vec{p}}(\mathbb{R}^n)$ norm of μ_Ω on a given ball $B(x_0, r)$.

Lemma 3.1 *[1, Lemma 3.1] Let Ω be satisfies the conditions* (1.1), (1.2) *and* $\Omega \in L^\infty(S^{n-1})$.
Then for $1 < \vec{p} < \infty$, the inequality

$$\|\mu_\Omega f\|_{L^{\vec{p}}(B(x_0,r))} \lesssim r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{2r}^\infty t^{-1-\sum_{i=1}^n \frac{1}{p_i}} \|f\|_{L^{\vec{p}}(B(x_0,t))}\, dt \tag{3.1}$$

holds for any ball $B(x_0, r)$ and all $f \in L^{\vec{p}}_{loc}(\mathbb{R}^n)$.

Now we can present the first main result in this section.

Theorem 3.1 *Let Ω be satisfies the conditions* (1.1), (1.2) *and $\Omega \in L^{\infty}(S^{n-1})$. Let also $1 < \vec{p} < \infty$, $0 \leq \lambda \leq n$ and $0 \leq \mu \leq n$. Then the operator μ_Ω is bounded on $L^{\vec{p},\lambda,\mu}$ Moreover,*

$$\|\mu_\Omega f\|_{L^{\vec{p},\lambda,\mu}} \leq \|f\|_{L^{\vec{p},\lambda,\mu},\mu}.$$

Proof. From the inequality (3.1) we get

$$\begin{aligned}
\|\mu_\Omega f\|_{L^{\vec{p},\lambda,\mu}} &= \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\frac{\lambda}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} [1/r]_1^{\frac{\mu}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} \|\mu_\Omega f\|_{L^{\vec{p}}(B(x,r))} \\
&\lesssim \sup_{x\in\mathbb{R}^n, r>0} [r]_1^{-\frac{\lambda}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} [1/r]_1^{\frac{\mu}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{2r}^{\infty} t^{-1-\sum_{i=1}^n \frac{1}{p_i}} \|f\|_{L^{\vec{p}}(B(x_0,t))}\, dt \\
&\lesssim \|f\|_{L^{\vec{p},\lambda,\mu}} \sup_{x\in\mathbb{R}^n, r>0} [r]_1^{-\frac{\lambda}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} [1/r]_1^{\frac{\mu}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} r^{\sum_{i=1}^n \frac{1}{p_i}} \\
&\times \int_r^{\infty} t^{-\sum_{i=1}^n \frac{1}{p_i}} [t]_1^{\frac{\lambda}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} [1/t]_1^{-\frac{\mu}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} \frac{dt}{t} \\
&\lesssim \|f\|_{L^{\vec{p},\lambda,\mu}} \sup_{x\in\mathbb{R}^n, r>0} [r]_1^{\frac{n-\lambda}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} [1/r]_1^{-\frac{n-\mu}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} \\
&\times \int_r^{\infty} [t]_1^{-\frac{n-\lambda}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} [1/t]_1^{\frac{n-\mu}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} \frac{dt}{t} \\
&= \|f\|_{L^{\vec{p},\lambda,\mu}} \int_1^{\infty} [t]_1^{-\frac{n-\lambda}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} [1/t]_1^{\frac{n-\mu}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} \frac{dt}{t} \\
&\lesssim \|f\|_{L^{\vec{p},\lambda,\mu}}.
\end{aligned}$$

By taking $\vec{p} = (p, \ldots, p)$ in Theorem 3.1, we obtain the boundedness of μ_Ω on the Morrey spaces.

4 Commutator of Marcinkiewicz operator $\mu_{\Omega,b}$ in total mixed Morrey spaces

In this section, we investigate the boundedness of commutator of Marcinkiewicz operator $\mu_{\Omega,b}$ satisfies the conditions (1.1), (1.2) and $\Omega \in L^{\infty}(S^{n-1})$ on the total mixed Morrey space $L^{\vec{p},\lambda,\mu}$. First, we review the definition of $BMO(\mathbb{R}^n)$, the bounded mean oscillation space. A function $f \in L^1_{loc}(\mathbb{R}^n)$ belongs to the bounded mean oscillation space $BMO(\mathbb{R}^n)$ if

$$\|f\|_{BMO} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}| dy < \infty. \tag{4.1}$$

If one regards two functions whose difference is a constant as one, then the space $BMO(\mathbb{R}^n)$ is a Banach space with respect to norm $\|.\|_{BMO}$. The John-Nirenberg ineqalitiy for BMO yields that for any $1 < q < \infty$ and $f \in BMO(\mathbb{R}^n)$, the BMO norm of f is equivalent to

$$\|f\|_{BMO^q} = \sup_{x\in\mathbb{R}^n, r>0} \Big(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}|^q dy \Big)^{\frac{1}{q}}$$

Recall that for any $\vec{p} = (p_1, \cdots, p_n) \in (1, \infty)^n$, the John-Nirenberg inequality for mixed norm space [19,20] shows that the BMO norm of all $f \in BMO(\mathbb{R}^n)$ is also equivalent to

$$\|f\|_{BMO^{\vec{p}}} = \sup_{x\in\mathbb{R}^n, r>0} \frac{\|(f - f_{B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{p}}}}{\|\chi_{B(x,r)}\|_{L^{\vec{p}}}}. \tag{4.2}$$

The following property for BMO functions is valid.

Lemma 4.1 *Let $f \in BMO(\mathbb{R}^n)$. Then for all $0 < 2r < t$, we have*

$$|f_{B(x,r)} - f_{B(x,t)}| \lesssim \|f\|_{BMO} \ln \frac{t}{r}. \tag{4.3}$$

We use one lemma, which give us the explicit estimates for the $L^{\vec{p}}(\mathbb{R}^n)$ norm of $\mu_{\Omega,b}$ on a given ball $B(x_0, r)$.

Lemma 4.2 *[1, Lemma 4.2] Let Ω be satisfies the conditions* (1.1), (1.2) *and $\Omega \in L^\infty(S^{n-1})$. Let also $1 < \vec{p} < \infty$ and $b \in BMO(\mathbb{R}^n)$. Then the inequality*

$$\begin{aligned} &\|\mu_{\Omega,b} f\|_{L^{\vec{p}}(B(x_0,r))} \\ &\lesssim \|b\|_{BMO}\, r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{2r}^{\infty} \left(1 + \ln\frac{t}{r}\right) t^{-1-\sum_{i=1}^n \frac{1}{p_i}} \|f\|_{L^{\vec{p}}(B(x_0,t))} dt \end{aligned} \tag{4.4}$$

holds for any ball $B(x_0, r)$ and all $f \in L^{\vec{p}}_{loc}(\mathbb{R}^n)$.

Now we give the boundedness of $\mu_{\Omega,b}$ on the mixed Morrey space.

Theorem 4.1 *Let Ω be satisfies the conditions* (1.1), (1.2) *and $\Omega \in L^\infty(S^{n-1})$. Let also $1 < \vec{p} < \infty$, $b \in BMO(\mathbb{R}^n)$, $0 \le \lambda \le n$ and $0 \le \mu \le n$. Then the operator $\mu_{\Omega,b}$ is bounded on $L^{\vec{p},\lambda,\mu}$ Moreover,*

$$\|\mu_{\Omega,b} f\|_{L^{\vec{p},\lambda,\mu}} \le \|b\|_{BMO} \|f\|_{L^{\vec{p},\lambda,\mu}}.$$

Proof. From the inequality (4.4) we get

$$\begin{aligned}
&\|\mu_{\Omega,b} f\|_{L^{\vec{p},\lambda,\mu}} = \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\frac{\lambda}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} [1/r]_1^{\frac{\mu}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} \|\mu_{\Omega,b} f\|_{L^{\vec{p}}(B(x,r))} \\
&\lesssim \|b\|_{BMO} \sup_{x\in\mathbb{R}^n, r>0} [r]_1^{-\frac{\lambda}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} [1/r]_1^{\frac{\mu}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} r^{\sum_{i=1}^n \frac{1}{p_i}} \\
&\times \int_{2r}^{\infty} \left(1 + \ln\frac{t}{r}\right) t^{-1-\sum_{i=1}^n \frac{1}{p_i}} \|f\|_{L^{\vec{p}}(B(x_0,t))}\, dt \\
&\lesssim \|b\|_{BMO} \|f\|_{L^{\vec{p},\lambda,\mu}} \sup_{x\in\mathbb{R}^n, r>0} [r]_1^{-\frac{\lambda}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} [1/r]_1^{\frac{\mu}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} r^{\sum_{i=1}^n \frac{1}{p_i}} \\
&\times \int_{r}^{\infty} \left(1 + \ln\frac{t}{r}\right) t^{-\sum_{i=1}^n \frac{1}{p_i}} [t]_1^{\frac{\lambda}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} [1/t]_1^{-\frac{\mu}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} \frac{dt}{t} \\
&\lesssim \|b\|_{BMO} \|f\|_{L^{\vec{p},\lambda,\mu}} \sup_{x\in\mathbb{R}^n, r>0} [r]_1^{\frac{n-\lambda}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} [1/r]_1^{-\frac{n-\mu}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} \\
&\times \int_{r}^{\infty} \left(1 + \ln\frac{t}{r}\right) [t]_1^{-\frac{n-\lambda}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} [1/t]_1^{\frac{n-\mu}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} \frac{dt}{t} \\
&= \|b\|_{BMO} \|f\|_{L^{\vec{p},\lambda,\mu}} \int_1^{\infty} (1 + \ln t)\, [t]_1^{-\frac{n-\lambda}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} [1/t]_1^{\frac{n-\mu}{n}\left(\sum_{i=1}^n \frac{1}{p_i}\right)} \frac{dt}{t} \\
&\lesssim \|b\|_{BMO} \|f\|_{L^{\vec{p},\lambda,\mu}}.
\end{aligned}$$

By taking $\vec{p} = (p, \dots, p)$ in Theorem 4.1, we obtain the boundedness of $\mu_{\Omega,b}$ on the total Morrey spaces.

5 Marcinkiewicz operators with rough kernel associated with the Schrödinger operators $\mu^L_{j,\Omega}$ and its commutator $\mu^L_{j,\Omega,b}$ in total mixed Morrey spaces

Let us consider the Schrödinger operator

$$L = -\Delta + V \ \text{ on } \ \mathbb{R}^n, \ n \geq 3,$$

where V is a non-negative, $V \neq 0$, and belongs to the reverse Hölder class B_q for some $q \geq n/2$, i.e., there exists a constant $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} V^q(y)dy\right)^{1/q} \leq \frac{C}{|B(x,r)|}\int_{B(x,r)} V(y)dy$$

holds for every $x \in \mathbb{R}^n$ and $0 < r < 1$, where $B(x,r)$ denotes the ball centered at x with radius r. In particular, if V is a nonnegative polynomial, then $V \in B_1$.

Obviously, $B_{q_2} \subset B_{q_1}$, if $q_2 > q_1$. The most important property of the class B_q is its self-improvement, that is, if $V \in B_q$, then $V \in B_{q+\epsilon}$ for some $\epsilon > 0$.

In this section, we prove the boundedness of the Marcinkiewicz operators with rough kernel associated with the Schrödinger operators $\mu^L_{j,\Omega}$ and its commutator $\mu^L_{j,\Omega,b}$ on total mixed Morrey space $L^{\vec{p},\lambda,\mu}$.

For $x \in \mathbb{R}^n$, the function $m_V(x)$ is defined by

$$\rho(x) = \sup_{r>0}\left\{r : \frac{1}{r^{n-2}}\int_{B(x,r)} V\ (y)dy \ \leq 1\right\}.$$

Lemma 5.1 *[26] Let $V \in B_q$ with $q \geq n/2$. Then there exists $l_0 > 0$ such that*

$$\frac{l}{C}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-l_0} \leq \frac{\rho(y)}{\rho(x)} \leq C\left(1+\frac{|x-y|}{\rho(x)}\right)^{l_0/(l_0+1)}.$$

In particular, $\rho(x) \sim \rho(y)$ if $|x-y| < C\rho(x)$.

Lemma 5.2 *[26] Let $V \in B_q$ with $q \geq n/2$. For any $l > 0$, there exists $C_l > 0$ such that*

$$\left|K^L_j(x,y)\right| \leq \frac{C_l}{\left(1+\frac{|x-y|}{\rho(x)}\right)^l}\frac{1}{|x-y|^{n-1}},$$

and

$$\left|K^L_j(x,y) - K_j(x-y)\right| \leq \ C\frac{\rho(x)}{|x-y|^{n-2}}.$$

Analogously proof of Lemma 3.1 and Theorem 3.1 the following results is valid.

Lemma 5.3 *Let Ω be satisfies the conditions* (1.1), (1.2)*, $\Omega \in L^\infty(S^{n-1})$ and $V \in B_n$. Then for $1 < \vec{p} < \infty$, the inequality*

$$\|\mu^L_{j,\Omega}f\|_{L^{\vec{p}}(B(x_0,r))} \lesssim r^{\sum_{i=1}^n \frac{1}{p_i}}\int_{2r}^\infty t^{-1-\sum_{i=1}^n \frac{1}{p_i}}\,\|f\|_{L^{\vec{p}}(B(x_0,t))}\,dt$$

holds for any ball $B(x_0,r)$ and all $f \in L^{\vec{p}}_{loc}(\mathbb{R}^n)$.

Theorem 5.1 *Let Ω be satisfies the conditions* (1.1), (1.2), $\Omega \in L^\infty(S^{n-1})$ *and* $V \in B_n$. *Let also* $1 < \vec{p} < \infty$, $0 \le \lambda \le n$ *and* $0 \le \mu \le n$. *Then the operator* $\mu^L_{j,\Omega}$ *is bounded on* $L^{\vec{p},\lambda,\mu}$ *Moreover,*

$$\|\mu^L_{j,\Omega} f\|_{L^{\vec{p},\lambda,\mu}} \le \|f\|_{L^{\vec{p},\lambda,\mu}}.$$

Analogously proof of Lemma 4.2 and Theorem 4.1 the following results is valid.

Lemma 5.4 *Let Ω be satisfies the conditions* (1.1), (1.2), $\Omega \in L^\infty(S^{n-1})$ *and* $V \in B_n$. *Then for* $1 < \vec{p} < \infty$ *and* $b \in BMO(\mathbb{R}^n)$, *the inequality*

$$\|\mu^L_{j,\Omega,b} f\|_{L^{\vec{p}}(B(x_0,r))} \lesssim \|b\|_{BMO}\, r^{\sum_{i=1}^n \frac{1}{p_i}} \int_{2r}^\infty \big(1 + \ln \frac{t}{r}\big)\, t^{-1-\sum_{i=1}^n \frac{1}{p_i}}\, \|f\|_{L^{\vec{p}}(B(x_0,t))}\, dt$$

holds for any ball $B(x_0, r)$ *and all* $f \in L^{\vec{p}}_{loc}(\mathbb{R}^n)$.

Theorem 5.2 *Let Ω be satisfies the conditions* (1.1), (1.2), $\Omega \in L^\infty(S^{n-1})$ *and* $V \in B_n$. *Let also* $1 < \vec{p} < \infty$, $b \in BMO(\mathbb{R}^n)$, $0 \le \lambda \le n$ *and* $0 \le \mu \le n$. *Then the operator* $\mu^L_{j,\Omega,b}$ *is bounded on* $L^{\vec{p},\lambda,\mu}$ *Moreover,*

$$\|\mu^L_{j,\Omega,b} f\|_{L^{\vec{p},\lambda,\mu}} \le \|b\|_{BMO}\, \|f\|_{L^{\vec{p},\lambda,\mu}}.$$

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