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Existence of Solutions to Mixed Boundary Value Problems for an Extended Class of Equations Beyond the Parabolic Type

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Abstract. *This paper investigates the existence and uniqueness of solutions to a mixed boundary value problem for a certain class of equations with complex-valued coefficients. These equations exhibit behavior characteristic of parabolic equations, although over the course of "time" their type may transition from parabolic to Schrödinger-type, or even to anti-parabolic.*

Keywords. fundamental solution · asymptotics · analytic function · continuous differentiation · asymptotic formula · parabolic equation · spectral problem · Cauchy problem · operator.

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1 Introduction

It is well known that second-order parabolic equations, when combined with knowledge of current parameters of thermal, diffusion, and other processes, allow for the prediction of future states. In contrast, anti-parabolic equations enable the analysis of past processes based on present parameters. For equations that are entirely of anti-parabolic type, classical initial, boundary, or mixed problems are typically ill-posed. This either makes it impossible to study past thermal or diffusion processes using current data or presents significant difficulties in such investigations. The results obtained in this study are significant in identifying, based on current parameters, which time intervals and spatial subregions of the domain may be revisited to reconstruct and analyze past states of heat transfer or diffusion processes.

The unique solvability and well-posedness of linear mixed problems have been extensively studied by numerous researchers [1, 5, 8, 9, 19, 20, 21, 22], among others. Various methods have been developed for studying such problems depending on how they are formulated. These include, for instance, the method of separation of variables (Fourier method), Laplace transforms, coefficient freezing, heat potential methods, a priori estimates, contour integration, the residue method, finite difference schemes, and others.

However, it is also known that each of these approaches may become inapplicable under certain conditions. Notably, the residue method and the contour integral method are among the most versatile techniques for addressing one-dimensional and multidimensional mixed problems [1], [9].

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In this study, both the residue method and the contour integral method are employed to investigate the unique solvability of a one-dimensional mixed problem exhibiting novel features that have not been explored previously. It is known that equations of the form

$$\frac{\partial U}{\partial t}=a\left(t,\,x\right)\frac{\partial^{2}U}{\partial x^{2}} \tag{1.1}$$

is said to be parabolic (or uniformly parabolic) in the sense of I.G. Petrovsky in a given domain of the space if, at every point of the domain, the following inequality is satisfied:

$$Re\,a\left(t,x\right)>0\left(Re\,a\left(t,x\right)\geq\delta>0\right).$$

Mixed problems for equations of the form (1.1) have been studied only under the assumption that they are parabolic [14],[16],[17], or when they belong to the Schrdinger type [13],[18],[19], that is, in cases where

$$Re\,a\left(t,x\right)=0. \tag{1.2}$$

At the same time, it is known [10] that if equation (1.1) is anti-parabolic (i.e., if $Rea\left(t,x\right)<0$ for $(t,x)\in Q$), then the corresponding mixed problem is not well-posed when the right-hand sides of the initial and boundary conditions possess only limited smoothness.

Definition 1.1. The equation (1.1) will be referred to as *generalized parabolic* in a given domain $Q_T=\{(t,x):\,0<t<T\leq\infty,0<x<1\}$ if the condition $Re\int\limits_0^t a\left(t,x\right)dt>0$ holds for all $(t,x)\in Q_T$.

It should be noted that any parabolic equation is generalized parabolic in the domain under consideration. However, not all generalized parabolic equations are genuinely parabolic: an equation may initially be parabolic up to a certain moment in time, and then transition into a Schrödinger-type or even an anti-parabolic type.

As shown in [7],[8], mixed problems may turn out to be ill-posed even for equations that are well-posed in the sense of I.G. Petrovsky, and conversely, equations that are not well-posed in that sense may still admit well-posed mixed problems.

The mixed problem under study exhibits similar behavior.

2 Problem Statement

This work investigates the solvability of a mixed boundary value problem...

$$M\left(t,\frac{\partial}{\partial t}\right)U=L\left(x,\frac{\partial}{\partial x}\right)U,\;0<t<T,\;0<x<1 \tag{2.1}$$

$$U(0,x)=\varphi(x) \tag{2.2}$$

$$l_1\left(u\right)=u\left(t,0\right)=0,$$

$$l_2\left(u\right)=u\left(t,1\right)=0, \tag{2.3}$$

where $M\left(t,\frac{\partial}{\partial t}\right)=P_1\left(t\right)\frac{\partial}{\partial t}+P_0\left(t\right),\quad L\left(x,\frac{\partial}{\partial x}\right)=\frac{1}{a(x)}\cdot\frac{\partial^2}{\partial x^2},\;P_1\left(t\right)=\frac{1}{(b-t)(d-t)},$ $P_0\left(t\right)=p_{01}\left(t\right)+ip_{02}\left(t\right)$ complex-valued functions, $p_{0j}\left(t\right)\in C[0,1]\;\;(j=1,2)$, $b=b_1+ib_2,\;d=d+id_2$, complex numbers, $a\left(x\right)>0$, $a\left(x\right)\in C[0,1]$, $\varphi\left(x\right)-$ given , and $U\left(t,x\right)-$ unknown function.

It is known [15] that the equation (2.1) is called parabolic in the sense of I.G. Petrovsky in the domain $D = \{(t,x) : 0 \le t \le T, 0 \le x \le 1\}$ of the t, x space if, for every point $(t,x) \in D$, the real part of the root γ of the characteristic equation

$$\frac{1}{(b-t)(d-t)} \cdot \gamma + a(x)\sigma^2 = 0$$

satisfies the inequality $Re\gamma \;(t,x,\sigma) < 0$ for every real $\sigma \ne 0$.

It follows from equation (2.1) that this equation changes its type from Petrovsky parabolic to anti-parabolic or even Schrdinger type within the domain under consideration.The final conditions for solvability are given by

1^0. $Re\int_0^t \frac{dt}{P_1(t)} = Ret\left(\frac{t^2}{3} - \frac{b+d}{2}t + bd\right) > 0, \quad 0 < t < T$;

2^0. $a(x) > 0,\ 0 < x < 1$

3^0. $\varphi(x) \in C^2[0,1], \varphi(0) = \varphi(1) = 0,$

4^0. $Re(b+d) > Imb \cdot Imd > 0, \quad T^2 - 2T \cdot Re(b+d) - 4Re(b\overline{d}) > 0$

It should be noted that the condition 1^0 allows the equation (2.1) to go beyond the scope of parabolicity (and even well-posedness) in the sense of I.G. Petrovsky. Obviously, under the fulfillment of the condition 2^0, equation (2.1) is parabolic in the sense of I.G. Petrovsky if and only if

$$Re\left(P_1^{-1}(t)\right) = Re(b-t)(d-t) > 0,\ \ 0 \le t \le T \tag{2.4}$$

Under condition 1^0, $Re\left(P_1^{-1}(t)\right)$ may be zero or negative on some subset of $(0,T]$. Since, under condition 4^0, both roots of the equation $Re(b-t)(d-t) = 0$ belong to the interval $[0,T]$, equation (2.1) changes its type from parabolic to anti-parabolic and vice versa.

Let us note that, for example, in the case of the equation

$$(x+1)^2\frac{\partial u}{\partial t} = (2+i-t)(1+2i-t)\frac{\partial^2 u}{\partial x^2}$$

Despite the fact that the given conditions are satisfied, due to the violation of inequality (2.4), the equation is not parabolic in the domain $[0,T]\times[0,1]$ in the sense of I.G. Petrovsky. This equation degenerates into a Schrödinger-type equation along the lines $t = 1, \ \ t = 2$. It is clear that this equation is not parabolic even in the sense of G.E. Shilov. Moreover, in part of the considered rectangle, it becomes anti-parabolic (for example, in the region $1 < t < 2,\ 0 \le x \le 1$ see the shaded area in the figure).

According to the Contour Integral Method scheme, the mixed problem (2.1)–(2.3) is associated with the following two auxiliary problems involving a complex parameter λ:

1) A spectral problem consisting of finding the solution to the equation

$$y'' - \lambda^2 a(x)\, y = -\phi(x), \tag{2.5}$$

under the boundary conditions

$$l_1(y) \equiv y(0) = 0$$
$$l_2(y) \equiv y(1) = 0 \tag{2.6}$$

2) The Cauchy problem consists in finding the solution of an equation involving a real parameter and a complex parameter λ.

$$M\left(t, \frac{d}{dt}\right) z - \lambda^2 z = 0,\ 0 < t < T, \tag{2.7}$$

under the initial conditions

$$z|_{t=0} = \varphi'(x),\ 0 < x < 1 \tag{2.8}$$

It is known [3], [16] that the fundamental system of particular solutions of the homogeneous equation corresponding to (2.5) admits the following asymptotic representation:

$$\frac{d^j y_k(x,\lambda)}{dx^j} = \left((-1)^{k-1}\cdot\lambda\sqrt{a(x)}\right)^j\cdot\left[1+\frac{E_{j,k}(x,\lambda)}{\lambda}\right]\cdot e^{(-1)^{k-1}\cdot\lambda\cdot\int_0^x\sqrt{a(\eta)}d\eta}\ (|\lambda|\to\infty)$$

$$(j=0,1;k=1,2;\lambda\in S_i;i=1,2)$$

where the functions $E_{j,k}(x,\lambda)$ are continuous and bounded for $\lambda\in S_i=\left\{\lambda\backslash\ (-1)^i Re\lambda\ <0\right\}$; $i=1,2; x\in[0,1]$.

It is also known [18] that if $a(x)$ and $\phi(x)$ are continuously differentiable functions on the interval $[0,1]$, then the problem (2.5), (2.6) has a unique solution that can be represented by the formula:

$$y(x,\lambda)=\int_0^1 G(x,\xi,\lambda)\phi(\xi)\,d\xi \tag{2.9}$$

here G(x,ξ,λ) is the Green's function of the spectral problem (2.5), (2.6), and has the form:

$$G(x,\xi,\lambda)=\frac{\Delta(x,\xi,\lambda)}{\Delta(\lambda)}$$

Where

$$\Delta(x,\xi,\lambda)=\begin{vmatrix} g(x,\xi,\lambda) & y_1(x,\lambda) & y_2(x,\lambda)\\ l_1(g(x,\xi,\lambda))_x & l_1(y_1(x,\lambda)) & l_1(y_2(x,\lambda))\\ l_2(g(x,\xi,\lambda))_x & l_2(y_1(x,\lambda)) & l_2(y_2(x,\lambda))\end{vmatrix} \tag{2.10}$$

And $\Delta(\lambda)$ is the characteristic determinant of the spectral problem (2.5), (2.6), and has the form:

$$\Delta(\lambda)=\begin{vmatrix} l_1(y_1(x,\lambda)) & l_1(y_2(x,\lambda))\\ l_2(y_1(x,\lambda)) & l_2(y_2(x,\lambda))\end{vmatrix} \tag{2.11}$$

The function $g(x,\xi,\lambda)$ is the Cauchy function and has the form:

$$g(x,\xi,\lambda)=\frac{\pm\begin{vmatrix} y_1(x,\lambda) & y_2(x,\lambda)\\ y_1(\xi,\lambda) & y_2(\xi,\lambda)\end{vmatrix}}{w(\xi,\lambda)},$$

"+" for $0\le\xi\le x\le1$, "-" for $0\le x\le\xi\le1$. Here, $w(\xi,\lambda)$ is the Wronskian determinant and has the following form:

$$w(\xi,\lambda)=\begin{vmatrix} y_1(\xi,\lambda) & y_2(\xi,\lambda)\\ y'^1(\xi,\lambda) & y'^2(\xi,\lambda)\end{vmatrix}=-2\lambda\sqrt{a(\xi)}\left(1+\frac{E_{11}(\xi,\lambda)}{\lambda}\right).$$

Therefore, we have that

$$g(x,\xi,\lambda)=\frac{1}{4\lambda\sqrt{a(\xi)}}\times$$

$$\times\begin{cases}\left(1+\frac{\tilde{E}_2(x,\xi,\lambda)}{\lambda}\right)e^{-\lambda\int_\xi^x\sqrt{a(\eta)}d\eta}-\left(1+\frac{\tilde{E}_1(x,\xi,\lambda)}{\lambda}\right)e^{\lambda\int_\xi^x\sqrt{a(\eta)}d\eta},\ 0\le\xi\le x\le1\\ \left(1+\frac{\tilde{E}_1(x,\xi,\lambda)}{\lambda}\right)e^{\lambda\int_\xi^x\sqrt{a(\eta)}d\eta}-\left(1+\frac{\tilde{E}_2(x,\xi,\lambda)}{\lambda}\right)e^{-\lambda\int_\xi^x\sqrt{a(\eta)}d\eta},\ 0\le x\le\xi\le1\end{cases} \tag{2.12}$$

It is easy to see that:

$$l_1(y_k(x,\lambda))=\left[1+\frac{E_{1,k}(0,\lambda)}{\lambda}\right],\ k=1,2,$$

$$l_2\left(y_k\left(x,\lambda\right)\right)=\left[1+\frac{E_{2,k}\left(1,\lambda\right)}{\lambda}\right]\cdot e^{(-1)^{k+1}\lambda\int_0^x\sqrt{a(\eta)}d\eta},\ k=1,2.$$

Consequently, according to formula (2.11), we have:

$$\Delta(\lambda)=\begin{vmatrix} 1+\frac{1}{\lambda}\cdot E_{11}\left(0,\lambda\right) & 1+\frac{1}{\lambda}\cdot E_{12}\left(0,\lambda\right) \\ \left(1+\frac{1}{\lambda}\cdot E_{21}\left(1,\lambda\right)\right)e^{\lambda\int_0^1\sqrt{a(\eta)}d\eta} & \left(1+\frac{1}{\lambda}\cdot E_{22}\left(1,\lambda\right)\right)e^{-\lambda\int_0^1\sqrt{a(\eta)}d\eta}\end{vmatrix} \quad (2.13)$$

It should be noted that the poles of the solution to the spectral problem (2.5)-(2.6) or of the Green's function are the zeros of the characteristic determinant $\Delta(\lambda)$. Substituting (2.12) into the boundary conditions (2.6), we obtain:

$$l_1\left(g\left(x,\xi,\lambda\right)\right)=\frac{1}{4\lambda\sqrt{a(\xi)}}\times$$

$$\times\left(\left(1+\frac{\tilde{E}_1\left(0,\xi,\lambda\right)}{\lambda}\right)e^{\lambda\int_\xi^0\sqrt{a(\eta)}d\eta}-\left(1+\frac{\tilde{E}_2\left(0,\xi,\lambda\right)}{\lambda}\right)e^{-\lambda\int_\xi^0\sqrt{a(\eta)}d\eta}\right),$$

$$l_2\left(g\left(x,\xi,\lambda\right)\right)=\frac{-1}{4\lambda\sqrt{a(\xi)}}\times$$

$$\times\left(\left(1+\frac{\tilde{E}_1\left(1,\xi,\lambda\right)}{\lambda}\right)e^{\lambda\int_\xi^1\sqrt{a(\eta)}d\eta}-\left(1+\frac{\tilde{E}_2\left(1,\xi,\lambda\right)}{\lambda}\right)e^{-\lambda\int_\xi^1\sqrt{a(\eta)}d\eta}\right).$$

Taking into account (2.10), we have:

$$\triangle\left(x,\xi,\lambda\right)=\frac{1}{4\lambda\sqrt{a(\xi)}}\times$$

$$\times\begin{vmatrix} \pm\left[\left(1+\frac{\tilde{E}_2(x,\xi,\lambda)}{\lambda}\right)e^{-\lambda\int_\xi^x\sqrt{a(\eta)}d\eta}-\left(1+\frac{\tilde{E}_1(x,\xi,\lambda)}{\lambda}\right)e^{\lambda\int_\xi^x\sqrt{a(\eta)}d\eta}\right] & \left(1+\frac{E_{01}(x,\lambda)}{\lambda}\right)e^{\lambda\int_0^x\sqrt{a(\eta)}d\eta} & \left(1+\frac{E_{02}(x,\lambda)}{\lambda}\right)e^{-\lambda\int_0^x\sqrt{a(\eta)}d\eta} \\ \left(1+\frac{\tilde{E}_1(0,\xi,\lambda)}{\lambda}\right)e^{\lambda\int_\xi^0\sqrt{a(\eta)}d\eta}-\left(1+\frac{\tilde{E}_2(0,\xi,\lambda)}{\lambda}\right)e^{-\lambda\int_\xi^0\sqrt{a(\eta)}d\eta} & 1+\frac{1}{\lambda}\cdot E_{01}\left(0,\lambda\right) & 1+\frac{1}{\lambda}\cdot E_{02}\left(0,\lambda\right) \\ \left(1+\frac{\tilde{E}_1(1,\xi,\lambda)}{\lambda}\right)e^{-\lambda\int_\xi^1\sqrt{a(\eta)}d\eta}-\left(1+\frac{\tilde{E}_2(1,\xi,\lambda)}{\lambda}\right)e^{\lambda\int_\xi^1\sqrt{a(\eta)}d\eta} & \left(1+\frac{E_{01}(1,\lambda)}{\lambda}\right)e^{\lambda\int_0^1\sqrt{a(\eta)}d\eta} & \left(1+\frac{1}{\lambda}\cdot E_{02}\left(1,\lambda\right)\right)e^{-\lambda\int_0^1\sqrt{a(\eta)}d\eta}\end{vmatrix} \quad (2.14)$$

From formulas (2.13)-(2.14), it follows that the solution of the problem (2.5), (2.6) is a meromorphic function of λ, provided that $\Delta(\lambda)\neq 0$, expressed as the ratio of two entire functions, and that the poles of (2.9) can only be the zeros of the characteristic determinant $\Delta(\lambda)$, if such zeros exist.

Lemma 2.1. Let the conditions 2^0 be satisfied. Then, for the eigenvalues of the problem (2.5)-(2.6), the following asymptotic representation holds [16], [19]:

$$\lambda_k=\frac{\pi k\sqrt{-1}}{\int_0^1\sqrt{a\left(\eta\right)}d\eta}+O\left(\frac{1}{k}\right),\left(|k|\to\infty\right)$$

and outside the δ -neighborhoods of the points λ_k, the Green's function of this problem satisfies the following estimates:

$$\left|\frac{\partial^k G(x,\xi,\lambda)}{\partial x^k}\right| \le c|\lambda|^{k-1}, (k=0,1,2), \lambda \in S_i\ , \quad i=1,2 \tag{2.15}$$

Proof. Consequently, according to formula (2.13), we have:

$$\Delta(\lambda) = \left(1 + \frac{E_{21}(1,\lambda) + E_{11}(0,\lambda)}{\lambda} + \frac{E_{11}(0,\lambda)\,E_{21}(1,\lambda)}{\lambda^2}\right) e^{-\lambda\int_0^1 \sqrt{a(\eta)}d\eta} -$$

$$-\left(1 + \frac{E_{21}(0,\lambda) + E_{11}(1,\lambda)}{\lambda} + \frac{E_{21}(0,\lambda)\,E_1(1,\lambda)}{\lambda^2}\right) e^{\lambda\int_0^1 \sqrt{a(\eta)}d\eta} \tag{2.16}$$

Let us introduce the following notations:

$$\delta_1 = \int_0^1 \sqrt{a(\eta)}d\eta, d_2 = -\int_0^1 \sqrt{a(\eta)}d\eta$$

The determination of the asymptotic representation of large zeros $\Delta(\lambda)$ and of the determinant $\Delta(\lambda)$ itself requires a more detailed study of the growth of the elements of this determinant. Let us consider the locus of values of λ satisfying the equation:

$$Re\delta_k\ \lambda = Re\,|\lambda|\ \cdot|\delta_k|\cdot e^{i(arg\lambda\ + arg\delta_k\ )} = |\lambda|\cdot|\delta_k|\cdot cos\,(arg\lambda\ + arg\delta_k\ )\ = 0, (k=1,2)$$

From this, we have:

$$arg\lambda\ = \pm\frac{\pi}{2} + arg\delta_k\ , (k=1,2)\,. \tag{2.17}$$

The equalities (2.17) define a number of straight lines passing through the origin of the λ -plane, each of which is divided into two rays originating at the coordinate origin. First, we isolate the principal part of the determinant of the characteristic matrix $\Delta(\lambda)$. We partition the complex plane into sectors such that in each sector the inequality $Re\lambda\delta_1 < 0 < Re\lambda\delta_2$ holds. It is easy to see that on the line $Re\lambda\int_0^1 \sqrt{a(\eta)}d\eta = 0$ the equality $Re\lambda\ \delta_1 = Re\lambda\ \delta_2$ holds. It is not difficult to show that in the strip $Re\lambda\ > h$, the inequalities $Re\lambda d_1 < 0, Re\lambda\ d_2 > 0$, and $Re\lambda\ \delta_1 > 0, Re\lambda\ \delta_2 < 0$ hold, whereas in the strip $Re\lambda\ < -h$, the inequalities $Re\lambda\delta_1 > 0, Re\lambda\delta_2 < 0$ hold. It is clear that in representation (2.16), the first term is exponentially growing as $|\lambda| \to \infty$, while the second term is exponentially decaying as $|\lambda| \to \infty$. If we factor out the exponential function $e^{\lambda\int_0^1 \sqrt{a(\eta)}d\eta}$ from the right-hand side of (2.16), then all remaining terms tend to zero in the strip $Re\lambda\ > h$ as $|\lambda| \to \infty$.

Therefore, from (2.16) we obtain:

$$e^{-\lambda\int_0^1 \sqrt{a(\eta)d\eta}}\Delta(\lambda) = \left(1 + \frac{E_{21}(1,\lambda) + E_{11}(0,\lambda)}{\lambda} + \frac{E_{11}(0,\lambda)\,E_{21}(1,\lambda)}{\lambda^2}\right) -$$

$$-\left(1 + \frac{E_{21}(0,\lambda) + E_{11}(1,\lambda)}{\lambda} + \frac{E_{21}(0,\lambda)\,E_1(1,\lambda)}{\lambda^2}\right) e^{2\lambda\int_0^1 \sqrt{a(\eta)d\eta}} \tag{2.18}$$

This implies that the characteristic determinant can be represented in the following form:

$$\Delta(\lambda) = e^{\lambda\int_0^1 \sqrt{a(\eta)d\eta}}\left(\Delta_0(\lambda) + N_1(\lambda)\right) \tag{2.19}$$

where

$$\Delta_0(\lambda) = -1 + e^{-2\lambda\int_0^1 \sqrt{a(\eta)}d\eta},$$

$$N_1(\lambda) = -\left(\frac{E_{21}(0,\lambda)+E_{11}(1,\lambda)}{\lambda} + \frac{E_{21}(0,\lambda)\,E_{11}(1,\lambda)}{\lambda^2}\right) +$$

$$+\left(\frac{E_{21}(1,\lambda)+E_{11}(0,\lambda)}{\lambda} + \frac{E_{11}(0,\lambda)\,E_{21}(1,\lambda)}{\lambda^2}\right) e^{-2\lambda\int_0^1 \sqrt{a(\eta)d\eta}} \tag{2.20}$$

From expression (2.20), it follows that the function $N_1(\lambda)$ satisfies the following estimate:

$$|N_1(\lambda)| \le \frac{c_1}{|\lambda|}, \text{ as } |\lambda| \to \infty \tag{2.21}$$

It is also evident that $\Delta_0(\lambda)$ constitutes the principal part of the determinant $\Delta(\lambda)$. On the other hand, by invoking Rouche's theorem, one concludes that the zeros of the characteristic determinant differ from those of $\Delta(\lambda)$ only by an infinitesimal term. Clearly, the roots of the equation $\Delta_0(\lambda) = 0$ take the following form:

$$\lambda_k = \frac{\pi k\sqrt{-1}}{\int\limits_0^1 \sqrt{a(\eta)\,d\eta}},$$

This implies that the zeros of the characteristic determinant $\Delta(\lambda)$ admit the following asymptotic representations:

$$\lambda_k = \frac{\pi k\sqrt{-1}}{\int\limits_0^1 \sqrt{a(\eta)}d\eta} + O\left(\frac{1}{k}\right), (|k| \to \infty) \tag{2.22}$$

We now determine the order of the zeros of the characteristic equation (2.19). To this end, we compute the derivative of the characteristic determinant $\Delta(\lambda)$.

$$\Delta'(\lambda) = \left\{e^{-\lambda\int_0^1 \sqrt{a(\eta)d\eta}}\left(\Delta_0(\lambda) + N_1(\lambda)\right)\right\}' =$$

$$= -e^{-\lambda\int_0^1 \sqrt{a(\eta)}d\eta}\left(\Delta_0(\lambda) + N_1(\lambda)\right)\int_0^1 \sqrt{a(\eta)}d\eta +$$

$$+e^{-\lambda\int_0^1 \sqrt{a(\eta)d\eta}}(\Delta_0(\lambda) + N_1(\lambda))'.$$

Taking into account that $\Delta_0(\lambda_k) + N_1(\lambda_k) = 0$, it follows that

$$\Delta'(\lambda_k) = e^{-\lambda\int_0^1 \sqrt{a(\eta)}d\eta}\left(\Delta_0'(\lambda_k) + N_1'(\lambda_k)\right).$$

This implies that $\Delta'(\lambda_k) \neq 0$. In other words, the roots determined by formula (2.22) of the characteristic equation (2.19) are simple zeros. On the other hand, the absolute difference between successive zeros has the following form:

$$|\lambda_{k+1} - \lambda_k| = \frac{\pi k}{\int_0^1 \sqrt{a(\eta)}d\eta}, \ (k = 1, 2, 3...).$$

This means that the roots of the characteristic equation do not accumulate. That is, for each root λ_k, there exists a neighborhood δ such that these roots lie within a strip of finite width containing the line $Re\left(\lambda\int_0^1 \sqrt{a(\eta)}d\eta\right) = 0$.

Thus, after removing the δ -neighborhoods of the roots λ_k , $(k = 1, 2, 3...)$ from the strip $\widetilde{E}_1 = \{\lambda : \ |Re\lambda| < h, \ \ h > 0\}$, it can be shown that the following estimate holds for the characteristic determinant in the remaining part of the strip.

$$\left| e^{\lambda \int_0^1 \sqrt{a(\eta) d\eta}} \Delta(\lambda) \right| \geq K_\delta > 0$$

Here, K_δ is a positive constant that depends solely on δ.

We now derive a lower bound for the characteristic determinant $\Delta(\lambda)$ in the strips $\widetilde{E}^+ = \{\lambda : \ \ Re\lambda > h, \ \ h > 0\}$ and $\widetilde{E}^- = \{\lambda : \ \ Re\lambda < -h, \ \ h > 0\}$.

Note that for sufficiently large values of the parameter λ, the characteristic determinant takes the form for $\lambda \in \ \widetilde{E}^+$: $\Delta(\lambda) = e^{\lambda \int_0^1 \sqrt{a(\eta) d\eta}} \left(e^{-2\lambda \int_0^1 \sqrt{a(\eta) d\eta}} - 1 + N_1(\lambda) \right)$, and for $\lambda \in \widetilde{E}^-$:

$\Delta(\lambda) = e^{-\lambda \int_0^1 \sqrt{a(\eta) d\eta}} \left(e^{2\lambda \int_0^1 \sqrt{a(\eta) d\eta}} - 1 + N_1(\lambda) \right)$. Consequently, for sufficiently large values of λ, the following inequalities can be established:

$$\left| e^{-\lambda \int_0^1 \sqrt{a(\eta) d\eta}} \Delta(\lambda) \right| \geq K_1 > 0 \text{ for } \lambda \in \ \widetilde{E}^+, \tag{2.23}$$

$$\left| e^{-\lambda \int_0^1 \sqrt{a(\eta) d\eta}} \Delta(\lambda) \right| \geq K_2 > 0, \text{ for } \lambda \in \ \widetilde{E}^-. \tag{2.24}$$

If we introduce the notation $K_0 = min\ \{K_1, \ \ K_2, \ K_\delta\}$, then we obtain the following estimate for the eigenvalues outside the δ -neighborhood $|\lambda| \geq K_0$.

We now estimate the Green's function outside the δ -neighborhood of the eigenvalue λ_k. From expression (2.14), it follows that for large values of $|\lambda|$ within the set E_1, the determinant $\Delta(x, \xi, \lambda)$ contains both exponentially increasing and exponentially decreasing terms.

As the parameter λ varies within the specified set, in order to partially eliminate the exponentially increasing terms in the first column of the determinant expression $\Delta(x, \xi, \lambda)$, we multiply the elements of the second and third columns of determinant (2.14), respectively, by $\left(1 + \frac{E_{02}(x,\lambda)}{\lambda}\right) \ e^{-\lambda \int_0^\xi \sqrt{a(\eta)} d\eta}$ and $\left(1 + \frac{E_1(x,\xi,\lambda)}{\lambda}\right) e^{\lambda \int_0^\xi \sqrt{a(\eta)} d\eta}$ and add the resulting expressions to the elements of the first column.After this transformation, the determinant $\Delta(x, \xi, \lambda)$ takes the following form:

$$\Delta(x, \xi, \lambda) = \frac{1}{4\lambda\sqrt{a(x)}} \times\times \begin{vmatrix} \left(1 + \frac{E_2(x,\xi,\lambda)}{\lambda}\right) e^{-\lambda \int_\xi^x \sqrt{a(\eta)} d\eta} & \left(1 + \frac{E_{01}(x,\lambda)}{\lambda}\right) e^{\lambda \int_0^x \sqrt{a(\eta)} d\eta} & \left(1 + \frac{E_{02}(x,\lambda)}{\lambda}\right) e^{-\lambda \int_0^x \sqrt{a(\eta)} d\eta} \\ \left(1 + \frac{E_1(0,\xi,\lambda)}{\lambda}\right) e^{\lambda \int_\xi^0 \sqrt{a(\eta)} d\eta} & 1 + \frac{E_{01}(0,\lambda)}{\lambda} & 1 + \frac{E_{02}(0,\lambda)}{\lambda} \\ \left(1 + \frac{E_1(1,\xi,\lambda)}{\lambda}\right) e^{-\lambda \int_\xi^1 \sqrt{a(\eta)} d\eta} & \left(1 + \frac{E_{01}(1,\lambda)}{\lambda}\right) e^{\lambda \int_0^1 \sqrt{a(\eta)} d\eta} & \left(1 + \frac{1}{\lambda} \cdot E_{02}(1,\lambda)\right) e^{-\lambda \int_0^1 \sqrt{a(\eta)} d\eta} \end{vmatrix}$$

If we expand the determinant $\Delta(x, \xi, \lambda)$ along the elements of the first row, we obtain the following expression:

$$\Delta(x, \xi, \lambda) = \frac{1}{4\lambda} \Big[\pm e^{-\lambda \int_\xi^x \sqrt{a(\eta)} d\eta} \cdot \Delta(\lambda) - y_1(x, \lambda) \left[(l_1(g_0) l_2(y_2) - l_2(g_0) l_1(y_2) \right] +$$

$$+ y_2(x, \lambda) \left[(l_1(g_0) l_2(y_1) - l_2(g_0) l_1(y_1) \right],$$

where $g_0(x,\xi,\lambda) = e^{-\lambda\int_\xi^x \sqrt{a(\eta)}d\eta}$, $l_1(g_0(x,\xi,\lambda)) = e^{\lambda\int_\xi^0 \sqrt{a(\eta)}d\eta}$, $l_2(g_0(x,\xi,\lambda)) = e^{-\lambda\int_\xi^1 \sqrt{a(\eta)}d\eta}$.

It follows from this that

$$\Delta(x,\xi,\lambda) = \frac{1}{4\lambda\sqrt{a(x)}}\left[\pm e^{-\lambda\int_\xi^x \sqrt{a(\eta)}d\eta}\cdot\Delta(\lambda) - \left(1+\frac{E_{01}(x,\lambda)}{\lambda}\right)e^{\lambda\int_0^x \sqrt{a(\eta)}d\eta}\times\right.$$

$$\left[e^{\lambda\int_\xi^0 \sqrt{a(\eta)}d\eta}\left(1+\frac{1}{\lambda}\cdot E_{02}(1,\lambda)\right)e^{-\lambda\int_0^1 \sqrt{a(\eta)}d\eta} -\right.$$

$$\left.-\left(1+\frac{1}{\lambda}\cdot E_{02}(0,\lambda)\right)e^{-\lambda\int_\xi^1 \sqrt{a(\eta)}d\eta}\right] + \left(1+\frac{1}{\lambda}\cdot E_{02}(1,\lambda)\right)e^{-\lambda\int_0^x \sqrt{a(\eta)}d\eta}\times$$

$$\left[e^{\lambda\int_\xi^0 \sqrt{a(\eta)}d\eta}\left(1+\frac{E_{01}(1,\lambda)}{\lambda}\right)e^{\lambda\int_0^1 \sqrt{a(\eta)}d\eta} -\right.$$

$$\left.\left.-\left(1+\frac{1}{\lambda}\cdot E_{01}(0,\lambda)\right)e^{-\lambda\int_\xi^1 \sqrt{a(\eta)}d\eta}\right]\right].$$

If both sides of this equation are divided by $\Delta(\lambda)$, we obtain:

$$G(x,\xi,\lambda) = \frac{\Delta(x,\xi,\lambda)}{\Delta(\lambda)} = \pm e^{-\lambda\int_\xi^x \sqrt{a(\eta)}d\eta} + \frac{B_1}{\Delta(\lambda)}e^{-\lambda\left(\int_x^1 \sqrt{a(\eta)}d\eta + \int_0^\xi \sqrt{a(\eta)}d\eta\right)} +$$

$$+\frac{B_2}{\Delta(\lambda)}e^{\lambda\left(\int_0^x \sqrt{a(\eta)}d\eta - \int_\xi^1 \sqrt{a(\eta)}d\eta\right)} + \frac{B_3}{\Delta(\lambda)}e^{\lambda\left(\int_x^1 \sqrt{a(\eta)}d\eta + \int_\xi^0 \sqrt{a(\eta)}d\eta\right)} +$$

$$+\frac{B_4}{\Delta(\lambda)}e^{-\lambda\left(\int_0^x \sqrt{a(\eta)}d\eta + \int_\xi^1 \sqrt{a(\eta)}d\eta\right)} \quad (2.25)$$

Here

$$B_1(x,\lambda) = 1+\frac{E_1(x,\lambda)}{\lambda}, B_2(x,\lambda) = 1+\frac{E_2(x,\lambda)}{\lambda}, B_3(x,\lambda) =$$

$$= 1+\frac{E_3(x,\lambda)}{\lambda}, B_4(x,\lambda) = 1+\frac{E_4(x,\lambda)}{\lambda}.$$

Where the functions $E_i(x,\lambda), (i=\overline{1,4})$ are continuous and bounded at $\lambda\in\widetilde{E}$, $x\in[0,1]$. It is not difficult to show that the Green's function $G(x,\xi,\lambda)$ corresponding to problem (2.5),(2.6) is analytic throughout the complex λ plane, except for a countable set of values $\lambda=\lambda_k\,(k=0,\pm1,\pm2,...)$.

By using the expression for the Green's function from (2.25), as well as inequalities (2.23) and (2.24) for the characteristic determinant, it is straightforward to show that outside the δ -neighborhoods of the points λ_ν, the derivatives of the Green's function $G(x,\xi,\lambda)$ corresponding to problem (2.5),(2.6) satisfy the following estimates:

$$\left|\frac{\partial^k G(x,\xi,\lambda)}{\partial x^k}\right| \le c\ |\lambda|^{k-1}, (k=0,1,2),\ \ \lambda\in\widetilde{E}_1\cup\widetilde{E}^{\pm}. \quad (2.26)$$

It is clear that the solution to the Cauchy problem (2.7)-(2.8) has the following form:

$$z(t,x,\lambda) = \varphi(x)\cdot e^{-\int_0^t \frac{P_0(\tau)-\lambda^2}{P_1(\tau t)}dt}$$

That is,

$$z\left(t,x,\lambda\right)=\varphi\left(x\right)\cdot e^{-\int_0^t\frac{(b-\tau)(d-\tau)P_0(\tau)}{P_1(\tau)}dt}\cdot e^{\lambda^2t\left(bd-\frac{b+d}{2}t+\frac{t^2}{3}\right)}$$

The following lemma holds:

Lemma. Suppose that conditions $1^0,2^0$ are satisfied. Then, for all $t\in[t_0,T]$ (where $\forall\, t_0\in(0,T)$) , the following estimate holds:

$$Re\left(\int_0^t\frac{\lambda_k^2}{P_1\left(t\right)}d\tau\right)\leq -c|k|^2 \tag{2.27}$$

where $c>0$.

Proof.

$$Re\left(\int_0^t\frac{\lambda_k^2}{P_1\left(\tau\right)}d\tau\right)=Re\int_0^t\frac{\frac{(k\pi i)^2}{\left(\int_0^1\sqrt{a(\eta)}d\eta\right)^2}}{P_1\left(\tau\right)}dt=$$

$$=Re\left[\int_0^t\frac{-k^2\pi^2}{\left(\int_0^1\sqrt{a\left(\eta\right)}d\eta\right)^2}\cdot\left(b-\tau\right)\left(d-\tau\right)d\tau\right]$$

$$=\frac{-k^2\pi^2}{\left(\int_0^1\sqrt{a\left(\eta\right)}d\eta\right)^2}\cdot Re\left(\int_0^t\left(b-\tau\right)\left(d-\tau\right)d\tau\right)=$$

$$=\frac{-k^2\pi^2t}{\left(\int_0^1\sqrt{a\left(\eta\right)}d\eta\right)^2}\cdot Re\left(bd-\frac{b+d}{2}t+\frac{t^2}{3}\right)$$

It follows that, in order to satisfy the estimate of the form (2.27), the following condition must be fulfilled:

$Re\left(bd-\frac{b+d}{2}t+\frac{t^2}{3}\right)>0$ for all $t\in[t_0,T]$ (for $\forall\, t_0\in(0,T)$). It is clear that $t=\frac{3Re(b+d)}{4}$ is a stationary point of the function$Re\left(bd-\frac{b+d}{2}t+\frac{t^2}{3}\right)$. This function attains its minimum value at this point. It follows that, in order for the inequality $Re\left(bd-\frac{b+d}{2}t+\frac{t^2}{3}\right)>0$ to hold for all $t\in[t_0,T]$, it is necessary that the condition $Re\left(16bd-3(b+d)^2\right)>0$ is satisfied.

Taking into account conditions $1^0,2^0$, it follows that:

$$Re\left(\int_0^t\frac{\lambda_k^2}{P_1\left(t\right)}dt\right)=\frac{-k^2\pi^2t}{\left(\int_0^1\sqrt{a\left(\eta\right)}d\eta\right)^2}\cdot Re\left(bd-\frac{b+d}{2}t+\frac{t^2}{3}\right)\leq -c\cdot|k|^2,$$

For $0<t_0\leq t\leq T$, where $c=\left(\frac{\pi}{\int_0^1\sqrt{a(\eta)}d\eta}\right)^2\cdot\frac{3}{64}Re\left(16bd-3\left(b+d\right)^2\right)$.

The lemma is proved

The following theorem holds:

Theorem 1. *Suppose that conditions* $1^0,2^0,3^0,4^0$. *are satisfied. If problem* (2.1)–(2.3) *admits a classical solution* $u(t,x)\in C^{1,2}\left((0,T]\times[0,1]\right)\cap C\left([0,T]\times[0,1]\right)$, then this solution is given by the following formula (for $t>0$).

$$U\left(t,x\right)=-\sum_{k=1}^{\infty}\underset{\lambda_k}{res}\,e^{\int_0^t\frac{P_0(\tau)-\lambda^2}{P_1(\tau)}d\tau}\cdot\int_0^1G\left(x,\xi,\lambda\right)\phi\left(\xi\right)d\xi \tag{2.28}$$

Proof: Let $u\,(t, x)$ be a classical solution to the mixed problem (2.1)–(2.3). We will show that, under the conditions of the theorem, this solution can be represented in the form (2.28).

Let χ_ν denote the multiplicity of the eigenvalue λ_ν of the spectral problem (2.5)–(2.6). In accordance with the general scheme described in [18], we introduce the following linear operators.

$$f_{\nu s}\,(x) = A_{\nu s} f\,(x) = \underset{\lambda_\nu}{res}\,\lambda^{1+2s} \int_0^1 G\,(x, \xi, \lambda) f\,(\xi)\, d\xi, \tag{2.29}$$
$$s = 0, \chi_\nu - 1; \nu = 1, 2, ...$$

Taking into account that the spectral problem (2.1)–(2.3) is regular and the function $f\,(x)$ has continuous derivatives ($f\,(x) \in C^2\,[0, 1]$, $f\,(0) = f\,(1) = 0$) on the interval [0, 1], the following expansion formula holds for every $x \in (0,\ 1)$ (see [16], [18]):

$$-\frac{1}{2\pi\sqrt{-1}} \sum_k \int_{c_k} \lambda d\lambda \cdot \int_0^1 G\,(x, \xi, \lambda)\, f\,(\xi)\, d\xi = f\,(x),$$

Here, c_k denotes a simple closed contour enclosing only one pole λ_k of the integrand function, and the summation over k extends over all poles. According to this expansion formula [16], [18] for functions $f\,(x)$ ($f\,(x) \in C^2\,[0, 1]$, $f\,(0) = f\,(1) = 0$), we have:

$$-\sum_{\nu=1}^{\infty} f_{\nu 0}\,(x) = f\,(x),\ x \in (0, 1)\,. \tag{2.30}$$

Applying operators (2.29) to equation (2.1), we obtain

$$\underset{\lambda_\nu}{res}\,\lambda^{1+2s} \int_0^1 G\,(x, \xi, \lambda) M \left(t, \frac{\partial}{\partial t}\right) U\,(t, \xi)\, d\xi \equiv$$
$$\equiv \underset{\lambda_\nu}{res}\,\lambda^{1+2s} \int_0^1 G\,(x, \xi, \lambda) L \left(\xi, \frac{\partial}{\partial \xi}\right) U\,(t, \xi)\, d\xi \tag{2.31}$$

By virtue of notation (2.29), we have:

$$\underset{\lambda_\nu}{res}\,\lambda^{1+2s} \int_0^1 G\,(x, \xi, \lambda) M \left(t, \frac{\partial}{\partial t}\right) U\,(t, \xi)\, d\xi \equiv$$
$$\equiv M \left(t, \frac{\partial}{\partial t}\right) \underset{\lambda_\nu}{res}\,\lambda^{1+2s} \int_0^1 G\,(x, \xi, \lambda) U\,(t, \xi)\, d\xi \equiv M \left(t, \frac{\partial}{\partial t}\right) U_{\nu s}\,(t, x)\,. \tag{2.32}$$

On the other hand, due to the evident equality

$$\int_0^1 G\,(x, \xi, \lambda)\, L \left(\xi, \frac{\partial}{\partial \xi}\right) \phi\,(\xi)\, d\xi = \phi\,(x) + \lambda^2 \int_0^1 G\,(x, \xi, \lambda) \phi\,(\xi)\, d\xi,$$

holds for any function $\phi\,(x)$, satisfying the condition 3^0 (as does the classical solution $U\,(t, x)$),). Taking into account notation (2.29), we obtain:

$$\underset{\xi,\lambda}{res}\,\lambda^{1+2s} \int_0^1 G\,(x, \xi, \lambda) L \left(\xi, \frac{\partial}{\partial \xi}\right) U\,(t, \xi)\, d\xi = \underset{\lambda_\nu}{res}\,\lambda^{1+2s} U\,(t, x) +$$
$$+ \underset{\lambda_\nu}{res}\,\lambda^{1+2s} \int_0^1 G\,(x, \xi, \lambda) U\,(t, \xi)\, d\xi = U_{\nu, s+1} \tag{2.33}$$

Taking into account (2.32) and (2.33) in (2.31), we obtain:

$$M\left(t,\frac{d}{dt}\right)U_{\nu s}\left(t,x\right)\equiv U_{\nu s+1}(s=0,1,...,\chi_\nu-1). \tag{2.34}$$

Similarly, applying operators (2.29) to the initial conditions (2.2), we obtain:

$$U_{\nu s}\left(0,x\right)=\phi_{\nu s}\left(x\right)\left(s=0,1,...,\chi_\nu-1\right). \tag{2.35}$$

Since χ_ν is the multiplicity of the eigenvalue λ_ν, the following equality holds:

$$\underset{\lambda_\nu}{res}\,\lambda^{1+2s}\left(\lambda^2-\lambda_\nu^2\right)^{\chi_\nu}\int_0^1 G\left(x,\xi,\lambda\right)U\left(t,\xi\right)d\xi\equiv 0,$$

which means the following

$$\sum_{k=0}^{\chi_\nu} C_{\lambda_\nu}^k\left(-\chi_\nu^2\right)^{\chi_\nu-k}\underset{\lambda_\nu}{res}\,\lambda^{1+2k}\left(\lambda^2-\lambda_\nu^2\right)^{\chi_\nu}\int_0^1 G\left(x,\xi,\lambda\right)U\left(t,\xi\right)d\xi\equiv 0.$$

Consequently, the following identity holds:

$$U_{\nu\chi_\nu}\left(t,x\right)=-\sum_{k=0}^{\chi_\nu-1}\left(-\lambda_\nu^2\right)^{\chi_\nu-k}C_{\chi_\nu}^k\cdot U_{\nu k}\left(t,x\right). \tag{2.36}$$

Taking into account (2.36), the identities (2.34) and (2.35) can be written as follows:

$$M\left(t,\frac{d}{dt}\right)V_\nu\equiv A_\nu V_\nu, \tag{2.37}$$

$$V_\nu\left(0,x\right)=\varphi_\nu\left(x\right), \tag{2.38}$$

where

$$V_\nu=V_\nu\left(t,x\right)=\begin{pmatrix} U_{\nu 0}\left(t,x\right)\\ U_{\nu 1}\left(t,x\right)\\ .\\ .\\ .\\ U_{\nu\chi_\nu-1}\left(t,x\right)\end{pmatrix}, \tag{2.39}$$

$$\varphi_\nu\left(x\right)=\begin{pmatrix} \phi_{\nu 0}\left(x\right)\\ \phi_{\nu 1}\left(x\right)\\ .\\ .\\ .\\ \phi_{\nu\chi_\nu-1}\left(x\right)\end{pmatrix},$$

$$A_\nu=\begin{pmatrix} 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ . & . & . & \dots & .\\ 0 & 0 & 0 & \dots & 1\\ -\left(-\lambda_\nu^2\right)^{\chi_\nu} & -\left(-\lambda_\nu^2\right)^{\chi_\nu-1}C_{\chi_\nu}^1 & -\left(-\lambda_\nu^2\right)^{\chi_\nu-2}C_{\chi_\nu}^2 & \dots & \lambda_\nu^2 C_{\chi_\nu}^{\chi_\nu-1}\end{pmatrix}.$$

From identities (2.37) and (2.38), we conclude that the vector function $V_\nu\left(t,x\right)$ is a solution to the Cauchy problem for a normal system of first-order ordinary differential equations. It is well known that such a problem has a unique solution. Consequently, the problem (2.34), (2.35) also has a unique solution. We will show that this solution is given by the following function:

$$U_{\nu s}(t,x) = -\underset{\lambda_\nu}{res}\lambda^{1+2s} \cdot e^{\int_0^t \frac{-P_0(\tau)+\lambda_\nu^2}{P_1(\tau)}} \cdot \int_0^1 G(x,\xi,\lambda)\,\phi(\xi)\,d\xi. \tag{2.40}$$

Indeed,

$$M\left(t,\frac{d}{dt}\right)U_{\nu s}(t,x) = -\underset{\lambda_\nu}{res}\lambda^{1+2s} \cdot e^{\int_0^t \frac{-P_0(\tau)+\lambda_\nu^2}{P_1(\tau)}} \cdot \int_0^1 G(x,\xi,\lambda)\,\phi(\xi)\,d\xi = U_{\nu s+1}(t,x,\varepsilon).$$

$$U_{\nu s}(0,x) = -\underset{\lambda_\nu}{res}\lambda^{1+2s} \cdot \int_0^1 G(x,\xi,\lambda)\,\phi(\xi)\,d\xi = \phi_{\nu s}(\xi).$$

Taking into account relations (2.30) and (2.40), the solution to problem (2.1)-(2.3) can be represented as:

$$U(t,x) = \sum_{\nu=1}^{\infty} U_{\nu 0}(t,x) =$$

$$= -\sum_{\nu=1}^{\infty} \underset{\lambda_\nu}{res} \cdot e^{\int_0^t \frac{-P_0(\tau)+\lambda_\nu^2}{P_1(\tau)}} \cdot \int_0^1 G(x,\xi,\lambda)\phi(\xi)\,d\xi. \tag{2.41}$$

Thus, we have proven that if the condition is satisfied and problem (2.1)-(2.3) admits a classical solution, then it can be represented in the form (2.41). From this, the validity of the following theorem directly follows.

Theorem 2. Under the conditions of the preceding theorem, if problem (2.1)-(2.3) admits a classical solution $u(t,x) \in C^{1,2}((0,T] \times [0,1]) \cap C([0,T] \times [0,1])$, then this solution is unique.

It is straightforward to show that under condition 3^0, the function defined by formulas (2.28) is a formal solution to the mixed problem (2.1)-(2.3). Therefore, to prove the theorem, it suffices to justify the validity of interchanging the operations $\frac{\partial}{\partial t}$, $\frac{\partial^2}{\partial x^2}$, $t \to +0$, $x \to +0$, $x \to 1-0$ under the improper integral sign in (2.28). For this purpose, it is enough to demonstrate the uniform convergence (on the corresponding sets) of the improper integrals obtained after applying these operations.

Let $\tau > 0$ be an arbitrary constant. Then, from estimate (2.27) and the lemma for the integrand functions obtained by formally differentiating the integral (2.28) once with respect to t and k $(k = 0,1,2)$ times with respect to x, for $0 \le x \le 1$, $t \ge \tau$, $\lambda \in S_i$, $i = 1,2$ and sufficiently large $|\lambda|$, we obtain the following estimates:

$$\left|\lambda e^{-\int_0^t \frac{P_0(\tau)-\lambda^2}{P_1(t)}} \cdot \frac{d^k y(x,\lambda)}{dx^k}\right| \le c|\lambda|^{k-1}e^{-c|\lambda|^2} \le \frac{c_1}{|\lambda|^2},\ (k = 0,\ 1,\ 2)$$

$$\left|\lambda \cdot \frac{P_0(t)-\lambda^2}{P_1(t)} e^{-\int_0^t \frac{P_0(\tau)-\lambda^2}{P_1(\tau)}} \cdot y(x,\lambda)\right| \le \tilde{c} \cdot |\lambda|^2 e^{-c|\lambda|^2} \le \frac{c_2}{|\lambda|^2}, \tag{2.42}$$

Consequently, the mentioned integrals converge uniformly in any rectangle $[t,T]\times[0,1]$, where $0 < t < T$. This implies that for $t > 0$, $0 \le x \le 1$ the operations $\frac{\partial}{\partial t}$, $\frac{\partial^k}{\partial x^k}$ $(k = 0,1,2)$ can be interchanged with the integral sign in (2.28). Moreover, for the function $U(t,x)$ defined by formula (2.28), we have:

$$u(t,x) \in C^{1,2}((0,T] \times [0,1]).$$

From the validity of the first estimate in (2.42) as $k = 0$, $0 \le x \le 1$ and $x \to +0$, $x \to 1-0$ can also be interchanged with the integral sign in (2.28). The theorem is thus proven.

3 Conclusion

This work deals with the existence and uniqueness of the solution to the mixed problem for a class of equations with complex-valued coefficients. Although initially parabolic, these equations may transition over time to Schrödinger or even antiparabolic type.

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