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Linear recursive schemes and an asymptotic expansion associated with the Kirchhoff-Carrier-Love equation

**Vo Thi Tuyet Mai, Nguyen Anh Triet, Le Thi Phuong Ngoc,
Nguyen Thanh Long**[⋆]

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Abstract. *In this paper, we consider the following nonlinear Kirchhoff-Carrier-Love equation*

$$\begin{cases} u_{tt} - B\left(\|u(t)\|^2, \|u_x(t)\|^2\right)(u_{xx} + \lambda u_{xxtt}) \\ \qquad\qquad\qquad = f(x,t,u,u_x,u_t,u_{xt}),\ 0 < x < 1,\ 0 < t < T, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x), \quad u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{0.1}$$

where $\lambda > 0$ *is a constant,* $\tilde{u}_0$, $\tilde{u}_1$, f, B *are given functions and* $\|u(t)\|^2 = \int_0^1 u^2(x,t)\,dx$, $\|u_x(t)\|^2 = \int_0^1 u_x^2(x,t)\,dx$. *First, combining the linearization method for nonlinear terms, the Faedo-Galerkin method and the weak compact method, a unique weak solution of the problem* (0.1) *is obtained. Next, by using Taylor's expansion of the functions* $B(y,z)$, $f(x,t,u,u_x,u_t,u_{xt})$ *up to order* $N+1$, *we establish an asymptotic expansion of high order in many small parameters of solution.*

Keywords. Faedo-Galerkin method; Linear recurrent sequence; Asymptotic expansion of order $N+1$ in p small parameters.

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⋆ Corresponding author

V.T.T. Mai
University of Natural Resources and Environment of Ho Chi Minh City, 236B Le Van Sy Str., Tan Son Hoa Ward, Ho Chi Minh City, Vietnam
E-mail: vttmai@hcmunre.edu.vn

N.A. Triet
University of Architecture Ho Chi Minh City, 196 Pasteur Str., Xuan Hoa Ward, Ho Chi Minh City, Vietnam
E-mail: triet.nguyenanh@uah.edu.vn

L.T.P. Ngoc
University of Khanh Hoa, 01 Nguyen Chanh Str., Nha Trang City, Vietnam
E-mail: ngoc1966@gmail.com

N.T. Long
Faculty of Mathematics and Computer Science, University of Science, Ho Chi Minh City, Vietnam, 227 Nguyen Van Cu Str., Cho Quan Ward, Ho Chi Minh City, Vietnam.
Vietnam National University, Ho Chi Minh City, Vietnam
E-mail: longnt2@gmail.com

1 Introduction

In this paper, we consider the following Dirichlet problem for a nonlinear Kirchhoff-Carrier-Love equation

$$
\begin{aligned}
&u_{tt} - B\left(\|u(t)\|^2, \|u_x(t)\|^2\right)(u_{xx} + \lambda u_{xxtt}) \qquad (1.1)\\
&= f(x,t,u,u_x,u_t,u_{xt}),\ x \in \Omega = (0,1),\ 0 < t < T,
\end{aligned}
$$

$$u(0,t) = u(1,t) = 0, \qquad (1.2)$$

$$u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \qquad (1.3)$$

where $\lambda > 0$ is a constant and $\tilde{u}_0$, $\tilde{u}_1$, f, B, are given functions.

When $\Omega = (0,L)$, $B = 1$, $f = 0$, Eq. (1.1) is related to the Love equation

$$u_{tt} - \frac{E}{\rho}u_{xx} - 2\mu^2k^2u_{xxtt} = 0, \qquad (1.4)$$

presented by V. Radochová in 1978 (see [34]). This equation describes the vertical oscillations of a rod, which was established from Euler's variational equation of an energy function

$$\int_0^T dt \int_0^L \left[\frac{1}{2}F\rho\left(u_t^2 + \mu^2k^2u_{tx}^2\right) - \frac{1}{2}F\left(Eu_x^2 + \rho\mu^2k^2u_xu_{xtt}\right)\right]dx, \qquad (1.5)$$

the parameters in (1.5) have the following meanings: u is the displacement, L is the length of the rod, F is the area of cross-section, k is the cross-section radius, E is the Young modulus of the material and ρ is the mass density. By using the Fourier method, Radochová [34] obtained a classical solution of Prob. (1.4) associated with initial conditions (1.3) and boundary conditions

$$u(0,t) = u(L,t) = 0, \qquad (1.6)$$

or

$$\begin{cases} u(0,t) = 0, \\ \lambda u_{xtt}(L,t) + c^2u_x(L,t) = 0, \end{cases} \qquad (1.7)$$

where $c^2 = \frac{E}{\rho}$, $\lambda = 2\mu^2k^2$. On the other hand, the asymptotic behaviour of solutions for Prob. (1.3), (1.4), (1.6) as $\lambda \to 0_+$ was also established by the method of small parameters.

Equations of Love waves or equations for waves of Love types have been studied by many authors, we refer to [3], [8], [22] and references therein.

On the other hand, in [37], a symmetric version of the regularized long wave equation (SRLW)

$$\begin{cases} u_{xxt} - u_t = \rho_x + uu_x, \\ \rho_t + u_x = 0, \end{cases} \qquad (1.8)$$

has been proposed to describe weakly nonlinear ion acoustic and space - charge waves. Eliminating ρ from (1.8), a class of SRLWE is obtained as follows

$$u_{tt} - u_{xx} - u_{xxtt} = -uu_{xt} - u_xu_t. \qquad (1.9)$$

Eq. (1.9) is explicitly symmetric in the x and t derivatives and it is very similar to the regularized long wave equation that describes shallow water waves and plasma drift waves [1], [2]. The SRLW equation also arises in many other areas of mathematical physics [4], [21], [30]. It is clear that Eq. (1.9) is a special form of Equation (1.1), in which $f(x,t,u,u_x,u_t,u_{xt}) = -uu_{xt} - u_xu_t$.

When $\Omega = (0, L)$, $B = B\left(\|u_x(t)\|^2\right)$, $\lambda = 0$, Eq. (1.1) is related to the Kirchhoff equation

$$\rho h u_{tt} = \left(P_0 + \frac{Eh}{2L}\int_0^L \left|\frac{\partial u}{\partial y}(y,t)\right|^2 dy\right) u_{xx}, \tag{1.10}$$

presented by Kirchhoff in 1876 (see [11]). This equation is an extension of the classical D'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations. The parameters in (1.10) have the following meanings: u is the lateral deflection, L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension.

One of the early classical studies dedicated to Kirchhoff equations was given by Pohozaev [31]. After the work of Lions, for example see [15], Eq. (1.10) received much attention where an abstract framework to the problem was proposed. We refer the reader to, e.g., Cavalcanti et al. [5] - [7], Ebihara, Medeiros and Miranda [9], Miranda et al. [26], Lasiecka and Ong [13], Hosoya, Yamada [10], Larkin [12], Medeiros [23], Menzala [27], Park et al. [32], [33], Rabello et al. [35], Santos et al. [36], for many interesting results and further references. A survey of the results about the mathematical aspects of Kirchhoff model can be found in Medeiros, Limaco and Menezes [24], [25], and the references therein.

Motivated by the problems in the above mentioned works, in this paper, we consider Prob. (1.1) - (1.3) with $f \in C^1([0,1] \times \mathbb{R}_+ \times \mathbb{R}^4)$, $B \in C^1(\mathbb{R}_+^2)$. Since f, B are arbitrary, the methods used in [34] or in [37] are no longer suitable, here we will combine the linearization method for a nonlinear term, the Faedo-Galerkin method and the weak compactness method.

The paper consists of four sections. At first, some preliminaries are done in Section 2. With the technique presented as above, we begin Section 3 by establishing a sequence of approximate solutions of Prob. (1.1) - (1.3) based on the Faedo-Galerkin's method. Thanks to a priori estimates, this sequence is bounded in an appropriate space, from which, using compact imbedding theorems and Gronwall's Lemma, one deduce the existence of a unique weak solution of Prob. (1.1) - (1.3). In particular, an asymptotic expansion of a weak solution $u = u(\varepsilon_1, \cdots, \varepsilon_p)$ of order $N+1$ in p small parameters $\varepsilon_1, \cdots, \varepsilon_p$ for the equation

$$\begin{aligned} u_{tt} - &\left[B\left(\|u(t)\|^2, \|u_x(t)\|^2\right) + \sum_{i=1}^p \varepsilon_i B_i\left(\|u(t)\|^2, \|u_x(t)\|^2\right)\right](u_{xx} + \lambda u_{xxt}) \\ &= f(x,t,u,u_x,u_t,u_{xt}) + \sum_{i=1}^p \varepsilon_i f_i(x,t,u,u_x,u_t,u_{xt}), \end{aligned} \tag{1.11}$$

$0 < x < 1$, $0 < t < T$, associated to (1.2), (1.3), with $B \in C^{N+1}(\mathbb{R}_+^2)$, $B_i \in C^N(\mathbb{R}_+^2)$, $B(y,z) \geq b_0 > 0$, $B_i(y,z) \geq 0$, $(i = 1, \cdots, p)$, for all $(y,z) \in \mathbb{R}_+^2$, $f \in C^{N+1}([0,1] \times \mathbb{R}_+ \times \mathbb{R}^4)$, $f_i \in C^N([0,1] \times \mathbb{R}_+ \times \mathbb{R}^4)$, $(i = 1, \cdots, p)$ is established in Section 4. This result is a relative generalization of [16] - [20], [28], [29].

2 Preliminaries

Without losing of generality, we consider Prob. (1.1) - (1.3) with $\lambda = 1$.

We put $\Omega = (0,1)$ and denote the usual function spaces used in this paper by the notations $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$. Let $\langle \cdot, \cdot \rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in L^2 and we denote by $\|\cdot\|_X$ the norm in the Banach space X. We call X' the dual space of X.

We denote by $L^p(0,T;X)$, $1 \leq p \leq \infty$ for the Banach space of real functions $u : (0,T) \to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt\right)^{1/p} < \infty \text{ for } 1 \leq p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \underset{0<t<T}{ess\sup} \|u(t)\|_X \text{ for } p = \infty.$$

On H^1, we shall use the following norm

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2\right)^{1/2}. \tag{2.1}$$

Then the following lemma is known.
Lemma 2.1. *The imbedding $H^1 \hookrightarrow C(\overline{\Omega})$ is compact and*

$$\|v\|_{C(\overline{\Omega})} \leq \sqrt{2}\, \|v\|_{H^1} \textit{ for all } v \in H^1. \tag{2.2}$$

Remark 2.1. On H_0^1, $v \longmapsto \|v\|_{H^1}$ and $v \longmapsto \|v_x\|$ are equivalent norms. Furthermore,

$$\|v\|_{C(\overline{\Omega})} \leq \|v_x\| \text{ for all } v \in H_0^1. \tag{2.3}$$

Let $u(t)$, $u'(t) = u_t(t) = \dot{u}(t)$, $u''(t) = u_{tt}(t) = \ddot{u}(t)$, $u_x(t) = \bigtriangledown u(t)$, $u_{xx}(t) = \Delta u(t)$, denote $u(x,t)$, $\dfrac{\partial u}{\partial t}(x,t)$, $\dfrac{\partial^2 u}{\partial t^2}(x,t)$, $\dfrac{\partial u}{\partial x}(x,t)$, $\dfrac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

With $f \in C^N([0,1] \times \mathbb{R}_+ \times \mathbb{R}^4)$, $f = f(x,t,u,v,w,z)$, we put $D_1 f = \dfrac{\partial f}{\partial x}$, $D_2 f = \dfrac{\partial f}{\partial t}$, $D_3 f = \dfrac{\partial f}{\partial u}$, $D_4 f = \dfrac{\partial f}{\partial v}$, $D_5 f = \dfrac{\partial f}{\partial w}$, $D_6 f = \dfrac{\partial f}{\partial z}$ and $D^\alpha f = D_1^{\alpha_1} \cdots D_6^{\alpha_6} f$; $\alpha = (\alpha_1, \cdots, \alpha_6) \in \mathbb{Z}_+^6$, $|\alpha| = \alpha_1 + \cdots + \alpha_6 = N$; $D^{(0,\cdots,0)} f = f$.

Similarly, with $B \in C^N(\mathbb{R}_+^2)$, $B = B(y,z)$, we put $D_1 B = \dfrac{\partial B}{\partial y}$, $D_2 B = \dfrac{\partial B}{\partial z}$ and $D^\beta B = D_1^{\beta_1} D_2^{\beta_2} B$, $\beta = (\beta_1, \beta_2) \in \mathbb{Z}_+^2$, $|\beta| = \beta_1 + \beta_2 = N$; $D^{(0,0)} B = B$.

3 The existence and uniqueness theorem

We make the following assumptions:
(H_1) $\tilde{u}_0, \tilde{u}_1 \in H^2 \cap H_0^1$;
(H_2) $B \in C^1(\mathbb{R}_+^2)$ and $B(y,z) \geq b_0 > 0$, for all $(y,z) \in \mathbb{R}_+^2$;
(H_3) $f \in C^1(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}^4)$ and
$f(0,t,0,v,0,z) = f(1,t,0,v,0,z) = 0$, for all $(t,v,z) \in \mathbb{R}_+ \times \mathbb{R}^2$.

The weak formulation of Prob. (1.1) - (1.3) can be stated in the following manner: Find $u \in W_T = \{u \in L^\infty\left(0,T;H^2 \cap H_0^1\right) : u',\, u'' \in L^\infty\left(0,T;H^2 \cap H_0^1\right)\}$, such that u satisfies the following variational equation

$$\langle u''(t), w\rangle + B[u](t)\langle u_x(t) + u_x''(t), w_x\rangle = \langle F[u](t), w\rangle, \tag{3.1}$$

for all $w \in H_0^1$, a.e., $t \in (0,T)$, together with the initial conditions

$$u(0) = \tilde{u}_0,\ u'(0) = \tilde{u}_1, \tag{3.2}$$

where

$$B[u](t) = B\left(\|u(t)\|^2, \|u_x(t)\|^2\right), \tag{3.3}$$
$$F[u](x,t) = F\left(x,t,u(x,t),u_x(x,t),u'(x,t),u'_x(x,t)\right).$$

Consider $T^* > 0$ fixed, let $M > 0$, we put

$$\tilde{K}_M(B) = \|B\|_{C^1([0,M]^2)} = \sup_{0\le y,z\le M^2}\left(B(y,z) + |D_1B(y,z)| + |D_2B(y,z)|\right), \tag{3.4}$$
$$K_M(f) = \|f\|_{C^1(A_M)} = \sup_{(x,t,u,v,w,z)\in A_M}\left(|f(x,t,u,v,w,z)| + \sum_{i=1}^{6}|D_if(x,t,u,v,w,z)|\right),$$

where

$$A_M = \{(x,t,u,v,w,z) \in [0,1]\times[0,T^*]\times\mathbb{R}^4 : |u|,\ |w| \le M,\ |v|,\ |z| \le \sqrt{2}M\}.$$

For each $T \in (0,T^*]$ and $M > 0$, we put

$$\begin{cases} W(M,T) = \Big\{v \in L^\infty(0,T;H^2\cap H_0^1) : v' \in L^\infty(0,T;H^2\cap H_0^1), \\ \qquad\qquad\qquad\qquad v'' \in L^\infty(0,T;H_0^1),\ \text{with } \|v\|_T \le M\Big\}, \\ W_1(M,T) = \Big\{v \in W(M,T) : v'' \in L^\infty(0,T;H^2\cap H_0^1)\Big\}, \end{cases} \tag{3.5}$$

with $\|v\|_T = \max\left\{\|v\|_{L^\infty(0,T;H^2\cap H_0^1)},\ \|v'\|_{L^\infty(0,T;H^2\cap H_0^1)},\ \|v''\|_{L^\infty(0,T;H_0^1)}\right\}$.

We establish the linear recurrent sequence $\{u_m\}$ as follows.

We choose the first term $u_0 \equiv \tilde{u}_0$, suppose that

$$u_{m-1} \in W_1(M,T), \tag{3.6}$$

and associate with Prob. (1.1) - (1.3) the following problem:

Find $u_m \in W_1(M,T)$ $(m \ge 1)$ which satisfies the linear variational problem

$$\begin{cases} \langle u''_m(t), w\rangle + B_m(t)\langle u_{mx}(t) + u''_{mx}(t), w_x\rangle = \langle F_m(t), w\rangle,\ \forall w \in H_0^1, \\ u_m(0) = \tilde{u}_0,\ u'_m(0) = \tilde{u}_1, \end{cases} \tag{3.7}$$

in which

$$\begin{cases} B_m(t) = B[u_{m-1}](t) = B\left(\|u_{m-1}(t)\|^2, \|\nabla u_{m-1}(t)\|^2\right), \\ F_m(x,t) = f[u_{m-1}](x,t) = f(x,t,u_{m-1}(t),\nabla u_{m-1}(t),u'_{m-1}(t),\nabla u'_{m-1}(t)). \end{cases} \tag{3.8}$$

Then we have the following theorem.

Theorem 3.1. *Let* $(H_1)-(H_3)$ *hold. Then there exist positive constants* $M, T > 0$ *such that, for* $u_0 \equiv \tilde{u}_0$, *there exists a recurrent sequence* $\{u_m\} \subset W_1(M,T)$ *defined by* (3.7), (3.8).

Proof. The proof consists of several steps.

Step 1. *The Faedo-Galerkin approximation* (introduced by Lions [14]). Consider a special orthonormal basis $\{w_j\}$ on H_0^1 : $w_j(x) = \sqrt{2}\sin(j\pi x)$, $j \in \mathbb{N}$, formed by the eigenfunctions of the Laplacian $-\Delta = -\frac{\partial^2}{\partial x^2}$. Put

$$u_m^{(k)}(t) = \sum_{j=1}^{k} c_{mj}^{(k)}(t)w_j, \tag{3.9}$$

where the coefficients $c_{mj}^{(k)}$ satisfy a system of linear differential equations

$$\begin{cases} \langle \ddot{u}_m^{(k)}(t), w_j\rangle + B_m(t)\langle u_{mx}^{(k)}(t) + \ddot{u}_{mx}^{(k)}(t), w_{jx}\rangle = \langle F_m(t), w_j\rangle, \, 1 \leq j \leq k, \\ u_m^{(k)}(0) = \tilde{u}_{0k}, \, \dot{u}_m^{(k)}(0) = \tilde{u}_{1k}, \end{cases} \tag{3.10}$$

in which

$$\begin{cases} \tilde{u}_{0k} = \sum_{j=1}^{k} \alpha_j^{(k)} w_j \to \tilde{u}_0 \text{ strongly in } H^2 \cap H_0^1, \\ \tilde{u}_{1k} = \sum_{j=1}^{k} \beta_j^{(k)} w_j \to \tilde{u}_1 \text{ strongly in } H^2 \cap H_0^1. \end{cases} \tag{3.11}$$

System (3.10) can be rewritten in form

$$\begin{cases} \ddot{c}_{mj}^{(k)}(t) + \mu_{mj}(t) c_{mj}^{(k)}(t) = f_{mj}(t), \\ c_m^{(k)}(0) = \alpha_j^{(k)}, \, \dot{c}_{mj}^{(k)}(0) = \beta_j^{(k)}, \, 1 \leq j \leq k, \end{cases} \tag{3.12}$$

where

$$\begin{aligned} f_{mj}(t) &= \frac{1}{1 + \lambda_j B_m(t)} \langle F_m(t), w_j\rangle, \\ \mu_{mj}(t) &= \frac{\lambda_j B_m(t)}{1 + \lambda_j B_m(t)}, \ \lambda_j = (j\pi)^2, \ 1 \leq j \leq k. \end{aligned} \tag{3.13}$$

Hence

$$\begin{aligned} c_{mj}^{(k)}(t) = \alpha_j^{(k)} + t\beta_j^{(k)} &+ \int_0^t dr \int_0^r f_{mj}(s) ds \\ &- \int_0^t dr \int_0^r \mu_{mj}(s) c_{mj}^{(k)}(s) ds, \ 1 \leq j \leq k. \end{aligned} \tag{3.14}$$

Note that by (3.6), it is not difficult to prove that the system (3.14) has a unique solution $c_{mj}^{(k)}(t), 1 \leq j \leq k$ on interval $[0, T]$, so let us omit the details.

Step 2. A priori estimates. Put

$$S_m^{(k)}(t) = p_m^{(k)}(t) + q_m^{(k)}(t) + r_m^{(k)}(t), \tag{3.15}$$

where

$$\begin{cases} p_m^{(k)}(t) = \left\| \dot{u}_m^{(k)}(t) \right\|^2 + B_m(t) \left(\left\| u_{mx}^{(k)}(t) \right\|^2 + \left\| \dot{u}_{mx}^{(k)}(t) \right\|^2 \right), \\ q_m^{(k)}(t) = \left\| \dot{u}_{mx}^{(k)}(t) \right\|^2 + B_m(t) \left(\left\| \Delta u_m^{(k)}(t) \right\|^2 + \left\| \Delta \dot{u}_m^{(k)}(t) \right\|^2 \right), \\ r_m^{(k)}(t) = \left\| \ddot{u}_m^{(k)}(t) \right\|^2 + B_m(t) \left(\left\| \dot{u}_{mx}^{(k)}(t) \right\|^2 + \left\| \ddot{u}_{mx}^{(k)}(t) \right\|^2 \right). \end{cases} \tag{3.16}$$

Then, it follows from (3.10), (3.15), (3.16) that

$$\begin{aligned} b_0\bar{S}_m^{(k)}(t) \le S_m^{(k)}(t) &= S_m^{(k)}(0) + 2\int_0^t \langle F_m(s), \dot{u}_m^{(k)}(s)\rangle ds \\ &+2\int_0^t \langle F_{mx}(s), \dot{u}_{mx}^{(k)}(s)\rangle ds + 2\int_0^t \langle F_m'(s), \ddot{u}_m^{(k)}(s)\rangle ds \\ &+ \int_0^t B_m'(s)\left[\left\|u_{mx}^{(k)}(s)\right\|^2 + 2\left\|\dot{u}_{mx}^{(k)}(s)\right\|^2 + \left\|\Delta u_m^{(k)}(s)\right\|^2 \right. \\ &\left. + \left\|\Delta\dot{u}_m^{(k)}(s)\right\|^2 - \left\|\ddot{u}_{mx}^{(k)}(s)\right\|^2 - 2\langle u_{mx}^{(k)}(s), \ddot{u}_{mx}^{(k)}(s)\rangle\right] ds \\ &= S_m^{(k)}(0) + \sum_{j=1}^4 I_j, \end{aligned} \tag{3.17}$$

where

$$\bar{S}_m^{(k)}(t) = \left\|u_m^{(k)}(t)\right\|^2_{H^2\cap H_0^1} + \left\|\dot{u}_m^{(k)}(t)\right\|^2_{H^2\cap H_0^1} + \left\|\ddot{u}_{mx}^{(k)}(t)\right\|^2. \tag{3.18}$$

First, we are going to estimate $\xi_m^{(k)} = \left\|\ddot{u}_m^{(k)}(0)\right\|^2 + B_m(0)\left\|\ddot{u}_{mx}^{(k)}(0)\right\|^2$.

Letting $t \to 0_+$ in Eq. $(3.10)_1$, multiplying the result by $\ddot{c}_{mj}^{(k)}(0)$, we get

$$\left\|\ddot{u}_m^{(k)}(0)\right\|^2 + B_m(0)\left\|\ddot{u}_{mx}^{(k)}(0)\right\|^2 + B_m(0)\left\langle \tilde{u}_{0kx}, \ddot{u}_{mx}^{(k)}(0)\right\rangle = \left\langle F_m(0), \ddot{u}_{mx}^{(k)}(0)\right\rangle. \tag{3.19}$$

This implies that

$$\begin{aligned} \xi_m^{(k)} &= \left\|\ddot{u}_m^{(k)}(0)\right\| \\ &\le B_m(0)\left\|\tilde{u}_{0kx}\right\|\left\|\ddot{u}_{mx}^{(k)}(0)\right\| + \left\|F_m(0)\right\|\left\|\ddot{u}_m^{(k)}(0)\right\| \\ &\le \left(B_m(0)\left\|\tilde{u}_{0kx}\right\| + \left\|F_m(0)\right\|\right)\left\|\ddot{u}_{mx}^{(k)}(0)\right\| \\ &\le \left(B_m(0)\left\|\tilde{u}_{0kx}\right\| + \left\|F_m(0)\right\|\right)\sqrt{\frac{\xi_m^{(k)}}{B_m(0)}} \\ &\le \frac{1}{B_m(0)}\left(B_m(0)\left\|\tilde{u}_{0kx}\right\| + \left\|F_m(0)\right\|\right)^2 \\ &\le \overline{X}_0, \text{ for all } m, k \in \mathbb{N}, \end{aligned} \tag{3.20}$$

where $\overline{X}_0$ is a constant depending only on f, $\tilde{u}_0$, $\tilde{u}_1$ and B.

By (3.11), (3.15), (3.16) and (3.20), we get

$$\begin{aligned} S_m^{(k)}(0) &= B\left(\|\nabla\tilde{u}_0\|^2, \|\nabla\tilde{u}_1\|^2\right)\left[\|\tilde{u}_{0kz}\|^2 + 2\|\tilde{u}_{1kz}\|^2 + \|\Delta\tilde{u}_{0k}\|^2\right. \\ &\left. + \|\Delta\tilde{u}_{1kz}\|^2 + \|\tilde{u}_{1k}\|^2 + \|\tilde{u}_{1kz}\|^2\right] + \xi_m^{(k)} \le S_0, \quad \text{for all } m \in \mathbb{N}, \end{aligned} \tag{3.21}$$

with a constant S_0 depending only on f, $\tilde{u}_0$, $\tilde{u}_1$ and B.

Next, we shall estimate three terms I_j on the right - hand side of (3.17) as follows.

First term I_1. By the Cauchy-Schwartz inequality, we have

$$I_1 = 2\int_0^t \langle F_m(s), \dot{u}_m^{(k)}(s)\rangle ds \le TK_M^2(f) + \int_0^t \bar{S}_m^{(k)}(s) ds. \tag{3.22}$$

Second term I_2. It is known that

$$F_{mx}(t) = D_1f[u_{m-1}] + D_3f[u_{m-1}]\nabla u_{m-1}(t) + D_4f[u_{m-1}]\Delta u_{m-1}(t) \quad (3.23)$$
$$+D_5f[u_{m-1}]\nabla u'_{m-1}(t) + D_6f[u_{m-1}]\Delta u'_{m-1}(t),$$

with $D_if[u_{m-1}] = D_if(x,t,u_{m-1}(t),\nabla u_{m-1}(t),u'_{m-1}(t),\nabla u'_{m-1}(t)),\ i = 1,\cdots,6.$ Combining (3.4), (3.6) and (3.23), we obtain

$$\begin{aligned}\|F_{mx}(t)\| \leq &\Big[1 + \|\nabla u_{m-1}(t)\| + \|\Delta u_{m-1}(t)\| \\ &+ \left\|\nabla u'_{m-1}(t)\right\| + \left\|\Delta u'_{m-1}(t)\right\|\Big]K_M(f) \\ \leq &\gamma_M K_M(f), \end{aligned} \quad (3.24)$$

where $\gamma_M = 1 + 4M$, so it implies that

$$\begin{aligned} I_2 &= 2\int_0^t \langle F_{mx}(s), \dot{u}_{mx}^{(k)}(s)\rangle ds \leq 2\int_0^t \|F_{mx}(s)\| \left\|\dot{u}_{mx}^{(k)}(s)\right\| ds \\ &\leq T\gamma_M^2 K_M^2(f) + \int_0^t \bar{S}_m^{(k)}(s)ds. \end{aligned} \quad (3.25)$$

Third term I_3. Similarly, based on the following equality

$$F'_m(t) = D_2f[u_{m-1}] + D_3f[u_{m-1}]u'_{m-1}(t) + D_4f[u_{m-1}]\nabla u'_{m-1}(t) \quad (3.26)$$
$$+D_5f[u_{m-1}]u''_{m-1}(t) + D_6f[u_{m-1}]\nabla u''_{m-1}(t),$$

we obtain

$$\begin{aligned}\left\|F'_m(t)\right\| \leq &\Big[1 + \left\|u'_{m-1}(t)\right\| + \left\|\nabla u'_{m-1}(t)\right\| \\ &+ \left\|u''_{m-1}(t)\right\| + \left\|\nabla u''_{m-1}(t)\right\| \Big]K_M(f) \\ \leq &\gamma_M K_M(f). \end{aligned} \quad (3.27)$$

Thus

$$I_3 = 2\int_0^t 2\langle F'_m(s), \ddot{u}_m^{(k)}(s)\rangle ds \leq T\gamma_M^2 K_M^2(f) + \int_0^t \bar{S}_m^{(k)}(s)ds. \quad (3.28)$$

Fourth term I_4. It is obviously that

$$\begin{aligned} B'_m(t) = &2D_1B[u_{m-1}]\langle u_{m-1}(t), u'_{m-1}(t)\rangle \\ &+ 2D_2B[u_{m-1}]\langle \nabla u_{m-1}(t), \nabla u'_{m-1}(t)\rangle, \end{aligned} \quad (3.29)$$

with $D_iB[u_{m-1}] = D_iB\left(\|\nabla u_{m-1}(t)\|^2, \left\|\nabla u'_{m-1}(t)\right\|^2\right), i = 1,2.$ Hence, by the Cauchy-Schwartz inequality, and (3.6), we have

$$\begin{aligned}\left|B'_m(t)\right| &\leq 2\left[\|u_{m-1}(t)\| \left\|u'_{m-1}(t)\right\| + \|\nabla u_{m-1}(t)\| \left\|\nabla u'_{m-1}(t)\right\| \right] \tilde{K}_M(B) \\ &\leq 4M^2\tilde{K}_M(B). \end{aligned} \quad (3.30)$$

Note that

$$-\left\|\ddot{u}_{mx}^{(k)}(s)\right\|^2 - 2\langle u_{mx}^{(k)}(s), \ddot{u}_{mx}^{(k)}(s)\rangle \leq \left\|u_{mx}^{(k)}(s)\right\|^2, \quad (3.31)$$

hence, from (3.18), (3.30), (3.31), we obtain

$$\begin{aligned} I_4 &= \int_0^t B_m'(s) \left[\left\| u_{mx}^{(k)}(s) \right\|^2 + 2 \left\| \dot{u}_{mx}^{(k)}(s) \right\|^2 + \left\| \Delta u_m^{(k)}(s) \right\|^2 \right. \\ &\qquad \left. + \left\| \Delta \dot{u}_m^{(k)}(s) \right\|^2 - \left\| \ddot{u}_{mx}^{(k)}(s) \right\|^2 - 2\langle u_{mx}^{(k)}(s), \ddot{u}_{mx}^{(k)}(s) \rangle \right] ds \\ &\le \int_0^t \left| B_m'(s) \right| \left[2 \left\| u_{mx}^{(k)}(s) \right\|^2 + 2 \left\| \dot{u}_{mx}^{(k)}(s) \right\|^2 + \left\| \Delta u_m^{(k)}(s) \right\|^2 + \left\| \Delta \dot{u}_m^{(k)}(s) \right\|^2 \right] ds \\ &\le 2 \int_0^t \left| B_m'(s) \right| \left[\left\| u_m^{(k)}(s) \right\|_{H^2 \cap H_0^1}^2 + \left\| \dot{u}_m^{(k)}(s) \right\|_{H^2 \cap H_0^1}^2 \right] ds \\ &\le 8M^2 \tilde{K}_M(B) \int_0^t \bar{S}_m^{(k)}(s) ds. \end{aligned} \tag{3.32}$$

Finally, from (3.17), (3.21), (3.22), (3.25), (3.28), (3.32), the following inequality is fulfilled

$$\bar{S}_m^{(k)}(t) \le \frac{S_0}{b_0} + \left(\frac{1 + 2\gamma_M^2}{b_0} \right) T K_M^2(f) + \frac{\left(3 + 8M^2 \tilde{K}_M(B) \right)}{b_0} \int_0^t \bar{S}_m^{(k)}(s) ds. \tag{3.33}$$

We can choose $M > 0$ sufficiently large such that

$$\frac{S_0}{b_0} \le \frac{1}{2} M^2, \tag{3.34}$$

next choice to get $T \in (0, T^*]$ small enough such that

$$\left[\frac{1}{2} M^2 + \left(\frac{1 + 2\gamma_M^2}{b_0} \right) T K_M^2(f) \right] \exp \left(\frac{T \left(3 + 8M^2 \tilde{K}_M(B) \right)}{b_0} \right) \le M^2, \tag{3.35}$$

and

$$k_T = 4 \sqrt{T \left(\frac{K_M^2(f) + 16M^4 \tilde{K}_M^2(B)}{b_0} \right)} \exp \left[T \left(\frac{1 + 2M^2 \tilde{K}_M(B)}{b_0} \right) \right] < 1. \tag{3.36}$$

It follows from (3.33) - (3.35) that

$$\bar{S}_m^{(k)}(t) \le M^2 \exp \left(\frac{-T \left(3 + 8M^2 \tilde{K}_M(B) \right)}{b_0} \right) + \frac{\left(3 + 8M^2 \tilde{K}_M(B) \right)}{b_0} \int_0^t \bar{S}_m^{(k)}(s) ds. \tag{3.37}$$

By using Gronwall's Lemma, (3.37) yields

$$\bar{S}_m^{(k)}(t) \le M^2 \exp \left(\frac{-T \left(3 + 8M^2 \tilde{K}_M(B) \right)}{b_0} \right) \exp \left(\frac{t \left(3 + 8M^2 \tilde{K}_M(B) \right)}{b_0} \right) \le M^2, \tag{3.38}$$

for all $t \in [0, T]$, for all m and k. Therefore

$$u_m^{(k)} \in W(M, T), \text{ for all } m \text{ and } k \in \mathbb{N}. \tag{3.39}$$

Step 3. Limiting process. From (3.38), we deduce the existence of a subsequence of $\{u_m^{(k)}\}$ denoted by the same symbol, such that

$$\begin{cases} u_m^{(k)} \to u_m & \text{in} \quad L^\infty(0,T;H^2 \cap H_0^1) \text{ weakly*}, \\ \dot{u}_m^{(k)} \to u_m' & \text{in} \quad L^\infty(0,T;H^2 \cap H_0^1) \text{ weakly*}, \\ \ddot{u}_m^{(k)} \to u_m'' & \text{in} \quad L^\infty(0,T;H_0^1) \text{ weakly*}, \\ u_m \in W(M,T). \end{cases} \tag{3.40}$$

Passing to limit in (3.10), (3.11), we have u_m satisfying (3.7), (3.8) in $L^2(0,T)$. On the other hand, it follows from $(3.7)_1$ and $(3.40)_4$ that

$$\Delta u_m'' = \frac{u_m'' - F_m}{B_m(t)} - \Delta u_m \in L^\infty(0,T;L^2). \tag{3.41}$$

Consequently

$$u_m'' \in L^\infty(0,T;H^2 \cap H_0^1), \tag{3.42}$$

so $u_m \in W_1(M,T)$ and the proof of Theorem 3.1 is complete. □

Remark 3.1. It follows from (3.40) and (3.42) that

$$u_m \in C^1([0,T];H^2 \cap H_0^1),\ u_m'' \in L^\infty(0,T;H^2 \cap H_0^1). \tag{3.43}$$

We use the result obtained in Theorem 3.1 and the compact imbedding theorems to get the existence and uniqueness of a weak solution of Prob. (1.1) - (1.3). The main result in this section as follows.

Theorem 3.2. *Let* $(H_1) - (H_3)$ *hold. Then*

(i) *Prob.* (1.1) - (1.3) *has a unique weak solution* $u \in W_1(M,T)$, *where the constants* $M > 0$ *and* $T > 0$ *are chosen as in Theorem 3.1.*

Furthermore,

(ii) *The linear recurrent sequence* $\{u_m\}$ *defined by* (3.7), (3.8) *converges to the solution* u *of Prob.* (1.1) - (1.3) *strongly in* $C^1([0,T];H_0^1)$.

And we have the estimate

$$\|u_m - u\|_{C^1([0,T];H_0^1)} \le C k_T^m, \textit{ for all } m \in \mathbb{N}, \tag{3.44}$$

where the constant $k_T \in (0,1)$ *is defined as in* (3.36) *and* C *is a constant only depending on* T, f, $\tilde{u}_0$, $\tilde{u}_1$ *and* k_T.

Proof. (a) *Existence*. First, we note that $C^1([0,T];H_0^1)$ is a Banach space with respect to the norm (see Lions [14]).

$$\|v\|_{C^1([0,T];H_0^1)} = \|v\|_{C([0,T];H_0^1)} + \|v'\|_{C([0,T];H_0^1)}. \tag{3.45}$$

We shall prove that $\{u_m\}$ is a Cauchy sequence in $C^1([0,T];H_0^1)$. Let $w_m = u_{m+1} - u_m$. Then w_m satisfies the variational problem

$$\begin{cases} \langle w_m''(t), w\rangle + B_{m+1}(t)\langle w_{mx}(t) + w_{mx}''(t), w_x\rangle \\ \qquad = -\left[B_{m+1}(t) - B_m(t)\right]\langle u_{mx}(t) + u_{mx}''(t), w_x\rangle \\ \qquad\quad + \langle F_{m+1}(t) - F_m(t), w\rangle,\ \forall w \in H_0^1, \\ w_m(0) = w_m'(0) = 0. \end{cases} \tag{3.46}$$

Taking $w = w'_m$ in (3.46), after integrating in t, we get

$$\begin{aligned} b_0 \bar{Z}_m(t) \leq Z_m(t) = &\int_0^t B'_{m+1}(s) \left(\|w_{mx}(s)\|^2 + \left\|w'_{mx}(s)\right\|^2 \right) ds \\ &-2\int_0^t \left[B_{m+1}(s) - B_m(s)\right] \left[\left\langle u_{mx}(s), w'_{mx}(s) \right\rangle + \left\langle u''_{mx}(s), w'_{mx}(s) \right\rangle \right] ds \\ &+2\int_0^t \left\langle F_{m+1}(s) - F_m(s), w'_m(s) \right\rangle ds \\ = &J_1 + J_2 + J_3, \end{aligned} \tag{3.47}$$

where

$$\begin{aligned} Z_m(t) &= \left\|w'_m(t)\right\|^2 + B_{m+1}(t) \left(\|w_{mx}(t)\|^2 + \left\|w'_{mx}(t)\right\|^2 \right), \\ \bar{Z}_m(t) &= \|w_{mx}(t)\|^2 + \left\|w'_{mx}(t)\right\|^2. \end{aligned} \tag{3.48}$$

We shall estimate three integrals J_1, J_2, J_3 on the right – hand side of (3.47) as follows.
Estimates J_1. By

$$\left|B'_{m+1}(t)\right| \leq 2\left[\|u_m(t)\| \left\|u'_m(t)\right\| + \|\nabla u_m(t)\| \left\|\nabla u'_m(t)\right\| \right] \tilde{K}_M(B) \leq 4M^2 \tilde{K}_M(B), \tag{3.49}$$

we have

$$\begin{aligned} J_1 &= \int_0^t B'_{m+1}(s) \left(\|w_{mx}(s)\|^2 + \left\|w'_{mx}(s)\right\|^2 \right) ds \\ &\leq 4M^2 \tilde{K}_M(B) \int_0^t \left(\|w_{mx}(s)\|^2 + \left\|w'_{mx}(s)\right\|^2 \right) ds \leq 4M^2 \tilde{K}_M(B) \int_0^t \bar{Z}_m(s). \end{aligned} \tag{3.50}$$

Estimates J_2. We have

$$\begin{aligned} &\left|B_{m+1}(t) - B_m(t)\right| \\ &= \left| B\left(\|u_m(t)\|^2, \|\nabla u_m(t)\|^2 \right) - B\left(\|u_{m-1}(t)\|^2, \|\nabla u_{m-1}(t)\|^2 \right) \right| \\ &\leq \tilde{K}_M(B) \left[\left| \|u_m(t)\|^2 - \|u_{m-1}(t)\|^2 \right| + \left| \|\nabla u_m(t)\|^2 - \|\nabla u_{m-1}(t)\|^2 \right| \right] \\ &\leq 2M\tilde{K}_M(B) \left[\|w_{m-1}(t)\| + \|\nabla w_{m-1}(t)\| \right] \\ &\leq 4M\tilde{K}_M(B) \|w_{m-1}\|_{C^1([0,T];H_0^1)}. \end{aligned} \tag{3.51}$$

Hence

$$\begin{aligned} J_2 &= -2\int_0^t \left[B_{m+1}(s) - B_m(s)\right] \left[\left\langle u_{mx}(s), w'_{mx}(s) \right\rangle + \left\langle u''_{mx}(s), w'_{mx}(s) \right\rangle \right] ds \\ &\leq 16M^2 \tilde{K}_M(B) \|w_{m-1}\|_{C^1([0,T];H_0^1)} \int_0^t \left\|w'_{mx}(s)\right\| ds \\ &\leq 16M^2 \tilde{K}_M(B) \|w_{m-1}\|_{C^1([0,T];H_0^1)} \int_0^t \sqrt{\bar{Z}_m(s)} ds \\ &\leq 64TM^4 \tilde{K}_M^2(B) \|w_{m-1}\|^2_{C^1([0,T];H_0^1)} + \int_0^t \bar{Z}_m(s) ds. \end{aligned} \tag{3.52}$$

Estimates J_3. From (H_3) we obtain from (3.4), (3.6), (3.8), $(3.40)_4$, that

$$\begin{aligned}
&\|F_{m+1}(t)-F_m(t)\| \qquad (3.53)\\
&=\left\|f(x,t,u_m(t),\nabla u_m(t),u_m'(t),\nabla u_m'(t))-f(x,t,u_{m-1}(t),\nabla u_{m-1}(t),u_{m-1}'(t),\nabla u_{m-1}'(t))\right\|\\
&\le K_M(f)\left[\|w_{m-1}(t)\|+\|\nabla w_{m-1}(t)\|+\left\|w_{m-1}'(t)\right\|+\left\|\nabla w_{m-1}'(t)\right\|\right]\\
&\le 2K_M(f)\left[\|\nabla w_{m-1}(t)\|+\left\|\nabla w_{m-1}'(t)\right\|\ \right]\le 2K_M(f)\,\|w_{m-1}\|_{C^1([0,T];H_0^1)}\,.
\end{aligned}$$

Hence

$$\begin{aligned}
J_3&=2\int_0^t\left\langle F_{m+1}(s)-F_m(s),w_m'(s)\right\rangle ds \qquad (3.54)\\
&\le 4K_M(f)\,\|w_{m-1}\|_{C^1([0,T];H_0^1)}\int_0^t\left\|w_m'(s)\right\|ds\\
&\le 4TK_M^2(f)\,\|w_{m-1}\|_{C^1([0,T];H_0^1)}^2+\int_0^t\bar{Z}_m(s)ds.
\end{aligned}$$

Combining (3.47), (3.50), (3.52) and (3.54), we obtain

$$\begin{aligned}
\bar{Z}_m(t)&\le 4T\left(\frac{K_M^2(f)+16M^4\tilde{K}_M^2(B)}{b_0}\right)\|w_{m-1}\|_{C^1([0,T];H_0^1)}^2 \qquad (3.55)\\
&\quad+2\left(\frac{1+2M^2\tilde{K}_M(B)}{b_0}\right)\int_0^t\bar{Z}_m(s).
\end{aligned}$$

Using Gronwall's Lemma, we deduce from (3.55) that

$$\|w_m\|_{C^1([0,T];H_0^1)}\le k_T\,\|w_{m-1}\|_{C^1([0,T];H_0^1)}\quad\forall m\in\mathbb{N}, \qquad (3.56)$$

where $0<k_T<1$ is defined as in (3.36), which implies that

$$\|u_m-u_{m+p}\|_{C^1([0,T];H_0^1)}\le\|u_0-u_1\|_{C^1([0,T];H_0^1)}\,(1-k_T)^{-1}k_T^m\ \ \forall m,p\in\mathbb{N}. \qquad (3.57)$$

It follows that $\{u_m\}$ is a Cauchy sequence in $C^1([0,T];H_0^1)$. Then there exists $u\in C^1([0,T];H_0^1)$ such that

$$u_m\to u \text{ strongly in } C^1([0,T];H_0^1). \qquad (3.58)$$

Note that $u_m\in W_1(M,T)$, then there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\left\{\begin{array}{lll}
u_{m_j}\to u & \text{in} & L^\infty(0,T;H^2\cap H_0^1)\text{ weakly*},\\
u_{m_j}'\to u' & \text{in} & L^\infty(0,T;H^2\cap H_0^1)\text{ weakly*},\\
u_{m_j}''\to u'' & \text{in} & L^\infty(0,T;H_0^1)\text{ weakly*},\\
u\in W(M,T). & &
\end{array}\right. \qquad (3.59)$$

By (3.4), (3.6), (3.8) and $(3.59)_4$, we obtain

$$\begin{aligned}
&\|F_m(t)-f[u](t)\|\le 2K_M(f)\,\|u_{m-1}-u\|_{C^1([0,T];H_0^1)}\,, \qquad (3.60)\\
&|B_m(t)-B[u](t)|\le 4M\tilde{K}_M(B)\,\|u_{m-1}-u\|_{C^1([0,T];H_0^1)}\,.
\end{aligned}$$

Hence, from (3.58) and (3.60), we obtain

$$\begin{aligned}
&F_m\to f[u]\text{ strongly in }L^\infty(0,T;L^2), \qquad (3.61)\\
&B_m\to B[u]\text{ strongly in }L^\infty(0,T).
\end{aligned}$$

Finally, passing to limit in (1.8), (3.8) as $m = m_j \to \infty$, it implies from (3.58), $(3.59)_{1,3}$, and (3.61) that there exists $u \in W(M,T)$ satisfying the equation

$$\langle u''(t), w\rangle + B[u](t)\langle u_x(t) + u'_x(t), w_x\rangle = \langle f[u](t), w\rangle \,, \tag{3.62}$$

for all $w \in H_0^1$ and the initial conditions

$$u(0) = \tilde{u}_0, \; u'(0) = \tilde{u}_1. \tag{3.63}$$

On the other hand, from assumptions (H_2), (H_3) we obtain from $(3.59)_4$, and (3.62) that

$$\Delta u'' = \frac{1}{B[u](t)}\big(u'' - f[u](t)\big) - \Delta u \in L^\infty(0,T;L^2). \tag{3.64}$$

Hence

$$u'' \in L^\infty(0,T;H^2 \cap H_0^1). \tag{3.65}$$

so $u \in W_1(M,T)$ and the existence follows.

(b) *Uniqueness*. Let u_1, u_2 be two weak solutions of Prob. (1.1) - (1.3), such that $u_i \in W_1(M,T)$, $i = 1,2$. Then $w = u_1 - u_2$ verifies

$$\begin{cases} \langle w''(t), v\rangle + B_1(t)\langle w_x(t) + w''_x(t), v_x\rangle = \langle \bar{F}_1(t) - \bar{F}_2(t), v\rangle \\ \qquad - \big[\bar{B}_1(t) - \bar{B}_2(t)\big]\,\langle u_{2x}(t) + u''_{2x}(t), v_x\rangle, \text{ for all } v \in H_0^1, \\ w(0) = w'(0) = 0. \end{cases} \tag{3.66}$$

where $\bar{B}_i(t) = B[u_i](t)$, $\bar{F}_i(x,t) = f[u_i](x,t)$, $i = 1,2$.

Taking $v = w' = u'_1 - u'_2$ in $(3.66)_1$ and integrating with respect to t, we obtain

$$\begin{aligned} \sigma(t) =& 2\int_0^t \big\langle \bar{F}_1(s) - \bar{F}_2(s), w'(s)\big\rangle\, ds \\ &+ \int_0^t \bar{B}'_1(s)\left(\|w_x(s)\|^2 + \|w'_x(s)\|^2\right) ds \\ &- 2\int_0^t \big[\bar{B}_1(s) - \bar{B}_2(s)\big]\big[\big\langle u_{2x}(s), w'_x(s)\big\rangle + \big\langle u''_{2x}(s), w'_x(s)\big\rangle\big]\, ds, \end{aligned} \tag{3.67}$$

where $\sigma(t) = \|w'(t)\|^2 + \bar{B}_1(t)\left(\|w_x(t)\|^2 + \|w'_x(t)\|^2\right)$.

Put $\hat{K}_M = \dfrac{4}{b_0}\left[\left(1 + 4\sqrt{2}\right)M^2\tilde{K}_M(B) + \sqrt{2b_0}K_M(f)\right]$, then it follows from (3.67) that

$$\sigma(t) \le \hat{K}_M \int_0^t \sigma(s)ds.$$

By Gronwall's Lemma, we deduce $\sigma(t) = 0$, i.e., $u_1 \equiv u_2$. Theorem 3.2 is proved completely. □

4 Asymptotic expansion of the solution with respect to p small parameters

In this section, let $(H_1)-(H_3)$ hold. We also make the following assumptions:

(H_4) $B_i \in C^1(\mathbb{R}_+^2)$, $B_i(y,z) \geq 0$, for all $(y,z) \in \mathbb{R}_+^2$, $(i=1,\cdots,p)$,

(H_5) $f_i \in C^1([0,1] \times \mathbb{R}_+ \times \mathbb{R}^4)$, and $f_i(0,t,0,v,0,z) = f_i(1,t,0,v,0,z) = 0$, for all $(t,v,z) \in \mathbb{R}_+ \times \mathbb{R}^2$, $(i=1,\cdots,p)$.

We consider the following perturbed problem, where $\varepsilon_1,\cdots,\varepsilon_p$ are p small parameters such that, $0 \leq \varepsilon_i < 1$, $i=1,\cdots,p$:

$$(P_{\boldsymbol{\varepsilon}}) \begin{cases} u_{tt} - B_{\boldsymbol{\varepsilon}}(\|u\|^2, \|u_x\|^2)Au = F_{\boldsymbol{\varepsilon}}(x,t,u,u_x,u_t,u_{xt}),\, 0<x<1,\, 0<t<T, \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\, u_t(x,0) = \tilde{u}_1(x), \\ Au = u_{xx} + u_{xxtt}, \\ B_{\boldsymbol{\varepsilon}}(\|u\|^2, \|u_x\|^2) = B(\|u\|^2, \|u_x\|^2) + \sum\limits_{i=1}^{p} \varepsilon_i B_i(\|u\|^2, \|u_x\|^2), \\ F_{\boldsymbol{\varepsilon}}(x,t,u,u_x,u_t,u_{xt}) = f(x,t,u,u_x,u_t,u_{xt}) + \sum\limits_{i=1}^{p} \varepsilon_i f_i(x,t,u,u_x,u_t,u_{xt}). \end{cases}$$

By theorem 3.2, Prob. $(P_{\boldsymbol{\varepsilon}})$ has a unique weak solution u depending on $\boldsymbol{\varepsilon} = (\varepsilon_1,\cdots,\varepsilon_p)$: $u_{\boldsymbol{\varepsilon}} = u(\varepsilon_1,\cdots,\varepsilon_p)$. When $\boldsymbol{\varepsilon} = (0,\cdots,0)$, $(P_{\boldsymbol{\varepsilon}})$ is denoted by (P_0). We shall study the asymptotic expansion of the solution of Prob. $(P_{\boldsymbol{\varepsilon}})$ with respect to p small parameters $\varepsilon_1,\cdots,\varepsilon_p$.

We use the following notations. For a multi-index $\alpha = (\alpha_1,\cdots,\alpha_p) \in \mathbb{Z}_+^p$, and $\boldsymbol{\varepsilon} = (\varepsilon_1,\cdots,\varepsilon_p) \in \mathbb{R}^p$, we put

$$\begin{cases} |\alpha| = \alpha_1 + \cdots + \alpha_p,\ \alpha! = \alpha_1! \cdots \alpha_p!, \\ \|\boldsymbol{\varepsilon}\| = \sqrt{\varepsilon_1^2 + \cdots + \varepsilon_p^2},\ \boldsymbol{\varepsilon}^\alpha = \varepsilon_1^{\alpha_1} \cdots \varepsilon_p^{\alpha_p}, \\ \alpha,\ \beta \in \mathbb{Z}_+^p,\ \alpha \leq \beta \Longleftrightarrow \alpha_i \leq \beta_i\ \ \forall i=1,\cdots,p. \end{cases} \tag{4.1}$$

First, we shall need the following lemma.

Lemma 4.1. *Let* $m,\, N \in \mathbb{N}$ *and* $u_\alpha \in \mathbb{R}$, $\alpha \in \mathbb{Z}_+^p$, $1 \leq |\alpha| \leq N$. *Then*

$$\left(\sum_{1 \leq |\alpha| \leq N} u_\alpha \boldsymbol{\varepsilon}^\alpha \right)^m = \sum_{m \leq |\alpha| \leq mN} T_N^{(m)}[u]_\alpha \boldsymbol{\varepsilon}^\alpha, \tag{4.2}$$

where the coefficients $T_N^{(m)}[u]_\alpha$, $m \leq |\alpha| \leq mN$ *depending on* $u = (u_\alpha)$, $\alpha \in \mathbb{Z}_+^p$, $1 \leq |\alpha| \leq N$ *defined by the recurrence formulas*

$$\begin{cases} T_N^{(1)}[u]_\alpha = u_\alpha,\ 1 \leq |\alpha| \leq N, \\ T_N^{(m)}[u]_\alpha = \sum\limits_{\beta \in A_\alpha^{(m)}(N)} u_{\alpha-\beta} T_N^{(m-1)}[u]_\beta,\ m \leq |\alpha| \leq mN,\ m \geq 2, \\ A_\alpha^{(m)}(N) = \{\beta \in \mathbb{Z}_+^p : \beta \leq \alpha,\ 1 \leq |\alpha-\beta| \leq N,\ m-1 \leq |\beta| \leq (m-1)N\}. \end{cases} \tag{4.3}$$

The proof of Lemma 4.1 can be found in [20].

Now, we assume that

(H_6) $B \in C^{N+1}(\mathbb{R}_+^2)$, $B_i \in C^N(\mathbb{R}_+^2)$,
$B(y,z) \geq b_0 > 0$, $B_i(y,z) \geq 0$, for all $(y,z) \in \mathbb{R}_+^2$, $(i=1,\cdots,p)$,

(H_7) $f \in C^{N+1}([0,1] \times \mathbb{R}_+ \times \mathbb{R}^4)$, $f_i \in C^N([0,1] \times \mathbb{R}_+ \times \mathbb{R}^4)$, and $f(0,t,0,v,0,z) = f(1,t,0,v,0,z) = f_i(0,t,0,v,0,z) = f_i(1,t,0,v,0,z) = 0$, for all $(t,v,z) \in \mathbb{R}_+ \times \mathbb{R}^2$, $(i = 1, \cdots, p)$.

We also use the notations $f[u] = f(x,t,u,u_x,u',u'_x)$, $B[u] = B(\|u\|^2, \|u_x\|^2)$.

Let u_0 be a unique weak solution of Prob. (P_0) (as in Theorem 3.2) corresponding to $\varepsilon = (0, \cdots, 0)$, i.e.,

$$(P_0) \begin{cases} u_0'' - B[u_0]Au_0 = f[u_0], 0 < x < 1, 0 < t < T, \\ u_0(0,t) = u_0(1,t) = 0, \\ u_0(x,0) = \tilde{u}_0(x), u_0'(x,0) = \tilde{u}_1(x), \\ u_0 \in W_1(M,T). \end{cases}$$

Considering the sequence of weak solutions u_ν, $\nu \in \mathbb{Z}_+^p$, $1 \le |\nu| \le N$, of the following problems:

$$(\tilde{P}_\nu) \begin{cases} u_\nu'' - B[u_0]Au_\nu = F_\nu, 0 < x < 1, 0 < t < T, \\ u_\nu(0,t) = u_\nu(1,t) = 0, \\ u_\nu(x,0) = u_\nu'(x,0) = 0, \\ u_\nu \in W_1(M,T), \end{cases}$$

where F_ν, $\nu \in \mathbb{Z}_+^p$, $1 \le |\nu| \le N$, are defined by the recurrence formulas

$$F_\nu = \begin{cases} f[u_0] \equiv f(x,t,u_0,\nabla u_0, u_0', \nabla u_0'), & |\nu| = 0, \\ \pi_\nu[f] + \sum\limits_{i=1}^{p} \pi_\nu^{(i)}[f_i] & \\ \quad + \sum\limits_{\substack{1 \le |\alpha| \le N, \\ |\nu - \alpha| \le N}} \left(\rho_\alpha[B] + \sum_{i=1}^p \rho_\alpha^{(i)}[B_i] \right) Au_{\nu-\alpha}, & 1 \le |\nu| \le N, \end{cases} \tag{4.4}$$

with $\rho_\nu[B] = \rho_\nu[B; \sigma^{(1)}, \sigma^{(2)}]$, $\rho_\nu^{(i)}[B] = \rho_\nu^{(i)}[B; \sigma^{(1)}, \sigma^{(2)}]$, $\pi_\nu[f] = \pi_\nu[f; \{u_\gamma\}_{\gamma \le \nu}]$, $\pi_\nu^{(i)}[f] = \pi_\nu^{(i)}[f; \{u_\gamma\}_{\gamma \le \nu}]$, $|\nu| \le N$, defined by the formulas

$$\begin{aligned} \rho_\nu[B] &= \rho_\nu[B, \sigma^{(1)}, \sigma^{(2)}] \\ &= \begin{cases} B[u_0], & |\nu| = 0, \\ \sum\limits_{|\gamma| \le |\nu|} \frac{1}{\gamma!} D^\gamma B[u_0] \sum\limits_{\substack{\gamma_1 \le |\alpha| \le 2\gamma_1 N, \\ \gamma_2 \le |\nu-\alpha| \le 2\gamma_2 N}} T_{2N}^{(\gamma_1)}[\sigma^{(1)}]_\alpha T_{2N}^{(\gamma_2)}[\sigma^{(2)}]_{\nu-\alpha}, & 1 \le |\nu| \le N, \end{cases} \end{aligned} \tag{4.5}$$

where $\sigma^{(1)} = \left(\sigma_\alpha^{(1)}\right)$, $\sigma^{(2)} = \left(\sigma_\alpha^{(2)}\right)$, $\alpha \in \mathbb{Z}_+^p$, $1 \le |\alpha| \le 2N$, are defined by

$$\sigma_\alpha^{(1)} = \begin{cases} 2\langle u_0, u_\alpha \rangle, & |\alpha| = 1, \\ 2\langle u_0, u_\alpha \rangle + \sum\limits_{\beta \le \alpha} \langle u_\beta, u_{\alpha-\beta} \rangle, & 2 \le |\alpha| \le N, \\ \sum\limits_{\beta \le \alpha} \langle u_\beta, u_{\alpha-\beta} \rangle, & N+1 \le |\alpha| \le 2N, \end{cases} \tag{4.6}$$

$$\sigma_\alpha^{(2)} = \begin{cases} 2\langle \nabla u_0, \nabla u_\alpha \rangle, & |\alpha| = 1, \\ 2\langle \nabla u_0, \nabla u_\alpha \rangle + \sum\limits_{\beta \le \alpha} \langle \nabla u_\beta, \nabla u_{\alpha-\beta} \rangle, & 2 \le |\alpha| \le N, \\ \sum\limits_{\beta \le \alpha} \langle \nabla u_\beta, \nabla u_{\alpha-\beta} \rangle, & N+1 \le |\alpha| \le 2N, \end{cases}$$

$$\rho_\nu^{(i)}[B] = \begin{cases} \rho_{\nu^{(i-)}}[B] = \rho_{\nu_1, \cdots, \nu_{i-1}, \nu_i - 1, \nu_{i+1}, \cdots, \nu_p}[B], & \text{if } \nu_i \ge 1, \\ \rho_{\nu_1, \cdots, \nu_{i-1}, -1, \nu_{i+1}, \cdots, \nu_p}[B] = 0, & \text{if } \nu_i = 0, \end{cases} \tag{4.7}$$

$$\nu = (\nu_1, \cdots, \nu_p) \in \mathbb{Z}_+^p, \ \nu^{(i-)} = (\nu_1, \cdots, \nu_{i-1}, \nu_i - 1, \nu_{i+1}, \cdots, \nu_p), \ i = 1, \cdots, p;$$

$$\pi_\nu[f]=\begin{cases} f[u_0], & |\nu|=0,\\ \sum\limits_{\substack{1\le|m|\le|\nu|\\ m=(m_1,\cdots,m_4)\in\mathbb{Z}_+^4}}\frac{1}{m!}D^mf[u_0]\sum\limits_{\substack{(\alpha,\beta,\gamma,\delta)\in A(m,N)\\ \alpha+\beta+\gamma+\delta=\nu}}T_N^{(m_1)}[u]_\alpha & \\ \qquad\times T_N^{(m_2)}[\nabla u]_\beta T_N^{(m_3)}[u']_\gamma T_N^{(m_4)}[\nabla u']_\delta, & 1\le|\nu|\le N,\end{cases}\tag{4.8}$$

in which $m=(m_1,\cdots,m_4)\in\mathbb{Z}_+^4$, $|m|=m_1+\cdots+m_4$, $m!=m_1!\cdots m_4!$, $D^mf=D_3^{m_1}D_4^{m_2}D_5^{m_3}D_6^{m_4}f$, $A(m,N)=\{(\alpha,\beta,\gamma,\delta)\in\left(\mathbb{Z}_+^p\right)^4: m_1\le|\alpha|\le m_1N,\ m_2\le|\beta|\le m_2N,\ m_3\le|\gamma|\le m_3N,\ m_4\le|\delta|\le m_4N\}$,

$$\begin{cases}\pi_\nu^{(i)}[f]=\pi_{\nu^{(i-)}}[f]=\pi_{\nu_1,\cdots,\nu_{i-1},\nu_i-1,\nu_{i+1},\cdots,\nu_p}[f],\ i=1,\cdots,p,\\ \pi_\nu^{(i)}[f]=\pi_{\nu_1,\cdots,\nu_{i-1},-1,\nu_{i+1},\cdots,\nu_p}[f]=0,\text{ if }\nu_i=0,\\ \nu=(\nu_1,\cdots,\nu_p)\in\mathbb{Z}_+^p,\ \nu^{(i-)}=(\nu_1,\cdots,\nu_{i-1},\nu_i-1,\nu_{i+1},\cdots,\nu_p).\end{cases}\tag{4.9}$$

Then, we have the following lemma.

Lemma 4.2. *Let $\rho_\nu[B]=\rho_\nu[B,\sigma^{(1)},\sigma^{(2)}]$, $\pi_\nu[f]$, $|\nu|\le N$, be the functions are defined by formulas* (4.5) *and* (4.8). *Put $h=\sum_{|\gamma|\le N}u_\gamma\varepsilon^\gamma$, then we have*

$$B[h]=\sum_{|\nu|\le N}\rho_\nu[B]\varepsilon^\nu+\|\varepsilon\|^{N+1}\widetilde{R}_N^{(1)}[B,\varepsilon],\tag{4.10}$$

$$f[h]=\sum_{|\nu|\le N}\pi_\nu[f]\varepsilon^\nu+\|\varepsilon\|^{N+1}\bar{R}_N^{(1)}[f,\varepsilon],\tag{4.11}$$

with $\left\|\widetilde{R}_N^{(1)}[B,\varepsilon]\right\|_{L^\infty(0,T)}+\left\|\bar{R}_N^{(1)}[f,\varepsilon]\right\|_{L^\infty(0,T;L^2)}\le C$, where C is a constant depending only on N, T, f, B, u_γ, $|\gamma|\le N$.

Proof of Lemma 4.2. (i) In the case of $N=1$, the proof of (4.10) is easy, hence we omit the details. We only prove it with $N\ge 2$. Put $h=u_0+\sum_{1\le|\alpha|\le N}u_\alpha\varepsilon^\alpha\equiv u_0+h_1$, we rewrite as below

$$B[h]=B(\|u_0+h_1\|^2,\|\nabla u_0+\nabla h_1\|^2)=B(\|u_0\|^2+\xi_1,\|\nabla u_0\|^2+\xi_2),\tag{4.12}$$

where $\xi_1=\|u_0+h_1\|^2-\|u_0\|^2$, $\xi_2=\|\nabla u_0+\nabla h_1\|^2-\|\nabla u_0\|^2$.

By using Taylor's expansion of the function $B(\|u_0\|^2+\xi_1,\|\nabla u_0\|^2+\xi_2)$ around the point $(\|u_0\|^2,\|\nabla u_0\|^2)$ up to order $N+1$, we obtain

$$\begin{aligned}B[h]&=B(\|u_0\|^2+\xi_1,\|\nabla u_0\|^2+\xi_2)\\&=B(\|u_0\|^2,\|\nabla u_0\|^2)+\sum_{1\le|\gamma|\le N}\frac{1}{\gamma!}D^\gamma B(\|u_0\|^2,\|\nabla u_0\|^2)\xi_1^{\gamma_1}\xi_2^{\gamma_2}+R_N[B,u_0,\xi_1,\xi_2]\\&=B[u_0]+\sum_{1\le|\gamma|\le N}\frac{1}{\gamma!}D^\gamma B[u_0]\xi_1^{\gamma_1}\xi_2^{\gamma_2}+R_N[B,u_0,\xi_1,\xi_2],\end{aligned}\tag{4.13}$$

where

$$\begin{aligned}&R_N[B,u_0,\xi_1,\xi_2]\\&=\sum_{|\gamma|=N+1}\frac{N+1}{\gamma!}\int_0^1(1-\theta)^ND^\gamma B(\|u_0\|^2+\theta\xi_1,\|\nabla u_0\|^2+\theta\xi_2)\xi_1^{\gamma_1}\xi_2^{\gamma_2}d\theta\\&\equiv\|\varepsilon\|^{N+1}R_N^{(1)}[B,u_0,\xi_1,\xi_2].\end{aligned}\tag{4.14}$$

On the other hand, we have

$$\xi_1 = \|u_0 + h_1\|^2 - \|u_0\|^2 = 2\langle u_0, h_1\rangle + \|h_1\|^2 \equiv \sum_{1\le|\alpha|\le 2N} \sigma_\alpha^{(1)} \boldsymbol{\varepsilon}^\alpha, \tag{4.15}$$

with $\sigma_\alpha^{(1)}$, $1 \le |\alpha| \le 2N$ are defined by $(4.6)_1$.

By the formula (4.2), it follows from (4.15) that

$$\xi_1^{\gamma_1} = \left(\sum_{1\le|\alpha|\le 2N} \sigma_\alpha^{(1)} \boldsymbol{\varepsilon}^\alpha\right)^{\gamma_1} = \sum_{\gamma_1\le|\alpha|\le 2\gamma_1 N} T_{2N}^{(\gamma_1)}[\sigma^{(1)}]_\alpha \boldsymbol{\varepsilon}^\alpha, \tag{4.16}$$

where $\sigma^{(1)} = (\sigma_\alpha^{(1)})$, $\alpha \in \mathbb{Z}_+^p$, $1 \le |\alpha| \le 2N$.

Similarly, we have

$$\xi_2^{\gamma_2} = \left(\sum_{1\le|\alpha|\le 2N} \sigma_\alpha^{(2)} \boldsymbol{\varepsilon}^\alpha\right)^{\gamma_2} = \sum_{\gamma_2\le|\alpha|\le 2\gamma_2 N} T_{2N}^{(\gamma_2)}[\sigma^{(2)}]_\alpha \boldsymbol{\varepsilon}^\alpha, \tag{4.17}$$

where $\sigma^{(2)} = (\sigma_\alpha^{(2)})$, $\alpha \in \mathbb{Z}_+^p$, $1 \le |\alpha| \le 2N$, are defined by $(4.6)_2$.

Therefore, it follows from (4.16), (4.17) that

$$\begin{aligned}
\xi_1^{\gamma_1}\xi_2^{\gamma_2} &= \sum_{|\gamma|\le|\nu|\le 2|\gamma|N} \left(\sum_{\substack{\gamma_1\le|\alpha|\le 2\gamma_1 N,\\ \gamma_2\le|\nu-\alpha|\le 2\gamma_2 N}} T_{2N}^{(\gamma_1)}[\sigma^{(1)}]_\alpha T_{2N}^{(\gamma_2)}[\sigma^{(2)}]_{\nu-\alpha}\right)\boldsymbol{\varepsilon}^\nu \\
&= \sum_{|\gamma|\le|\nu|\le 2|\gamma|N} \Phi_\nu[N,\sigma^{(1)},\sigma^{(2)},\gamma_1,\gamma_2,\alpha]\boldsymbol{\varepsilon}^\nu \\
&= \sum_{|\gamma|\le|\nu|\le N} \Phi_\nu[N,\sigma^{(1)},\sigma^{(2)},\gamma_1,\gamma_2,\alpha]\boldsymbol{\varepsilon}^\nu + \sum_{N+1\le|\nu|\le 2|\gamma|N} \Phi_\nu[N,\sigma^{(1)},\sigma^{(2)},\gamma_1,\gamma_2,\alpha]\boldsymbol{\varepsilon}^\nu \\
&= \sum_{|\gamma|\le|\nu|\le N} \Phi_\nu[N,\sigma^{(1)},\sigma^{(2)},\gamma_1,\gamma_2,\alpha]\boldsymbol{\varepsilon}^\nu + \|\boldsymbol{\varepsilon}\|^{N+1} R_N[N,\sigma^{(1)},\sigma^{(2)},\gamma_1,\gamma_2,\alpha,\boldsymbol{\varepsilon}],
\end{aligned} \tag{4.18}$$

where

$$\begin{cases}
\Phi_\nu[N,\sigma^{(1)},\sigma^{(2)},\gamma_1,\gamma_2,\alpha] = \sum\limits_{\substack{\gamma_1\le|\alpha|\le 2\gamma_1 N,\\ \gamma_2\le|\nu-\alpha|\le 2\gamma_2 N}} T_{2N}^{(\gamma_1)}[\sigma^{(1)}]_\alpha T_{2N}^{(\gamma_2)}[\sigma^{(2)}]_{\nu-\alpha}, \\
\|\boldsymbol{\varepsilon}\|^{N+1} R_N[N,\sigma^{(1)},\sigma^{(2)},\gamma_1,\gamma_2,\alpha,\boldsymbol{\varepsilon}] = \sum\limits_{N+1\le|\nu|\le 2|\gamma|N} \Phi_\nu[N,\sigma^{(1)},\sigma^{(2)},\gamma_1,\gamma_2,\alpha]\boldsymbol{\varepsilon}^\nu.
\end{cases} \tag{4.19}$$

Hence, we deduce from (4.13), (4.18), (4.19) that

$$B[h] = \sum_{|\nu|\le N} \rho_\nu[B,\sigma^{(1)},\sigma^{(2)}]\boldsymbol{\varepsilon}^\nu + \|\boldsymbol{\varepsilon}\|^{N+1} \widehat{R}_N^{(1)}[B,u_0,\sigma^{(1)},\sigma^{(2)},\xi_1,\xi_2], \tag{4.20}$$

where $\rho_\nu[B] = \rho_\nu[B;\sigma^{(1)},\sigma^{(2)}]$, $\nu \in \mathbb{Z}_+^p$, $|\nu| \le N$, are defined by (4.5) and

$$\begin{aligned}
&\widehat{R}_N^{(1)}[B,u_0,\sigma^{(1)},\sigma^{(2)},\xi_1,\xi_2] \\
&= \sum_{1\le|\gamma|\le N} \frac{1}{\gamma!} D^\gamma B[u_0] R_N[N,\sigma^{(1)},\sigma^{(2)},\gamma_1,\gamma_2,\alpha,\boldsymbol{\varepsilon}] + R_N^{(1)}[B,u_0,\xi_1,\xi_2].
\end{aligned} \tag{4.21}$$

By the boundedness of the functions $u_\gamma, u'_\gamma, |\gamma| \le N$ in the function space $L^\infty(0,T;H_0^1 \cap H^2)$, we obtain from (4.14), (4.19), (4.21) that $\left\| \widehat{R}_N^{(1)}[B, u_0, \sigma^{(1)}, \sigma^{(2)}, \xi_1, \xi_2] \right\|_{L^\infty(0,T)} \le C$, where C is a constant depending only on N, T, B, $\|u_\gamma\|_{L^\infty(0,T;L^2)}$, $\|\nabla u_\gamma\|_{L^\infty(0,T;L^2)}$, $|\gamma| \le N$. Hence, the part 1 of Lemma 4.2 is proved.

(ii) We only prove (4.11) with $N \ge 2$. By using Taylor's expansion of the function $f[u_0 + h_1]$ around the point u_0 up to order $N+1$, we obtain from (4.2), that

$$\begin{aligned} f[u_0 + h_1] &= f[u_0] + D_3 f[u_0] h_1 + D_4 f[u_0] \nabla h_1 + D_5 f[u_0] h'_1 + D_6 f[u_0] \nabla h'_1 \\ &\quad + \sum_{\substack{2 \le |m| \le N \\ m=(m_1,\cdots,m_4) \in \mathbb{Z}_+^4}} \frac{1}{m!} D^m f[u_0] h_1^{m_1} (\nabla h_1)^{m_2} (h'_1)^{m_3} \left(\nabla h'_1\right)^{m_4} + R_N^{(1)}[f, h_1] \\ &= f[u_0] + D_3 f[u_0] h_1 + D_4 f[u_0] \nabla h_1 + D_5 f[u_0] h'_1 + D_6 f[u_0] \nabla h'_1 \\ &\quad + \sum_{\substack{2 \le |m| \le N \\ m \in \mathbb{Z}_+^4}} \frac{1}{m!} D^m f[u_0] \sum_{|m| \le |\nu| \le N} \tilde{\Phi}_\nu[m, N, f, u, \nabla u, u', \nabla u'] \boldsymbol{\varepsilon}^\nu \\ &\quad + \sum_{\substack{2 \le |m| \le N \\ m \in \mathbb{Z}_+^4}} \frac{1}{m!} D^m f[u_0] \sum_{N+1 \le |\nu| \le |m| N} \tilde{\Phi}_\nu[m, N, f, u, \nabla u, u', \nabla u'] \boldsymbol{\varepsilon}^\nu + R_N^{(1)}[f, h_1], \end{aligned} \tag{4.22}$$

where

$$R_N^{(1)}[f, h_1] = \sum_{\substack{|m| = N+1 \\ m=(m_1,\cdots,m_4) \in \mathbb{Z}_+^4}} \frac{N+1}{m!} \int_0^1 (1-\theta)^N D^m f[u_0 + \theta h_1] h_1^{m_1} (\nabla h_1)^{m_2} (h'_1)^{m_3} \left(\nabla h'_1\right)^{m_4} d\theta,$$

$$\begin{aligned} &\tilde{\Phi}_\nu[m, N, f, u, \nabla u, u', \nabla u'] \\ &= \sum_{\substack{(\alpha,\beta,\gamma,\delta) \in A(m,N) \\ \alpha+\beta+\gamma+\delta=\nu}} T_N^{(m_1)}[u]_\alpha T_N^{(m_2)}[\nabla u]_\beta T_N^{(m_3)}[u']_\gamma T_N^{(m_4)}[\nabla u']_\delta, \; |m| \le |\nu| \le |m| N. \end{aligned} \tag{4.23}$$

We note that

$$\begin{aligned} &f[u_0] + D_3 f[u_0] h_1 + D_4 f[u_0] \nabla h_1 + D_5 f[u_0] h'_1 + D_6 f[u_0] \nabla h'_1 \\ &+ \sum_{\substack{2 \le |m| \le N \\ m \in \mathbb{Z}_+^4}} \frac{1}{m!} D^m f[u_0] \sum_{|m| \le |\nu| \le N} \tilde{\Phi}_\nu[m, N, f, u, \nabla u, u', \nabla u'] \boldsymbol{\varepsilon}^\nu \\ &= \sum\nolimits_{|\nu| \le N} \pi_\nu[f] \boldsymbol{\varepsilon}^\nu, \end{aligned} \tag{4.24}$$

where $\pi_\nu[f]$, $1 \le |\nu| \le N$ are defined by (4.8).

Similarly,

$$\begin{aligned} &\sum_{\substack{2 \le |m| \le N \\ m \in \mathbb{Z}_+^4}} \frac{1}{m!} D^m f[u_0] \sum_{N+1 \le |\nu| \le |m| N} \tilde{\Phi}_\nu[m, N, f, u, \nabla u, u', \nabla u'] \boldsymbol{\varepsilon}^\nu + R_N^{(1)}[f, h_1] \\ &= \|\boldsymbol{\varepsilon}\|^{N+1} \bar{R}_N^{(1)}[f, \boldsymbol{\varepsilon}], \end{aligned} \tag{4.25}$$

with $\left\|\bar{R}_N^{(1)}[f,\boldsymbol{\varepsilon}]\right\|_{L^\infty(0,T;L^2)} \le C$, C is a constant depending only on N, T, f, u_γ, $|\gamma| \le N$.

Then (4.11) holds. Lemma 4.2 is proved.

Remark 4.1. Lemma 4.2 is a generalization of a formula contained in ([17], p.262, formula (4.38)) and it is useful to obtain the following Lemma 4.3. These Lemmas are the key to the asymptotic expansion of a weak solution $u_{\boldsymbol{\varepsilon}} = u(\varepsilon_1, \cdots, \varepsilon_p)$ of order $N+1$ in p small parameters $\varepsilon_1, \cdots, \varepsilon_p$ as it will be said below.

Let $u_{\boldsymbol{\varepsilon}} = u(\varepsilon_1, \cdots, \varepsilon_p) \in W_1(M,T)$ be a unique weak solution of the problem $(P_{\boldsymbol{\varepsilon}})$. Then $v = u_{\boldsymbol{\varepsilon}} - \sum_{|\gamma|\le N} u_\gamma \boldsymbol{\varepsilon}^\gamma \equiv u_{\boldsymbol{\varepsilon}} - h$ satisfies the problem

$$\begin{cases} v'' - B_{\boldsymbol{\varepsilon}}[v+h]Av = F_{\boldsymbol{\varepsilon}}[v+h] - F_{\boldsymbol{\varepsilon}}[h] + (B_{\boldsymbol{\varepsilon}}[v+h] - B_{\boldsymbol{\varepsilon}}[h])Ah \\ \qquad\qquad + E_{\boldsymbol{\varepsilon}}(x,t), \quad 0<x<1,\ 0<t<T, \\ v(0,t) = v(1,t) = 0, \\ v(x,0) = v'(x,0) = 0, \\ Av = \Delta v + \Delta v_{tt}, \\ B_{\boldsymbol{\varepsilon}}[v] = B[v] + \sum_{i=1}^p \varepsilon_i B_i[v] = B(\|v\|^2, \|v_x\|^2) + \sum_{i=1}^p \varepsilon_i B_i(\|v\|^2, \|v_x\|^2), \\ F_{\boldsymbol{\varepsilon}}[v] = f[v] + \sum_{i=1}^p \varepsilon_i f_i[v] = f(x,t,v,v_x,v',v_x') + \sum_{i=1}^p \varepsilon_i f_i(x,t,v,v_x,v',v_x'). \end{cases} \tag{4.26}$$

where

$$E_{\boldsymbol{\varepsilon}}(x,t) = f[h] - f[u_0] + \sum_{i=1}^p \varepsilon_i f_i[h] + \left(B[h] - B[u_0] + \sum_{i=1}^p \varepsilon_i B_i[h]\right)Ah - \sum_{1\le|\nu|\le N} F_\nu \boldsymbol{\varepsilon}^\nu. \tag{4.27}$$

Then, we have the following lemma.

Lemma 4.3. *Let (H_1), (H_6), and (H_7) hold. Then there exists a constant $\bar{C}_*$ such that*

$$\|E_{\boldsymbol{\varepsilon}}\|_{L^\infty(0,T;L^2)} \le \bar{C}_* \|\boldsymbol{\varepsilon}\|^{N+1}, \tag{4.28}$$

where $\bar{C}_*$ is a constant depending only on N, T, f, f_i, B, B_i, u_γ, $|\gamma| \le N$, $1 \le i \le p$.

Proof of Lemma 4.3. In the case of $N=1$, the proof of Lemma 4.3 is easy, hence we omit the details. We only consider $N \ge 2$.

By using formulas (4.10), (4.11) for the functions $B_i[h]$ and $f_i[h]$, we obtain

$$\begin{cases} B_i[h] = \sum_{|\nu|\le N-1} \rho_\nu[B_i]\boldsymbol{\varepsilon}^\nu + \|\boldsymbol{\varepsilon}\|^N \widetilde{R}_{N-1}^{(1)}[B_i,\boldsymbol{\varepsilon}], \\ f_i[h] = \sum_{|\nu|\le N-1} \pi_\nu[f_i]\boldsymbol{\varepsilon}^\nu + \|\boldsymbol{\varepsilon}\|^N \bar{R}_{N-1}^{(1)}[f_i,\boldsymbol{\varepsilon}],\ 1\le i\le p. \end{cases} \tag{4.29}$$

By (4.7), $(4.29)_1$, we rewrite $\varepsilon_i B_i[h]$ as follows

$$\begin{aligned} \varepsilon_i B_i[h] &= \sum_{1\le|\nu|\le N,\ \nu_i\ge1} \rho_{\nu_1,\nu_2,\cdots,\nu_{i-1},\nu_i-1,\nu_{i+1},\cdots,\nu_p}[B_i]\boldsymbol{\varepsilon}^\nu \\ &\qquad + \varepsilon_i \|\boldsymbol{\varepsilon}\|^N \widetilde{R}_{N-1}^{(1)}[B_i,\boldsymbol{\varepsilon}] \\ &= \sum_{1\le|\nu|\le N} \rho_\nu^{(i)}[B_i]\boldsymbol{\varepsilon}^\nu + \varepsilon_i \|\boldsymbol{\varepsilon}\|^N \widetilde{R}_{N-1}^{(1)}[B_i,\boldsymbol{\varepsilon}]. \end{aligned} \tag{4.30}$$

Similarly, with $f_i[h]$, $1 \leq i \leq p$, we also obtain

$$\begin{aligned}
\varepsilon_i f_i[h] &= \sum_{|\nu| \leq N-1} \pi_\nu[f_i]\varepsilon_i \boldsymbol{\varepsilon}^\nu + \varepsilon_i \left\|\boldsymbol{\varepsilon}\right\|^N \bar{R}^{(1)}_{N-1}[f_i, \boldsymbol{\varepsilon}] \\
&= \sum_{1 \leq |\nu| \leq N,\, \nu_i \geq 1} \pi_{\nu_1,\nu_2,\cdots,\nu_{i-1},\nu_i-1,\nu_{i+1},\cdots,\nu_p}[f_i]\boldsymbol{\varepsilon}^\nu + \varepsilon_i \left\|\boldsymbol{\varepsilon}\right\|^N \bar{R}^{(1)}_{N-1}[f_i, \boldsymbol{\varepsilon}] \\
&= \sum_{1 \leq |\nu| \leq N} \pi^{(i)}_\nu[f_i]\boldsymbol{\varepsilon}^\nu + \varepsilon_i \left\|\boldsymbol{\varepsilon}\right\|^N \bar{R}^{(1)}_{N-1}[f_i, \boldsymbol{\varepsilon}].
\end{aligned} \tag{4.31}$$

First, we deduce from (4.11) and (4.31), that

$$\begin{aligned}
&f[h] - f[u_0] + \sum\nolimits_{i=1}^{p} \varepsilon_i f_i[h] \\
&= \sum_{1 \leq |\nu| \leq N} \left[\pi_\nu[f] + \sum\nolimits_{i=1}^{p} \pi^{(i)}_\nu[f_i]\right] \boldsymbol{\varepsilon}^\nu + \left\|\boldsymbol{\varepsilon}\right\|^{N+1} \bar{R}^{(1)}_N[f, f_1, \cdots, f_p, \boldsymbol{\varepsilon}],
\end{aligned} \tag{4.32}$$

where $\bar{R}^{(1)}_N[f, f_1, \cdots, f_p, \boldsymbol{\varepsilon}] = \bar{R}^{(1)}_N[f, \boldsymbol{\varepsilon}] + \sum_{i=1}^{p} \frac{\varepsilon_i}{\|\boldsymbol{\varepsilon}\|} \bar{R}^{(1)}_{N-1}[f_i, \boldsymbol{\varepsilon}]$ is bounded in the function space $L^\infty(0, T; L^2)$ by a constant depending only on N, T, f, f_i, u_γ, $|\gamma| \leq N$, $1 \leq i \leq p$.

On the other hand, we deduce from (4.10) and (**??**) that

$$\begin{aligned}
&\left(B[h] - B[u_0] + \sum_{i=1}^{p} \varepsilon_i B_i[h]\right) Ah \\
&= \sum_{1 \leq |\nu| \leq 2N} \sum_{\substack{1 \leq |\alpha| \leq N, \\ |\nu-\alpha| \leq N}} \left(\rho_\alpha[B] + \sum_{i=1}^{p} \rho^{(i)}_\alpha[B_i]\right) Au_{\nu-\alpha}\boldsymbol{\varepsilon}^\nu \\
&+ \left\|\boldsymbol{\varepsilon}\right\|^{N+1} \widetilde{R}^{(1)}_N[B, B_1, \cdots, B_p, \boldsymbol{\varepsilon}],
\end{aligned} \tag{4.33}$$

where

$$\widetilde{R}^{(1)}_N[B, B_1, \cdots, B_p, \boldsymbol{\varepsilon}] = \left[\widetilde{R}^{(1)}_N[B, \boldsymbol{\varepsilon}] + \sum_{i=1}^{p} \frac{\varepsilon_i}{\|\boldsymbol{\varepsilon}\|} \widetilde{R}^{(1)}_{N-1}[B_i, \boldsymbol{\varepsilon}]\right] Ah. \tag{4.34}$$

We decompose the sum $\sum_{1 \leq |\nu| \leq 2N}$ into the sum of two the sums $\sum_{1 \leq |\nu| \leq N}$ and $\sum_{N+1 \leq |\nu| \leq 2N}$. Therefore, we deduce from (4.33), (4.34) that

$$\begin{aligned}
&\left(B[h] - B[u_0] + \sum_{i=1}^{p} \varepsilon_i B_i[h]\right) Ah \\
&= \sum_{1 \leq |\nu| \leq N} \sum_{\substack{1 \leq |\alpha| \leq N, \\ |\nu-\alpha| \leq N}} \left(\rho_\alpha[B] + \sum_{i=1}^{p} \rho^{(i)}_\alpha[B_i]\right) Au_{\nu-\alpha}\boldsymbol{\varepsilon}^\nu \\
&+ \left\|\boldsymbol{\varepsilon}\right\|^{N+1} \widetilde{R}^{(2)}_N[B, B_1, \cdots, B_p, \boldsymbol{\varepsilon}],
\end{aligned} \tag{4.35}$$

in which

$$\begin{aligned}
&\|\varepsilon\|^{N+1}\, \widetilde{R}_N^{(2)}[B, B_1, \cdots, B_p, \varepsilon] \\
&= \|\varepsilon\|^{N+1}\, \widetilde{R}_N^{(1)}[B, B_1, \cdots, B_p, \varepsilon] \\
&\quad + \sum_{N+1\le|\nu|\le 2N} \sum_{\substack{1\le|\alpha|\le N,\\ |\nu-\alpha|\le N}} \left(\rho_\alpha[B] + \sum_{i=1}^{p} \rho_\alpha^{(i)}[B_i]\right) A u_{\nu-\alpha}\varepsilon^{\nu}.
\end{aligned} \tag{4.36}$$

Combining (4.4), (4.5), (4.8), (4.27), (4.32) and (4.35), we then obtain

$$E_\varepsilon = \|\varepsilon\|^{N+1}\left[\bar{R}_N^{(1)}[f, f_1, \cdots, f_p, \varepsilon] + \widetilde{R}_N^{(2)}[B, B_1, \cdots, B_p, \varepsilon]\right]. \tag{4.37}$$

By the functions $u_\nu \in W_1(M,T)$, $|\nu| \le N$, we obtain from (4.32) and (4.36), that

$$\begin{aligned}
&\|E_\varepsilon\|_{L^\infty(0,T;L^2)} \\
&= \|\varepsilon\|^{N+1}\left\|\bar{R}_N^{(1)}[f, f_1, \cdots, f_p, \varepsilon] + \widetilde{R}_N^{(2)}[B, B_1, \cdots, B_p, \varepsilon]\right\|_{L^\infty(0,T;L^2)} \\
&\le \bar{C}_*\, \|\varepsilon\|^{N+1},
\end{aligned} \tag{4.38}$$

where $\bar{C}_*$ is a constant depending only on N, T, f, f_i, B, B_i, u_γ, $|\gamma| \le N$, $1 \le i \le p$.

The proof of Lemma 4.3 is complete.

Now, we consider the sequence of functions $\{v_m\}$ defined by

$$\left\{\begin{array}{l}
v_0 \equiv 0, \\
v_m'' - B_{\vec{\varepsilon}}[v_{m-1}+h]Av_m = F_\varepsilon[v_{m-1}+h] - F_\varepsilon[h] + \left(B_\varepsilon[v_{m-1}+h] - B_\varepsilon[h]\right)Ah \\
\qquad\qquad\qquad\qquad +E_\varepsilon(x,t),\ 0<x<1,\ 0<t<T, \\
v_m(0,t) = v_m(1,t) = 0, \\
v_m(x,0) = v_m'(x,0) = 0,\ m \ge 1.
\end{array}\right. \tag{4.39}$$

With $m = 1$, we have the problem

$$\left\{\begin{array}{l}
v_1'' - B_\varepsilon[h]Av_1 = E_\varepsilon(x,t),\ 0<x<1,\ 0<t<T, \\
v_1(0,t) = v_1(1,t) = 0, \\
v_1(x,0) = v_1'(x,0) = 0.
\end{array}\right. \tag{4.40}$$

By multiplying the two sides of (4.40) by v_1', we verify without difficulty from (4.28) that

$$\begin{aligned}
&||v_1'(t)||^2 + \bar{B}_{1,\varepsilon}(t)\left(\|v_{1x}(t)\|^2 + \left\|v_{1x}'(t)\right\|^2\right) \\
&= \int_0^t \bar{B}_{1,\varepsilon}'(s)\left(\|v_{1x}(s)\|^2 + \left\|v_{1x}'(s)\right\|^2\right)ds + 2\int_0^t \langle E_\varepsilon(s), v_1'(s)\rangle ds \\
&\le T\bar{C}_*^2\,\|\varepsilon\|^{2N+2} + \int_0^t \left\|v_1'(s)\right\|^2 ds + \int_0^t \left|\bar{B}_{1,\varepsilon}'(s)\right|\left(\|v_{1x}(s)\|^2 + \left\|v_{1x}'(s)\right\|^2\right)ds,
\end{aligned} \tag{4.41}$$

where $\bar{B}_{1,\varepsilon}(t) = B_\varepsilon[h](t) = B_\varepsilon(\|h(t)\|^2, \|\nabla h(t)\|^2)$. By

$$\bar{B}_{1,\varepsilon}'(t) = 2D_1B_\varepsilon[h]\langle h(t), h'(t)\rangle + 2D_2B_\varepsilon[h]\langle \nabla h(t), \nabla h'(t)\rangle, \tag{4.42}$$

we have

$$\left|\bar{B}_{1,\varepsilon}'(t)\right| \le 4M_*^2\left(\tilde{K}_{M_*}(B) + \sum_{i=1}^{p}\tilde{K}_{M_*}(B_i)\right) \equiv \zeta_1,\ \text{for all}\ \|\varepsilon\| < 1, \tag{4.43}$$

with $M_* = N_1 M$, and $N_1 = card\{\gamma \in \mathbb{Z}_+^p : |\gamma| \leq N\}$. It follows from (4.41), (4.43) that

$$||v_1'(t)||^2 + b_0 \left(\|v_{1x}(t)\|^2 + \left\|v_{1x}'(t)\right\|^2 \right) \tag{4.44}$$
$$\leq T\bar{C}_*^2 \|\boldsymbol{\varepsilon}\|^{2N+2} + (1+\zeta_1) \int_0^t \left(\|v_{1x}(s)\|^2 + \left\|v_{1x}'(s)\right\|^2 \right) ds.$$

By Gronwall's lemma we obtain from (4.44) that

$$\|v_{1x}(t)\|^2 + \left\|v_{1x}'(t)\right\|^2 \leq \frac{1}{b_0} T\bar{C}_*^2 \|\boldsymbol{\varepsilon}\|^{2N+2} \exp\left[(1+\zeta_1)\, T\right]. \tag{4.45}$$

Hence

$$||v_1||_{C^1([0,T];H_0^1)} \leq \frac{2}{\sqrt{b_0}} \sqrt{T} \bar{C}_* \|\boldsymbol{\varepsilon}\|^{N+1} \exp\left[\frac{1}{2}(1+\zeta_1)\, T\right]. \tag{4.46}$$

We shall prove that there exists a constant C_T, independent of m and $\boldsymbol{\varepsilon}$, such that

$$\|v_m\|_{C^1([0,T];H_0^1)} \leq C_T \|\boldsymbol{\varepsilon}\|^{N+1}, \text{ with } \|\boldsymbol{\varepsilon}\| < 1, \text{ for all } m. \tag{4.47}$$

By multiplying the two sides of (4.39) with v_m' and after integrating in t, we obtain from (4.28) that

$$||v_m'(t)||^2 + \bar{B}_{m,\boldsymbol{\varepsilon}}(t) \left(\|v_{mx}(t)\|^2 + \left\|v_{mx}'(t)\right\|^2 \right) \tag{4.48}$$
$$\begin{aligned}
\leq & T\bar{C}_*^2 \|\boldsymbol{\varepsilon}\|^{2N+2} + \int_0^t \left\|v_m'(s)\right\|^2 ds \\
& + \int_0^t \bar{B}_{m,\boldsymbol{\varepsilon}}'(s) \left(\|v_{mx}(s)\|^2 + \left\|v_{mx}'(s)\right\|^2 \right) ds \\
& + 2\int_0^t \langle F_{\boldsymbol{\varepsilon}}[v_{m-1}+h] - F_{\boldsymbol{\varepsilon}}[h], v_m'(s)\rangle ds \\
& + 2\int_0^t \left(B_{\boldsymbol{\varepsilon}}[v_{m-1}+h] - B_{\boldsymbol{\varepsilon}}[h] \right) \langle Ah(s), v_m'(s)\rangle ds \\
\equiv & T\bar{C}_*^2 \|\boldsymbol{\varepsilon}\|^{2N+2} + \int_0^t \left\|v_m'(s)\right\|^2 ds + \widehat{J}_1 + \widehat{J}_2 + \widehat{J}_3,
\end{aligned}$$

with $\bar{B}_{m,\boldsymbol{\varepsilon}}(t) = B_{\boldsymbol{\varepsilon}}[v_{m-1}+h](t) = B_{\boldsymbol{\varepsilon}}(\|v_{m-1}(t)+h(t)\|^2), \|\nabla v_{m-1}(t) + \nabla h(t)\|^2)$.

We now estimate the integrals on the right - hand side of (4.48) as follows.

Estimating $\widehat{J}_1$. We have

$$\begin{aligned}
\bar{B}_{m,\boldsymbol{\varepsilon}}'(t) = & 2D_1 B_{\boldsymbol{\varepsilon}}[v_{m-1}+h](t)\langle v_{m-1}+h, v_{m-1}'+h'\rangle \\
& + 2D_2 B_{\boldsymbol{\varepsilon}}[v_{m-1}+h](t)\langle \nabla v_{m-1}+\nabla h, \nabla v_{m-1}'+\nabla h'\rangle,
\end{aligned} \tag{4.49}$$

hence

$$\left|\bar{B}_{m,\boldsymbol{\varepsilon}}'(t)\right| \leq 4\bar{M}_*^2 \left(\tilde{K}_{\bar{M}_*}(B) + \sum_{i=1}^p \tilde{K}_{\bar{M}_*}(B_i) \right) \equiv \bar{\zeta}_1, \text{ for all } \|\boldsymbol{\varepsilon}\| < 1, \tag{4.50}$$

with $\bar{M}_* = (1+N_1)M$.

It follows from (4.50), that

$$\widehat{J}_1 = \int_0^t \bar{B}'_{m,\varepsilon}(s)\left(\|v_{mx}(s)\|^2 + \|v'_{mx}(s)\|^2\right)ds \leq \bar{\zeta}_1 \int_0^t \left(\|v_{mx}(s)\|^2 + \|v'_{mx}(s)\|^2\right)ds. \tag{4.51}$$

Estimating $\widehat{J}_2$. Note that

$$\|f(v_{m-1}+h) - f[h]\| \leq 2K_{\bar{M}_k}(f)\,\|v_{m-1}\|_{C^1(0,T;H_0^1)},$$
$$\|f_i(v_{m-1}+h) - f_i[h]\| \leq 2K_{\bar{M}_k}(f_i)\,\|v_{m-1}\|_{C^1(0,T;H_0^1)},$$

hence, we have

$$\|F_{\varepsilon}[v_{m-1}+h] - F_{\varepsilon}[h]\| \leq \bar{\zeta}_2\,\|v_{m-1}\|_{C^1([0,T];H_0^1)}\,, \tag{4.52}$$

where $\bar{\zeta}_2 = \bar{\zeta}_2(M, f, f_1, \cdots, f_p) = 2K_{\bar{M}_*}(f) + 2\sum_{i=1}^p K_{\bar{M}_*}(f_i)$. Therefore, we deduce from (4.52) that

$$\widehat{J}_2 = 2\int_0^t \|F_{\varepsilon}[v_{m-1}+h] - F_{\varepsilon}[h]\|\,\|v'_m(s)\|\,ds \leq T\bar{\zeta}_2^2\,\|v_{m-1}\|^2_{C^1([0,T];H_0^1)} + \int_0^t \|v'_m(s)\|^2\,ds. \tag{4.53}$$

Estimating $\widehat{J}_3$. First, we need an estimation of $|B_{\varepsilon}[v_{m-1}+h] - B_{\varepsilon}[h]|$.
From the inequalities

$$|B[v_{m-1}+h] - B[h]| \leq 4\bar{M}_*\tilde{K}_{\bar{M}_*}(B)\,\|v_{m-1}\|_{C^1([0,T];H_0^1)}\,,$$
$$|B_i[v_{m-1}+h] - B_i[h]| \leq 4\bar{M}_*\tilde{K}_{\bar{M}_*}(B_i)\,\|v_{m-1}\|_{C^1([0,T];H_0^1)}\,,\ i = 1, \cdots, p,$$

it follows that

$$|B_{\varepsilon}[v_{m-1}+h] - B_{\varepsilon}[h]| \leq 4\bar{M}_*\left(\tilde{K}_{\bar{M}_*}(B) + \sum_{i=1}^p \tilde{K}_{\bar{M}_*}(B_i)\right)\|v_{m-1}\|_{C^1([0,T];H_0^1)}\,. \tag{4.54}$$

We remark that

$$\|Ah(s)\| \leq \sum_{1\leq|\alpha|\leq N} \|Au_{\alpha}(s)\|\,\|\varepsilon^{\alpha}\| \leq \sum_{1\leq|\alpha|\leq N} \|Au_{\alpha}(s)\| \leq 2N_1 M = 2M_*. \tag{4.55}$$

Hence, we deduce from (4.54) and (4.55) that

$$\widehat{J}_3 = 2\int_0^t (B_{\varepsilon}[v_{m-1}+h] - B_{\varepsilon}[h])\,\langle Ah(s), v'_m(s)\rangle ds \leq T\bar{\zeta}_3^2\,\|v_{m-1}\|^2_{C^1([0,T];H_0^1)} + \int_0^t \|v'_m(s)\|^2\,ds, \tag{4.56}$$

in which $\bar{\zeta}_3 = \bar{\zeta}_3(M, B, B_1, \cdots, B_p) = 8M_*\bar{M}_*\left(\tilde{K}_{\bar{M}_*}(B) + \sum_{i=1}^p \tilde{K}_{\bar{M}_*}(B_i)\right)$.

Combining (4.48), (4.51), (4.53), (4.56), we then obtain

$$\begin{aligned} & \|v_m'(t)\|^2 + \bar{B}_{m,\varepsilon}(t)\left(\|v_{mx}(t)\|^2 + \left\|v_{mx}'(t)\right\|^2\right) \\ \leq\; & T\bar{C}_*^2 \|\varepsilon\|^{2N+2} + T\left(\bar{\zeta}_2^2 + \bar{\zeta}_3^2\right)\|v_{m-1}\|^2_{C^1([0,T];H_0^1)} \\ & + \left(3+\bar{\zeta}_1\right)\int_0^t \left(\|v_{mx}(s)\|^2 + \left\|v_{mx}'(s)\right\|^2\right) ds. \end{aligned} \tag{4.57}$$

By using Gronwall's lemma we deduce from (4.57) that

$$\|v_m\|_{C^1([0,T];H_0^1)} \leq \sigma_T \|v_{m-1}\|_{C^1([0,T];H_0^1)} + \delta, \text{ for all } m \geq 1, \tag{4.58}$$

with $\sigma_T = \eta_T\sqrt{\bar{\zeta}_2^2 + \bar{\zeta}_3^2}$, $\delta = \eta_T \bar{C}_* \|\varepsilon\|^{N+1}$, $\eta_T = \sqrt{\frac{T}{b_0}} \exp\left(\frac{1}{2b_0} T\left(3+\bar{\zeta}_1\right)\right)$.

Assuming that

$$\sigma_T < 1, \text{ with the suitable constant } T > 0. \tag{4.59}$$

We can prove the following lemma easily.

Lemma 4.4. *Let the sequence $\{z_m\}$ satisfy*

$$z_m \leq \sigma z_{m-1} + \delta \textit{ for all } m \geq 1,\ z_0 = 0, \tag{4.60}$$

where $0 \leq \sigma < 1$, $\delta \geq 0$ are the given constants. Then

$$z_m \leq \delta/(1-\sigma) \textit{ for all } m \geq 1. \ \Box \tag{4.61}$$

Applying Lemma 4.4 with $z_m = \|v_m\|_{C^1([0,T];H_0^1)}$, $\sigma = \sigma_T = \eta_T\sqrt{\bar{\zeta}_2^2 + \bar{\zeta}_3^2} < 1$, $\delta = \eta_T \bar{C}_* \|\varepsilon\|^{N+1}$, it follows from (4.61), that

$$\|v_m\|_{C^1([0,T];H_0^1)} \leq \delta/(1-\sigma_T) = C_T \|\varepsilon\|^{N+1}, \tag{4.62}$$

where $C_T = \frac{\eta_T \bar{C}_*}{1-\eta_T\sqrt{\bar{\zeta}_2^2+\bar{\zeta}_3^2}}$. On the other hand, the linear recurrent sequence $\{v_m\}$ defined by (4.39) converges strongly in the space $C^1([0,T];H_0^1)$ to the solution v of Prob. (4.26). Hence, as $m \to +\infty$ in (4.62), it gives $\|v\|_{C^1([0,T];H_0^1)} \leq C_T \|\varepsilon\|^{N+1}$, or

$$\left\| u_\varepsilon - \sum\nolimits_{|\gamma| \leq N} u_\gamma \varepsilon^\gamma \right\|_{C^1([0,T];H_0^1)} \leq C_T \|\varepsilon\|^{N+1}. \tag{4.63}$$

Thus, we have the following theorem.

Theorem 4.5. *Let (H_1), (H_6) and (H_7) hold. Then there exist constants $M > 0$ and $T > 0$ such that, for every ε, with $\|\varepsilon\| < 1$, Prob. (P_ε) has a unique weak solution $u_\varepsilon \in W_1(M,T)$ satisfying an asymptotic estimation up to order $N+1$ as in* (4.63), *where the functions u_ν, $|\nu| \leq N$ are weak solutions of Prob. $(\tilde{P}_\nu)$, $|\nu| \leq N$, respectively.* $\Box$

Remark 4.2. Typical examples about asymptotic expansion of solutions in a small parameter can be found in the researches of many authors, such as [16] - [18], [28]. In the case of many small parameters, there is only partial results, for example, we refer to [19], [20], [29], $\cdots$ for the asymptotic expansion of solutions in two or three small parameters.

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