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Two-weight norm inequality for multidimensional Hausdorff operator on Lebesgue spaces

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Abstract. *In this paper we give a sufficient condition for the boundedness of the multidimensional Hausdorff operator from one weighted Lebesgue spaces to anther weighted Lebesgue spaces to anther. Also, we get similar results for important operators of harmonic analysis which are special cases of the multidimensional Hausdorff operator. The results are illustrated by a number of corollaries.*

Keywords. Multidimensional Hausdorff operator, weighted Lebesgue spaces, two-weight norm inequality, boundedness

Mathematics Subject Classification (2010): 28C99; 46E30

1 Introduction

It is well known that one-dimensional Hausdorff operator has a deep root in the study of the Fourier analysis and it has a long history in the study of real and complex analysis. In particular, it is closely related to the summability of the classical Fourier series. The reader is referred to [19] for a survey of some historic background and recent developments on one-dimensional Hausdorff operators. Modern theory of Hausdorff operators started with the work of Siskakis [22] in complex analysis setting and with the work of Georgakis [14] and Liflyand-Móricz [21] in the Fourier transform setting.

Let $\mathbb{R}_+ = (0, \infty)$ and let ϕ be a locally integrable function on $\mathbb{R}_+$. Then the one-dimensional Hausdorff operator is defined by

$$H_\phi(f)(x) = \int_0^\infty \frac{\phi\left(\frac{x}{y}\right)}{y} f(y)\, dy.$$

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Many important operators in analysis are special cases of the one-dimensional Hausdorff operators, by taking suitable choice of ϕ. For example, the Hardy operator, the adjoint Hardy operator, the Hardy-Littlewood-Pólya operator, the Cesàro operator, fractional Riemann-Liouville operator are special cases of the one-dimensional Hausdorff operators or fractional one-dimensio-nal Hausdorff operators. In the last two decades various problems related to one-dimensional Hausdorff operators attracted much attention. The Hausdorff operators has been extensively studied in recent years, particularly its boundedness on the Lebesgue space as well as on the Hardy space [1], [9], [15], [17], [18]-[21] and so on. We also refer to [4], [5], [7] and [10]-[13] for some recent works in this vein. Recently, two-weight inequalities in the framework of one-dimensional Hausdorff operators were studied in [8] and [20] (see, also [4]). We note that in [20] the obtained necessary conditions differ from the sufficient conditions and coincide for the Hardy and Bellman operators. Also, in [5] and [8] the obtained necessary conditions differ from the sufficient conditions. Moreover, in [8], the corresponding boundedness of one-dimensional Hausdorff operators has been studied in the framework of other function spaces as well, namely, grand Lebesgue spaces and variable exponent Lebesgue spaces. Next, in [8], the authors establish necessary and sufficient conditions on monotone weight functions for the boundedness of special kind Hausdorff operators on weighted Lebesgue spaces. In [16] the boundedness of Dunkl-Hausdorff operator on weighted Lebesgue spaces has been characterized, and for a power type weight function the corresponding operator norm has been obtained. Very recently, in [16], the results of [5] were extended for the Dunkl-Hausdorff operator and the boundedness of one-dimensional Hausdorff operator on the cone of non-increasing functions was proved. We refer to [2], [3], [6] and [15] for more results on two-weight inequalities and its applications for different type integral operators.

It is well-known that Hausdorff operators, particularly in high dimension, are important operators in harmonic analysis and they were attracted extensive research by many authors. These observations motivate us to study the multidimensional Hausdorff operators and their boundedness in various function spaces, while the Hardy operators are our model case. For multidimensional Hausdorff operators, there are many kinds of definitions [1], [9], [10], [18], [19] and so on. One of the interesting definitions of the Hausdorff operators is

$$\mathcal{H}_{\Phi,A} f(x) = \int\limits_{\mathbb{R}^n} \Phi(y)\, f\left(xA(y)\right) dy, \tag{1.1}$$

where $A(u) = \left(a_{ij}(y)\right)_{i,j=1}^{n}$ is an $n \times n$ matrix with the entries $a_{ij}(y)$ being Lebesgue measurable functions of y. This matrix is nonsingular almost everywhere in the support of Φ. And also, it is assumed that $\Phi(y)\det A^{-1}(y) \in L_1\left(\mathbb{R}^n\right)$. The operator $\mathcal{H}_{\Phi,A}$ was defined and studied by Lerner and Liflyand [18]. The definition (1.1) suggests much wider range of the Hausdorff type operators than those for which efficient results were obtained.

This is how Hausdorff operators are defined in [9] and [22] for the Borel measures. In [22] the boundedness of such operators in $H^1\left(\mathbb{R}^n\right)$ is proved for a very special case of diagonal matrices A with all entries on the diagonal equal to one another.

Other multidimensional extension of the Hausdorff operator is the following operator

$$\mathcal{H}_{\Phi,\Omega} f(x) = \int\limits_{\mathbb{R}^n} \frac{\Phi\left(\frac{x}{|y|}\right)}{|y|^n}\, \Omega\left(y'\right)\, f(y)\, dy, \tag{1.2}$$

where Φ is a radial function on $\mathbb{R}^n$ and Ω is an integrable function defined on the unit sphere $\mathbb{S}^{n-1}$. We denote $\mathcal{H}_{\Phi,\Omega} = \mathcal{H}_{\Phi}$, if $\Omega = 1$. The operator $\mathcal{H}_{\Phi,\Omega}$ was defined and studied by Chen, Fan and Li [10].

We observe that for a radial function Φ the boundedness of the Hausdorff operator $\mathcal{H}_{\Phi,\Omega}$ on the Lebesgue spaces was proved in [10]. For a general function Φ the boundedness of the Hausdorff operator $\mathcal{H}_{\Phi}$ on the Lebesgue spaces was proved in [24]. Later for a general function Φ the boundedness of the Hausdorff operator $\mathcal{H}_{\Phi}$ on the Lebesgue space with power weight was proved in [12] and [13]. In [13], the authors give some sufficient conditions for the boundedness of several types of Hausdorff operators on the Lebesgue spaces with power weights. In some cases, these conditions are also necessary, and the corresponding operator norms are worked out in [13].

The main goal of the paper is to study the boundedness of multidimensional Hausdorff operator $\mathcal{H}_{\Phi}$ on weighted Lebesgue spaces.

Let us denote by χ_E the characteristic function of $E \subset \mathbb{R}^n$. In particular, we get the n-dimensional Hardy operator

$$Hf(x) = \frac{1}{|x|^n} \int\limits_{|y|<|x|} f(y)\,dy$$

and the n-dimensional adjoint Hardy operator

$$H^{\star}f(x) = \int\limits_{|y|>|x|} \frac{f(y)}{|y|^n}\,dy,$$

if we choose $\Phi(y) = \dfrac{\chi_{\{|y|>1\}}(y)}{|y|^n}$ and $\Phi(y) = \chi_{\{|y|<1\}}(y)$, respectively. If $\Phi(y) = \min\left\{1, \dfrac{1}{|y|^n}\right\}$, we get the n-dimensional version of Calderón operator

$$Cf(x) = \frac{1}{|x|^n} \int\limits_{|y|<|x|} f(y)\,dy + \int\limits_{|y|>|x|} \frac{f(y)}{|y|^n}\,dy.$$

Also, if $\Phi(y) = \dfrac{\chi_{\mathbb{R}^n_+}(y)}{(|y|+1)^n}$ or $\Phi(y) = \dfrac{\chi_{\mathbb{R}^n_+}(y)}{|y|^n+1}$ then we have n-dimensional version of the Hardy-Hilbert type operators

$$T_1f(x) = \int\limits_{\mathbb{R}^n_+} \frac{f(y)}{(|x|+|y|)^n}\,dy$$

or

$$T_2f(x) = \int\limits_{\mathbb{R}^n_+} \frac{f(y)}{|x|^n+|y|^n}\,dy,$$

respectively.

The remainder of the paper is structured as follows. Section 2 contains some preliminaries along with the standard ingredients used in the proofs. Our principal assertions, concerning the continuity of the multidimensional Hausdorff operator $\mathcal{H}_{\Phi}$ from one weighted Lebesgue spaces to another are formulated and proved in Section 3. In particular, we establish a sufficient condition for the boundedness of the multidimensional Hausdorff operator on weighted Lebesgue spaces for radial weight functions.

2 Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and let $\mathbb{R}^n_+ = \{x : x \in \mathbb{R}^n, \ x_n > 0\}$. By $B(0,t)$ we denote the open ball centered at the origin of radius t. Suppose that $\mathbb{S}^{n-1}$ is the unit sphere centered at the origin in $\mathbb{R}^n$. Throughout this paper, $\left|\mathbb{S}^{n-1}\right|$ denotes the surface area of unit sphere $\mathbb{S}^{n-1}$ and $|B(0,1)|$ is the volume of the unit ball in $\mathbb{R}^n$. Let $1 \le p < \infty$ and let p' denote the conjugate exponent defined by $p' = \dfrac{p}{p-1}$. By a weight function, we shall mean a function which is Lebesgue measurable, positive and finite almost everywhere on $\mathbb{R}^n$. A function f is called a radial function, if $f(x) = f(|x|)$ for all $x \in \mathbb{R}^n$. Throughout this paper, C is a positive constant, whose value can be different at different places.

We recall the definition of the weighted Lebesgue spaces.

We shall denote by $L_{p,\omega}(\mathbb{R}^n)$, the weighted Lebesgue space which is the space of all Lebesgue measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L_{p,\omega}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx\right)^{\frac{1}{p}} < \infty.$$

Observe that for $\omega \equiv 1$, $L_{p,\omega}(\mathbb{R}^n)$ means the usual Lebesgue space $L_p(\mathbb{R}^n)$.

3 Main results

In this section, we state and prove our principal results of the multidimensional Hausdorff operator $\mathcal{H}_\Phi$ for a general function Φ.

Theorem 3.1 *Let $1 \le q < p < \infty$, and let u and v be weight functions defined on $\mathbb{R}^n$. Suppose Φ is a locally summable function on $\mathbb{R}^n$. Let Ψ be a positive radial function on $\mathbb{R}^n$ satisfying the following conditions:*

$i)$ there exists a constant $C > 0$ such that

$$|\Phi(t\xi)| \le C\,\Psi(t) \quad \textit{for all} \quad t > 0 \quad \textit{and} \quad \xi \in \mathbb{S}^{n-1},$$

$$ii) \quad B(u,v) = \int\limits_0^\infty \frac{\Psi(t)}{t}\, t^{\frac{n}{q}} \left(\int\limits_{\mathbb{R}^n} \left(\int\limits_{\mathbb{S}^{n-1}} [v(|x|\xi)]^{-\frac{q}{p-q}}\, d\sigma(\xi)\right) [u(tx)]^{\frac{p}{p-q}}\, dx\right)^{\frac{p-q}{pq}} dt < \infty.$$

Then the inequality

$$\|\mathcal{H}_\Phi f\|_{L_{q,u}(\mathbb{R}^n)} \le C\, \left|\mathbb{S}^{n-1}\right|^{\frac{1}{q'}+\frac{1}{p}}\, B(u,v)\, \|f\|_{L_{p,v}(\mathbb{R}^n)} \qquad (3.1)$$

holds.

Proof.

Using polar coordinates in $\mathbb{R}^n$, by condition $i)$ and by change of variable, we have

$$\|\mathcal{H}_\Phi f\|_{L_{q,u}(\mathbb{R}^n)} = \left(\int\limits_{\mathbb{R}^n} |\mathcal{H}_\Phi f(x)|^q\, u(x)\, dx\right)^{\frac{1}{q}}$$

$$= \left(\int\limits_{\mathbb{R}^n} \left|\int\limits_{\mathbb{R}^n} \frac{\Phi\left(\frac{x}{|y|}\right)}{|y|^n}\, f(y) dy\right|^q u(x)\, dx\right)^{\frac{1}{q}} dx$$

$$\leq \left(\int\limits_{\mathbb{R}^n}\left(\int\limits_{\mathbb{R}^n}\frac{\left|\Phi\left(\frac{x}{|y|}\right)\right|}{|y|^n}|f(y)|dy\right)^q u(x)\,dx\right)^{\frac{1}{q}}$$

$$=\left(\int\limits_{\mathbb{R}^n}\left(\int\limits_0^\infty\int\limits_{\mathbb{S}^{n-1}}\frac{\left|\Phi\left(\frac{x}{t}\right)\right|}{t}|f(t\xi)|d\sigma(\xi)dt\right)^q u(x)\,dx\right)^{\frac{1}{q}}$$

$$=\left(\int\limits_{\mathbb{R}^n}\left(\int\limits_0^\infty\int\limits_{\mathbb{S}^{n-1}}\frac{\left|\Phi\left(\frac{|x|}{t}\cdot\frac{x}{|x|}\right)\right|}{t}|f(t\xi)|d\sigma(\xi)dt\right)^q u(x)\,dx\right)^{\frac{1}{q}}$$

$$\leq C\left(\int\limits_{\mathbb{R}^n}\left(\int\limits_0^\infty\int\limits_{\mathbb{S}^{n-1}}\frac{\Psi\left(\frac{|x|}{t}\right)}{t}|f(t\xi)|d\sigma(\xi)dt\right)^q u(x)\,dx\right)^{\frac{1}{q}}$$

$$=C\left(\int\limits_{\mathbb{R}^n}\left(\int\limits_0^\infty\frac{\Psi(t)}{t}\left(\int\limits_{\mathbb{S}^{n-1}}\left|f\left(|x|t^{-1}\xi\right)\right|d\sigma(\xi)\right)dt\right)^q u(x)\,dx\right)^{\frac{1}{q}}$$

$$=C\left(\int\limits_{\mathbb{R}^n}\left(\int\limits_0^\infty\frac{\Psi(t)}{t}\left(\int\limits_{\mathbb{S}^{n-1}}\left|f\left(|x|t^{-1}\xi\right)\right|d\sigma(\xi)\right)dt\right)^q u(x)\,dx\right)^{\frac{1}{q}}. \tag{3.2}$$

By Minkowski's inequality and change of variables, we have

$$\left(\int\limits_{\mathbb{R}^n}\left(\int\limits_0^\infty\frac{\Psi(t)}{t}\left(\int\limits_{\mathbb{S}^{n-1}}\left|f\left(|x|t^{-1}\xi\right)\right|d\sigma(\xi)\right)dt\right)^q u(x)\,dx\right)^{\frac{1}{q}}$$

$$\leq\int\limits_0^\infty\frac{\Psi(t)}{t}\left(\int\limits_{\mathbb{R}^n}\left(\int\limits_{\mathbb{S}^{n-1}}\left|f\left(|x|t^{-1}\xi\right)\right|d\sigma(\xi)\right)^q u(x)\,dx\right)^{\frac{1}{q}}dt$$

$$=\int\limits_0^\infty\frac{\Psi(t)}{t}\,t^{\frac{n}{q}}\left(\int\limits_{\mathbb{R}^n}\left(\int\limits_{\mathbb{S}^{n-1}}\left|f\left(|x|\xi\right)\right|d\sigma(\xi)\right)^q u(tx)\,dx\right)^{\frac{1}{q}}dt.$$

By Hölder inequality, one has

$$\int\limits_0^\infty\frac{\Psi(t)}{t}\,t^{\frac{n}{q}}\left(\int\limits_{\mathbb{R}^n}\left(\int\limits_{\mathbb{S}^{n-1}}\left|f\left(|x|\xi\right)\right|d\sigma(\xi)\right)^q u(tx)\,dx\right)^{\frac{1}{q}}dt$$

$$\leq\left|\mathbb{S}^{n-1}\right|^{\frac{1}{q'}}\int\limits_0^\infty\frac{\Psi(t)}{t}\,t^{\frac{n}{q}}\left(\int\limits_{\mathbb{R}^n}\left(\int\limits_{\mathbb{S}^{n-1}}\left|f\left(|x|\xi\right)\right|^q d\sigma(\xi)\right)u(tx)\,dx\right)^{\frac{1}{q}}dt.$$

Using Hölder inequality with exponents $\dfrac{p}{q}$ and $\dfrac{p}{p-q}$, one has

$$\int_0^\infty \frac{\Psi(t)}{t}\, t^{\frac{n}{q}} \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{S}^{n-1}} |f(|x|\xi)|^q \, d\sigma(\xi)\right) u(tx)\, dx\right)^{\frac{1}{q}} dt$$

$$= \int_0^\infty \frac{\Psi(t)}{t}\, t^{\frac{n}{q}} \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{S}^{n-1}} |f(|x|\xi)|^q \, [v(|x|\xi)]^{\frac{q}{p}} \, [v(|x|\xi)]^{-\frac{q}{p}} \, d\sigma(\xi)\right) u(tx)dx\right)^{\frac{1}{q}} dt \le$$

$$\int_0^\infty \frac{\Psi(t)}{t}\, t^{\frac{n}{q}} \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{S}^{n-1}} |f(|x|\xi)|^p \, v(|x|\xi)\, d\sigma(\xi)\right)^{\frac{q}{p}} \left(\int_{\mathbb{S}^{n-1}} [v(|x|\xi)]^{-\frac{q}{p-q}} \, d\sigma(\xi)\right)^{\frac{p-q}{p}} u(tx)dx\right)^{\frac{1}{q}} dt.$$

By Hölder inequality with similar exponents in variable x, we get

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{S}^{n-1}} |f(|x|\xi)|^p \, v(|x|\xi)\, d\sigma(\xi)\right)^{\frac{q}{p}} \left(\int_{\mathbb{S}^{n-1}} [v(|x|\xi)]^{-\frac{q}{p-q}} \, d\sigma(\xi)\right)^{\frac{p-q}{p}} u(tx)dx \le$$

$$\left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{S}^{n-1}} |f(|x|\xi)|^p \, v(|x|\xi)\, d\sigma(\xi)\right) dx\right)^{\frac{q}{p}} \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{S}^{n-1}} [v(|x|\xi)]^{-\frac{q}{p-q}} \, d\sigma(\xi)\right) [u(tx)]^{\frac{p}{p-q}} \, dx\right)^{\frac{p-q}{p}}.$$

By condition $ii)$, we get

$$\int_0^\infty \frac{\Psi(t)}{t}\, t^{\frac{n}{q}} \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{S}^{n-1}} |f(|x|\xi)|^p \, v(|x|\xi)\, d\sigma(\xi)\right)^{\frac{q}{p}} \left(\int_{\mathbb{S}^{n-1}} [v(|x|\xi)]^{-\frac{q}{p-q}} \, d\sigma(\xi)\right)^{\frac{p-q}{p}} u(tx)dx\right)^{\frac{1}{q}} dt$$

$$\le \left(\int_0^\infty \frac{\Psi(t)}{t}\, t^{\frac{n}{q}} \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{S}^{n-1}} [v(|x|\xi)]^{-\frac{q}{p-q}} \, d\sigma(\xi)\right) [u(tx)]^{\frac{p}{p-q}} \, dx\right)^{\frac{p-q}{pq}} dt\right) \times$$

$$\times \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{S}^{n-1}} |f(|x|\xi)|^p \, v(|x|\xi)\, d\sigma(\xi)\right) dx\right)^{\frac{1}{p}}$$

$$= B(u,v) \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{S}^{n-1}} |f(|x|\xi)|^p \, v(|x|\xi)\, d\sigma(\xi)\right) dx\right)^{\frac{1}{p}}. \tag{3.3}$$

Again, using polar coordinates in $\mathbb{R}^n$ and Hölder's inequality, we get

$$\left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{S}^{n-1}} |f(|x|\xi)|^p \, v(|x|\xi)\, d\sigma(\xi)\right) dx\right)^{\frac{1}{p}}$$

$$= \left|\mathbb{S}^{n-1}\right|^{\frac{1}{p}} \left(\int\limits_0^\infty s^{n-1} \left(\int\limits_{\mathbb{S}^{n-1}} |f(s\xi)|^p \, v(s\xi) \, d\sigma(\xi) \right) ds \right)^{\frac{1}{p}} = \left|\mathbb{S}^{n-1}\right| \, \|f\|_{L_{p,v}(\mathbb{R}^n)}.$$

This completes the proof.

From Theorem 2.1 we have the following corollaries.

Corollary 3.1 *Let $1 \leq q < p < \infty$. Suppose u, v are weight functions defined on $\mathbb{R}^n$ and let v be a radial function. Suppose $\Phi \in L_1^{loc}(\mathbb{R}^n)$ is a radial function satisfying the following conditions:*

$$B_1(u,v) = \int\limits_0^\infty \frac{|\Phi(t)|}{t} t^{\frac{n}{q}} \left(\int\limits_{\mathbb{R}^n} \frac{[u(tx)]^{\frac{p}{p-q}}}{[v(|x|)]^{\frac{q}{p-q}}} dx \right)^{\frac{p-q}{pq}} dt < \infty.$$

Then the inequality

$$\|\mathcal{H}_\Phi f\|_{L_{q,u}(\mathbb{R}^n)} \leq \left|\mathbb{S}^{n-1}\right| \, B_1(u,v) \, \|f\|_{L_{p,v}(\mathbb{R}^n)} \tag{3.4}$$

holds.

Corollary 3.2 *Let $1 \leq q < p < \infty$ and let $u(x) = |x|^\alpha$ and $v(x) = (1+|x|)^\beta$. Suppose that $-\dfrac{n(p-q)}{p} < \alpha < n(q-1)$ and $\beta > \dfrac{(\alpha+n)p}{q} - n$. Let $\Phi(x) = \dfrac{\chi_{\{|x|>1\}}(x)}{|x|^n}$*

Then

$$\|Hf\|_{L_{q,u}(\mathbb{R}^n)} \leq \frac{q \left|\mathbb{S}^{n-1}\right|^{\frac{1}{p'}+\frac{1}{q}}}{n(q-1)-\alpha} B^{\frac{p-q}{pq}} \left(\frac{\beta q - \alpha p}{p-q} - n, \ \frac{\alpha p}{p-q} + n \right) \|f\|_{L_{p,v}(\mathbb{R}^n)}.$$

Corollary 3.3 *Let $1 \leq q < p < \infty$ and let $u(x) = |x|^\alpha$ and $v(x) = (1+|x|)^\beta$. Suppose that $\alpha > -\dfrac{n(p-q)}{p}$ and $\beta > \dfrac{(\alpha+n)p}{q} - n$. Let $\Phi(x) = \chi_{\{|x|\leq 1\}}(x)$*

Then

$$\|H^\star f\|_{L_{q,u}(\mathbb{R}^n)} \leq \frac{q \left|\mathbb{S}^{n-1}\right|^{\frac{1}{p'}+\frac{1}{q}}}{\alpha+n} B^{\frac{p-q}{pq}} \left(\frac{\beta q - \alpha p}{p-q} - n, \ \frac{\alpha p}{p-q} + n \right) \|f\|_{L_{p,v}(\mathbb{R}^n)}.$$

Remark 3.1 We observe that in the one-dimensional case Theorem 3.1 was proved in [5]. In the non-weighted Lebesgue spaces Theorem 3.1 for a radial function Φ was proved in [10]. Also, in the non-weighted Lebesgue spaces Theorem 3.1 for a general function Φ was proved in [24]. Later for a general function Φ the boundedness of the n-dimensional Hausdorff operator $\mathcal{H}_\Phi$ on the Lebesgue space with power weight was proved in [12] and [13]. In [13], authors give some sufficient conditions for the boundedness of several types of Hausdorff operators on the Lebesgue spaces with power weights.

4 Competing interests

The authors declare that they have no competing interests.

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6 Authors contributions

All authors contributed equally to this article. They read and approved the final manuscript.

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