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On strong solvability of one nonlocal boundary value problem for Poisson's equation in grand Sobolev space in rectangle

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Abstract. *We consider a nonlocal boundary value problem for the Poisson's equation on a rectangle in Sobolev spaces generated by the norm of the grand Lebesgue space. The concept of strong solvability of this problem is introduced and it's correct solvability is proved. The basis property of the eigen and associated functions of the corresponding spectral problem in separable grand Lebesgue spaces is proved, and this fact is used to establish correct solvability.*

Keywords. Poisson's equation, nonlocal problem, grand Lebesgue space, strong solvability.

Mathematics Subject Classification (2010): 35A01, 35J05, 35K05

1 Introduction

The theory of the strong and weak solvability of linear elliptic equations in the Sobolev spaces is well developed and can be found in the classical monographs. In spite of this, a lot of problems, arising in the mechanics and the mathematical physics do not fit to this theory. An example, of such a problem is the following degenerate elliptic equation, studied by Moiseev in [14] (see, also [10])

Consider the following (formal for now) nonlocal boundary value problem for the Poisson's equation

$$u_{xx} + u_{yy} = f(x,y), \quad 0 < x < 2\pi, \quad 0 < y < h, \tag{1.1}$$

$$u(x,0) = 0, \quad u(x,h) = 0, \qquad 0 < x < 2\pi, \tag{1.2}$$

$$u_x(0,y) = 0, \quad u(0,y) = u(2\pi, y), \; 0 < y < h. \tag{1.3}$$

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Such problems have specific features in comparison with problems with local conditions. Earlier F. I. Frankl [5]; [6, p. 453-456] considered a problem with a nonlocal boundary condition for a shifted type equation. The Bitsadze-Samarskii problem [4] for elliptic equations is also nonlocal with supports on a part of the boundary of the domain and, moreover, the supports are free from other boundary conditions. In the work of N. I. Ionkin and E. I. Moiseev [9], for multidimensional parabolic equations, a boundary value problem was solved with nonlocal conditions supported by the characteristic and improper parts of the domain boundary. It was considered a nonlocal boundary value problem for the Laplace equation in an unbounded domain studding the weak and strong solvability of that problem in the framework of the weighted Sobolev space in the work of B.Bilalov [19]

2 Auxiliary concepts and facts

21 Notations

We will use standard notations. N will be the set of positive integers, while $\alpha = (\alpha_1; \alpha_2) \in Z^+ \times Z^+$ will denote a multi-index, where $Z^+ = N \cup \{0\}$. Denote $\partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^\alpha} \partial y_2^\alpha$, where $|\alpha| = \alpha_1 + \alpha_2$. By $|M|$ we will denote the Lebesgue measure of the set M; $\overline{M}$ will be the closure of M. $C^\infty\left(\overline{M}\right)$ will stand for the infinitely differentiable functions on $\overline{M}$, and $C_0^\infty(M)$ will denote the infinitely differentiable and finite functions on M. Throughout this paper we will assume that p' is a conjugate number of p, $1 < p < +\infty$: $\frac{1}{p'} + \frac{1}{p} = 1$, $d\sigma$ is an area element. We also accept $p_\varepsilon = p - \varepsilon$.

22 Grand Sobolev space $W^2_{p)}(\mathit{\Pi})$

Let us define grand Sobolev space. Let $\mathit{\Pi} = (0, 2\pi) \times (0, h)$. Denote by $L_{p)}(\mathit{\Pi})$ a Banach space of functions on $\mathit{\Pi}$ with the mixed norm

$$\|f\|_{L_{p)}(\mathit{\Pi})} = \sup_{0<\varepsilon<p-1} \int_0^h \left(\varepsilon \int_0^{2\pi} |f(.;y)|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}} dy, 1 < p < +\infty.$$

Denote by $W^2_{p)}(\mathit{\Pi})$ a grand Sobolev space generated by the norm

$$\|u\|_{W^2_{p)}(\mathit{\Pi})} = \sum_{|\alpha| \le 2} \|\partial^\alpha u\|_{L_{p)}(\mathit{\Pi})}.$$

Also denote by $L_{p)}(I)$, where $I = (0, 2\pi)$, a grand Lebesgue space generated by the norm

$$\|f\|_{L_{p)}(I)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_I |f(x)|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}}.$$

These spaces are nonseperable and therefore the method of biorthogonal expansion (essentially the spectral method) is not applicable for studying the solvability of differential equations with respect to these spaces. In this regard we select the subspace $N_{p)}(\mathit{\Pi}) \subset L_{p)}(\mathit{\Pi})$ (seperable) based on the shift operator T_δ:

$$(T_\delta u)(x; y) = \begin{cases} u(x + \delta : y), & (x + \delta : y) \in \mathit{\Pi}, \\ 0, & (x + \delta : y) \notin \mathit{\Pi}. \end{cases}$$

So let us assume

$$N^2_{p)}(\Pi) = \left\{ W^2_{p)}(\Pi) : \sum_{|\alpha|\le 2} \|T_\delta(\partial^\alpha u) - \partial^\alpha u\|_{L_{p)}(\Pi)} \to 0, \delta \to 0 \right\}.$$

$$N^{1,0}_{p)}(\Pi) = \left\{ W^1_{p)}(\Pi) : \left\| T_\delta(\frac{\partial f}{\partial x}) - \frac{\partial f}{\partial x} \right\|_{L_{p)}(\Pi)} \to 0, \delta \to 0 \right\}.$$

23 Basicity of one system for $N_{p)(I)}$

Definition 2.1 *A function* $u \in N^2_{p)}(\Pi)$ *is called a strong solution of the problem* (1.1)-(1.3) *if the equality* (1.1) *is satisfied for a.e.* $(x; y) \in \Pi$ *and its trace* $u|_{\partial \Pi}$ *satisfies the relations* (1.2), (1.3).

Introduce the systems of functions $\{u_n(x)\}_{n\in Z^+}$ and $\{\vartheta_n(x)\}_{n\in Z^+}$, where

$$u_{2n}(x) = cosnx\,, n \in Z^+, \qquad u_{2n-1}(x) = xsinnx\,, n \in N, \tag{2.1}$$

$$\vartheta_0(x) = \frac{1}{2\pi^2}(2\pi - x)\,, \vartheta_{2n}(x) = \frac{1}{\pi^2}(2\pi - x)\,cosnx\,, \vartheta_{2n-1}(x) = \frac{1}{\pi^2} sinnx\,, n \in N. \tag{2.2}$$

Note that these systems are biorthogonal conjugate, which can be verified directly. To obtain our main result, we will significantly use the following theorem.

Theorem 2.1 *The system* (2.1) *forms a basis for* $N_{p)}(I)$.

Proof. From the embedding $L_p(I) \subset N_{p)}(I) \subset L_{p-\varepsilon}(I)$ it follows that (2.1) is complete and consequently, complete and minimal in $N_{p)}(I)$.

Let's prove the basicity of the system (2.1) for $N_{p)}(I)$. Consider the projectors

$$P_n(f) = \sum_{k=0}^{n} \langle f, \vartheta_k \rangle u_k, \forall n \in Z^+, \forall f \in N_{p)}(I),$$

where

$$\langle f, g \rangle = \int_0^{2\pi} f(x)\,\overline{g(x)} dx.$$

By constructing the Green's function similarly to [1], it is easy to show that system (2.1) forms a basis with brackets for $N_{p)}(I)$ for any $p \in (1, +\infty)$. The basis properties of the eigenfunctions of differential operators were studied in works [2,3,7,15,19]. From the basicity with brackets of the system (2.1) for $N_{p)}(I)$ it follows that

$$\exists C > 0 : \|P_{2n}(f)\|_{L_{p)}(I)} \le C \|f\|_{L_{p)}(I)}, \forall n \in N. \tag{2.3}$$

On the other hand, from (2.1), (2.2) we have

$$\exists M > 0 : \|u_n\|_{L_{p)}(I)} \le M, \qquad \|\vartheta_n\|_{L_{p')}} \le M, \qquad \forall n \in N. \tag{2.4}$$

Considering the relations (2.3), (2.4), we obtain

$$\begin{aligned} \|P_{2n+1}(f)\|_{L_{p)}(I)} &= \|P_{2n}(f) + \langle f, \vartheta_{2n+1}\rangle u_{2n+1}\|_{L_{p)}(I)} \le \\ &\le \|P_{2n}(f)\|_{L_{p)}(I)} + \|\langle f, \vartheta_{2n+1}\rangle u_{2n+1}\|_{L_{p)}(I)} \le \\ &\le C\|f\|_{L_{p)}(I)} + \|f\|_{L_{p)}(I)} \|u_n\|_{L_p(I)} \|\vartheta_n\|_{L_{p')}(I)} \le \left(C + M^2\right) \|f\|_{L_{p)}(I)}. \end{aligned} \tag{2.5}$$

3 Main Results

In this section, we will study the existence and uniqueness of strong solution of the problem (3.1)-(3.2) in the sense of Definition 2.1. First, denote $\Gamma_0 = \{(0; y) : 0 < y < h\}$ and $\Gamma_{2\pi} = \{(2\pi; y), 0 < y < h\}$. Consider the following nonlocal problem

$$\Delta u = f(x; y) \in \Pi, \tag{3.1}$$

$$u|_{I_0} = 0, u|_{I_h} = 0, u|_{\Gamma_0} = u|_{\Gamma_{2\pi}}, u_x|_{\Gamma_0} = 0. \tag{3.2}$$

By the solution of this problem, we mean a function $u \in N^2_{p)}(\Pi)$, which satisfies the equality (3.1) a.e. in Π and whose traces satisfy the relations (3.2) on the boundary $\partial\Pi = I_0 \cup I_h \cup \Gamma_0 \cup \Gamma_{2\pi}$.

$$\left.\begin{array}{r} F_0(y) = \frac{1}{2\pi^2}\int_0^{2\pi} f(x,y)(2\pi - x)dx, \\ F_{2n}(y) = \frac{1}{\pi^2}\int_0^{2\pi} f(x,y)(2\pi - x)cosnx\, dx\ , \\ F_{2n-1}(y) = \frac{1}{\pi^2}\int_0^{2\pi} f(x,y)sinnx\, dx, n \in N. \end{array}\right\} \tag{3.3}$$

Let's first examine the uniqueness of the solution. We obtain the uniquenss of solution from results of the works [17, 18].

Theorem 3.1 *Let the function* $f(x,y) \in N^{1,0}_{p)}(\Pi)$ *and satisfies the following condition* $f(0,y) = f(2\pi, y)$ *. Then problem* (3.1)-(3.2) *has a unique solution in* $N^2_{p)}(\Pi)$ *and moreover it is valid the following estimate* $\|u\|_{N^2_{p)}(\Pi)} \le c\|f\|_{N^{1,0}_{p)}(\Pi)}$*, where* $c > 0$ *is a constant independent of* $f(x,y)$ *.*

Suppose $u(x,y) \in N^2_{p)}(\Pi)$ is a solution of the problem (3.1)-(3.2). Consider $U_n(y) = \langle u(\cdot, y), \vartheta_n(\cdot)\rangle$, i.e.

$$\left.\begin{array}{r} U_0(y) = \frac{1}{2\pi^2}\int_0^{2\pi} u(x,y)(2\pi - x)dx \\ U_{2n}(y) = \frac{1}{\pi^2}\int_0^{2\pi} u(x,y)(2\pi - x)cosnx\, dx\ , \\ U_{2n-1}(y) = \frac{1}{\pi^2}\int_0^{2\pi} u(x,y)sinnx\, dx, n \in N. \end{array}\right\} \tag{3.4}$$

From Theorems 1.1.1-1.1.3 of [12, pp.13-15] it follows that the functions $U_n(y)$ are twice differentiable they can be differentiated under the integral sign. Since the function $u(x; y)$ satisfies the equation (3.1), multiplying it by $sinnx$ (by $2\pi - x)cosnx$) and integrating over I, we obtain the following relations for $U_{2n-1}(y)$ (respectively, for $U_{2n}(y)$):

$$U''_{2n-1}(y) - n^2 U_{2n-1}(y) = F_{2n-1}(y), y \in (0,h), \tag{3.5}$$

$$U''_{2n}(y) - n^2 U_{2n}(y) = -2nU_{2n-1}(y) + F_{2n}(y), y \in (0,h) \tag{3.6}$$

From the boundedness of the trace operator we immediately obtain that

$$U_n(0) = 0, \qquad U_n(h) = 0, \qquad \forall n \in Z^+, \tag{3.7}$$

The solution of the problem (3.5), (3.7) is

$$U_{2n-1}(y) = -\frac{1}{n}\frac{sinhn(h-y)}{sinhnh}\int_0^y sinhnt\, F_{2n-1}(t)\, dt$$

$$-\frac{1}{n}\frac{sinhny}{sinhnh}\int_y^h sinhn(h-t\,)F_{2n-1}\left(t\right)dt,\forall n\in N \quad (3.8)$$

and the solution of the problem (3.6), (3.7) is

$$U_0\left(y\right)=-\frac{y}{h}+\int_0^h\left(h-t\right)F_0\left(t\right)dt+\int_0^y\left(y-t\right)F_0\left(t\right)dt, \quad (3.9)$$

$$U_{2n}\left(y\right)=-\frac{1}{n}\frac{sinhn\left(h-y\right)}{sinhn\,h}\int_0^y F_{2n}\left(t\right)sinhnt\;dt-\frac{1}{n}\frac{sinhny}{sinhnh}\int_0^y F_{2n}\left(t\right)sinhnt\;dt-$$

$$-\frac{2}{n}\frac{sinhn\left(h-y\right)}{sinhnh}\int_0^y\frac{sinhn\left(h-t\right)}{sinhnh}\int_0^t F_{2n-1}\left(\tau\right)sinhn\tau\;d\tau dt-$$

$$-\frac{2}{n}\frac{sinhn\left(h-y\right)}{sinhnh}\int_0^y\frac{sinhnt}{sinhnh}\int_t^h F_{2n-1}\left(\tau\right)sinhn\left(h-\tau\right)\;d\tau dt-$$

$$-\frac{2}{n}\frac{sinhny}{sinhnh}\int_y^h\frac{sinhn(h-t\,)}{sinhnh}\int_0^t F_{2n-1}\left(\tau\right)sinhn\tau\;d\tau dt\;-$$

$$-\frac{2}{n}\frac{sinhny}{sinhnh}\int_y^h\frac{sinhnt}{sinhnh}\int_t^h F_{2n-1}\left(\tau\right)sinhn\left(h-\tau\right)d\tau dt,\;\forall\,n\in N, \quad (3.10)$$

Consider the function

$$u\left(x,y\right)=U_0\left(y\right)+\sum_{n=1}^{\infty}U_n\left(y\right)u_n\left(x\right)=U_0\left(y\right)+$$

$$+\sum_{k=1}^{\infty}\left(U_{2k}\left(y\right)coskx+U_{2k-1}\left(y\right)xsinkx\right),\left(x,y\right)\in\Pi, \quad (3.11)$$

where the coefficients $U_0\left(y\right),U_{2k}\left(\cdot\right),U_{2k-1}\left(\cdot\right),k\in N$, are defined by (3.8)-(3.10). Let's show that the function $u\left(x,y\right)$ belongs to $N^2_{p)}\left(\Pi\right)$. Denote by $u_{\alpha_1,\alpha_2}\left(x,y\right)$ the sum of the series obtained by the formal differentiation of the series (3.11), i.e.

$$u_{\alpha_1,\alpha_2}\left(x,y\right)=U_0^{(\alpha_2)}\left(y\right)+\sum_{n=1}^{\infty}U_n^{(\alpha_2)}\left(y\right)u_n^{(\alpha_1)}\left(x\right), \quad (3.12)$$

where

$$\alpha_1,\alpha_2\in Z^+,\alpha_1+\alpha_2\leq 2;u_{0,0}\left(x,y\right)=u\left(x,y\right)$$

and

$$U_n^{(\alpha_2)}\left(y\right)=\frac{d^{\alpha_2}U_n}{dy^{\alpha_2}};\;\;U_n^{(\alpha_1)}\left(x\right)=\frac{d^{\alpha_1}U_n}{dx^{\alpha_1}}.$$

Let us first consider the following member of series (3.11).

$$u_1\left(x,y\right)=\sum_{k=1}^{\infty}U_{2k-1}\left(y\right)\;xsinkx\;.$$

So, differentiating this series formally term-by-term, we have

$$\frac{\partial^2 u_1}{\partial y^2}=\sum_{k=1}^{\infty}U''_{2k-1}\left(y\right)\;\;xsink\,x=\sum_{k=1}^{\infty}k^2U_{2k-1}\left(y\right)\;xsinkx\;, \quad (3.13)$$

$$\frac{\partial u_1}{\partial x}=\sum_{k=1}^{\infty}U_{2k-1}\left(y\right)sink\;x+\sum_{k=1}^{\infty}kU_{2k-1}\left(y\right)\;xcosk\;x, \tag{3.14}$$

$$\frac{\partial^2 u_1}{\partial x^2}=2\sum_{k=1}^{\infty}kU_{2k-1}\left(y\right)cosk\;x-\sum_{k=1}^{\infty}k^2U_{2k-1}\left(y\right)\;xsink\;x. \tag{3.15}$$

Denote

$$w\left(x,y\right)=\sum_{k=1}^{\infty}k^2U_{2k-1}\left(y\right)xsink\;x.$$

Let's show that the function $w(x;y)$ belongs to $N_{p)}(\mathit{\Pi})$. Let $F_{2k}^{'}$=$\frac{1}{\pi^2}\int_0^{2\pi}f^{'}\left(x;y\right)coskx\;dx$. From (3.3), integrating by parts, we obtain

$$F_{2k-1}=\int_0^{2\pi}f(x;y)sinkx\;dx=-\frac{1}{\pi^2k}\left(f\left(2\pi;y\right)-f\left(0;y\right)\right)-$$

$$-\int_0^{2\pi}f^{'}\left(x;y\right)coskx\;dx=\frac{1}{\pi^2k}\int_0^{2\pi}f^{'}\left(x;y\right)coskx\;dx=\frac{1}{k}F_{2k}^{'}.$$

I. $p\geq 2$.

$$\left\|w(.;y)\right\|_{L_{p)(I)}}\leq c\bigg(\int_0^{2\pi}\left|w\left(x,y\right)\right|^p dx\bigg)^{\frac{1}{p}}\leq$$

$$\leq c\Bigg(\sum_{k=1}^{\infty}\left|k^2U_{2k-1}\left(y\right)\right|^{p^{'}}\Bigg)^{\frac{1}{p^{'}}}\leq c_1\sum_{k=1}^{\infty}k^2\left|U_{2k-1}\left(y\right)\right|\leq$$

$$\leq c_1\sum_{k=1}^{\infty}\frac{k}{sinhkh}\left(sinhk\left(h-y\right)\;\int_0^y sinhkt\;\left|F_{2k-1}\left(t\right)\right|dt+\right.$$

$$\left.+sinhky\;\int_y^h sinhk(h-t\;)\left|F_{2k-1}\left(t\right)\right|dt\right)$$

Hence, first integrating with respect to $y\in(0,h)$ and let's apply integration by parts formula and then applying Holder's inequality for any $\beta\in(1,\infty)$, we obtain

$$\left\|w\right\|_{L_{p)}(\mathit{\Pi})}\leq c_1\int\limits_0^h\left(\sum_{k=1}^{\infty}\frac{k}{\sinh kh}\left(\sinh k(h-y)\int\limits_0^y\sinh kt\left|F_{2k-1}(t)\right|dt+\right.\right.$$

$$\left.\left.+\sinh ky\int\limits_y^h\sinh k(h-t)\left|F_{2k-1}(t)\right|dt\right)\right)dy\leq$$

$$\leq c_2\sum_{k=1}^{\infty}\int_0^h\left|F_{2k-1}\left(t\right)\right|dt\leq c_2\int_0^h\left(\sum_{k=1}^{\infty}\left|F_{2k-1}\left(t\right)\right|\right)dt\leq$$

$$\leq c_2\int_0^h\left(\sum_{k=1}^{\infty}\frac{1}{k}\left|F_{2k}^{'}\left(t\right)\right|\right)dt\leq c_2\left(\sum_{k=1}^{\infty}\frac{1}{k^{\beta^{'}}}\right)^{\frac{1}{\beta^{'}}}\int_0^h\left(\left(\sum_{k=1}^{\infty}\left|F_{2k}^{'}\left(t\right)\right|^{\beta}\right)^{\frac{1}{\beta}}\right)dt,$$

where, we choose $\beta^{'}{=}\frac{\varepsilon}{p}$. Now, assuming $\beta \geq 2$ and applying classical Hausdorff-Young inequality (see, e.g. $[16, p.154]$. We have

$$\|w\|_{L_{p)}(\Pi)}{\le}c_3\int_0^h\left(\left\|\frac{\partial f}{\partial x}\right\|_{L_{\beta'}(I)}\right)dt. \tag{3.16}$$

Then, the last inequality means and

$$\|g\|_{L_r(I)}{\le}c\|g\|_{L_{p_\varepsilon}(I)}, \tag{3.17}$$

where $c > 0$ is a constant independent of g. Also note that the continuous embedding $L_{p)}\left(I\right){\subset}L_\alpha\left(I\right)$ is true for every $\alpha{\in}\left(1, r\right)$. Let us choose β big enough to satisfy the condition $1 <\beta^{'}< r \Rightarrow\|g\|_{L_{\beta'}(I)}{\le}c\|g\|_{L_r(I)}$ is satisfied. Then from inequalities (3.16),(3.17) we obtain

$$\|w\|_{L_{p)}(\Pi)}{\le}c\left(\left\|\frac{\partial f}{\partial x}\right\|_{L_{p)}(\Pi)}\right).$$

II. $p{\in}\left(1, 2\right)$. Therefore, choosing $\alpha > 1$ close enough to 1, we can provide that $p_1{=}\,p\alpha^{'}{>}\,2$ (this is possible, because $\alpha^{'}{\to}{+}\infty$ as $\alpha{\to}1 + 0$). With this, further considerations are carried out similar to the previous case.

Other series from (3.13)-(3.15), and, consequently, all series from (3.11) are estimated in a similar way. So, as a result, we obtain $\|u\|_{W^2_{p)}(\Pi)} \le c\|f\|_{N^{1,0}_{p)}(\Pi)}$, where $c > 0$ is a constant independent of f. The fulfillment of equation (3.1) by $u\left(\cdot;\cdot\right)$ can be verified directly. Let's verify the fulfillment of boundary conditions. Denote the trace operators on $\Gamma_0, \Gamma_{2\pi}, I_0$ and I_h by θ_0, $\theta_{2\pi}, T_0$ and T_h, respectively. Let's show that $T_0u{=}0$. It is clear that, $T_0u{\in}L_1\left(I\right)\ and\ 0{\in}L_1\left(I\right)$. From the boundedness of the operator $T_0{\in}\left[W^2_p\left(\Pi\right);L_p\left(I\right)\right]$, $\forall\, p \geq 1$, it follows that if $u_m{\to}u$ in $W^2_{p)}\left(\Pi\right)$, then $u_m/_I{\to}u/_I$ in $L_{p)}\left(I\right)$.

Now, let's consider the following functions:

$$u_m\left(x, y\right){=}U_0\left(y\right) + \sum_{n=1}^{m}\left(U_{2n}\left(y\right)cosnx\ {+}U_{2n-1}\left(y\right)xsinnx\ \right), (x;y)\in\Pi, m{\in}N.$$

We have

$$T_0u_m{=}u_m\left(x,0\right){=}U_0\left(0\right) + \sum_{n=1}^{m}\left(U_{2n}\left(0\right)cosnx\ {+}U_{2n-1}\left(0\right)\ xsinnx\ \right){=}$$

$$= \frac{1}{2\pi^2}\int_0^{2\pi}\varphi\left(\tau\right)\left(2\pi-\tau\right)d\tau + \sum_{n=1}^{m}\left(\frac{1}{\pi^2}\int_0^{2\pi}\varphi\left(\tau\right)\left(2\pi-\tau\right)cosn\tau\ d\tau cosnx\ + \right.$$

$$\left. +\frac{1}{\pi^2}\int_0^{2\pi}\varphi\left(\tau\right)sinn\tau\ d\tau sinnx\ \right). \tag{3.18}$$

It is clear that, $T_0u_m{\to}T_0u$. On the other hand, the basicity of the system (2.1) for $N_{p)}\left(I\right)$ implies $T_0u_m{\to}0, m{\to}\infty$, in $L_{p)}\left(I\right)$. Consequently, $T_0u = 0$, a.e. on I.

Absolutely similar we can show that $T_hu_m{\to}0, m{\to}\infty$, in $L_{p)}\left(I\right)$. Consequently, $T_hu = 0$, a.e. on I.

Consider the operators θ_0 and $\theta_{1\pi}$. It is clear that $\theta_0u_m{=}\theta_{2\pi}u_m$, $\forall m{\in}N$. Obviously, $\theta_0u_m{\to}\theta_0u$ and $\theta_{2\pi}u_m{=}\theta_{2\pi}u{\Rightarrow}\theta_0u\ {=}\theta_{2\pi}u$. Thus, the boundary conditions (3.2) are fulfilled.

The theorem is proved.

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